Musical Rhythms: a mathematical investigation

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LA TROBE UNIVERSITY

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AMSI AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

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Outline

melody $=$ rhythm $+$ pitch

This is not actually the mathematical part.

melody = rhythm + pitch

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Rhythm

Rhythm: a regular, recurring motion. Some pattern of beats, to be repeated.

Ostinato (Beethoven, Symphony No. 7, 2nd movement.)

- Rhythm can form the underlying structure over which a melody (with its own rhythm) sits.
  - (Such as the bass line and also the time signature.)
- This often gives a second layer of rhythm: rhythm within rhythm.
  - (Emphasis at start of phrasing.)

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Onset rhythm: a simplification, for mathematical study

- We record only the onset of a beat.
- There is an underlying “timespan” of possible beats.
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Example. The *paradiddle* as a series of pulses.

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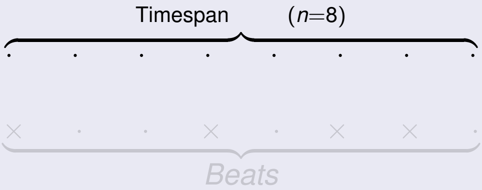

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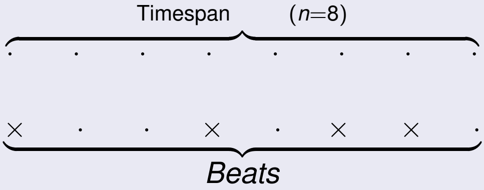

Rhythm abstraction

The paradiddle again

$$\times \cdot \cdot \times \cdot \times \times \cdot$$

We could also think of this as:

- As a subset of $\{0,1,2,3,4,5,6,7\}$: $\{0,3,5,6\}$
- As a sequence of lengths: $3+2+1+2$
- As a binary sequence: 10010110 (or *LRRLRLLR*?)

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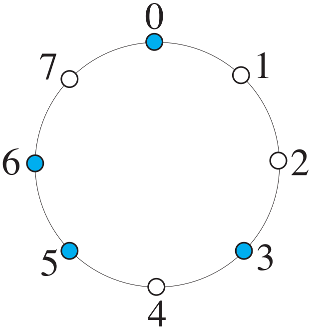

Represent the beats as coloured beads on a necklace.

Rhythm abstraction

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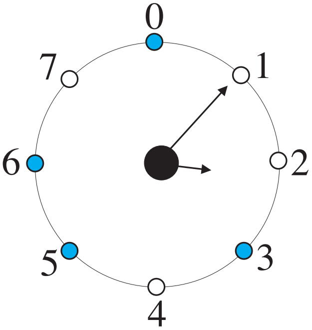
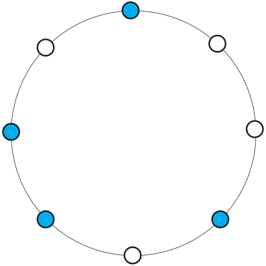


Represent the beats as coloured beads on a necklace. Or on a "clockface": here with only 8 hours!

Rhythm abstraction

The paradiddle again

× · · × · × × ·

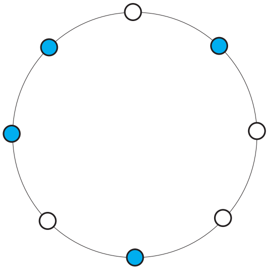


We could even forget about the numbers (and the clock hands!)

Rhythm abstraction

The paradiddle again

× · · × · × × ·

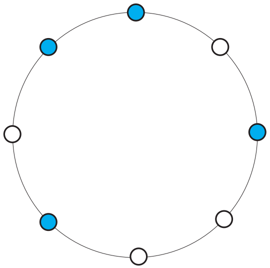


Here it is syncopated by one beat (shifted). This corresponds to adding 1 “hour” on the clock-face.

Rhythm abstraction

The paradiddle again

× · · × · × × ·



Here it is syncopated by two beats.

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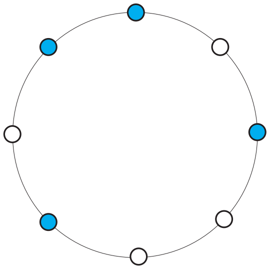
$\times \cdot \cdot \times \cdot \times \times \cdot$



We'll consider two rhythms as the same if they correspond to the same necklace: if they agree up to syncopation.

Outline

The number of rhythms

There are a lot of rhythms, even allowing for our simplifications.

No. of beats:	1	2	3	4	5	6	7	8
16	1	8	35	116	273	1505	715	810

Burnside counting. These numbers can be obtained by analysing the symmetries of the necklace and counting the number of beat placements that are unchanged by each symmetry.

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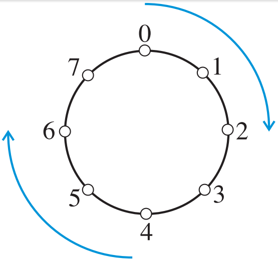
An example symmetry: rotate clockwise by $1/4$



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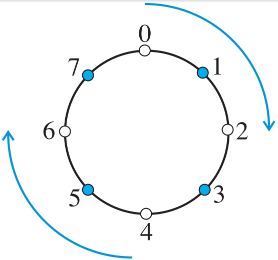
A beat placement fixed by this symmetry.



More counting

Asymmetric

A rhythm is *asymmetric* if it cannot be split into two equal length patterns, each with an onset on the first position. (Generalises.)

Cyclic shifts of the paradiddle with pulse on first beat

×	·	·	×	·	×	×	·
×	·	×	×	·	×	·	·
×	·	×	·	·	×	·	×
×	·	·	×	·	×	×	·

The paradiddle is asymmetric

Assymmetry is a common feature of exotic rhythms in world music. Assymmetric rhythms are inherently syncopated.

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Assymmetry is a common feature of exotic rhythms in world music. Assymmetric rhythms are inherently syncopated.

Counting asymmetric rhythms and other restricted forms of rhythm is a substantially harder task. This and other properties are examined in the article

> Assymmetric rhythms and Tiling cannons, by
>
> R.W. Hall and P. Klingsberg, 113 *American Mathematical Monthly* (2006), 887–896.

Outline

Spacing beats evenly

Polyrhythm

Play a onsets over the top of b onsets.

- Find lowest common multiple m of a and b. “Stretch” the timespan to m, and merge the two patterns

Example 3 against 5

Lowest common multiple is 15.

5’s	:×	·	·	·	·	×	·	·	·	·	×	·	·	·	·
3’s	:×	·	·	×	·	·	×	·	·	×	·	·	×	·	·
Both:	×	·	·	×	·	×	×	·	·	×	×	·	×	·	·

- Interesting, widely used effect.
- Mathematical, but not so deep, mathematically.

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Spacing beats in some time span

What if we want to stay within the original timespan?

- For some $k \leq n$, place k beats as *evenly as possible* but keeping to positions in the n beat timespan.
- Obviously easy if k divides n, but otherwise, there has to be some unevenness.

Example: 2 in 4

$\times \quad \cdot \quad \times \quad \cdot$ (When repeated, the "same" as just $\times\ \cdot$)

Example 2 in 5?

Choices:

$\times \quad \cdot \quad \cdot \quad \times \quad \cdot$

$\times \quad \times \quad \cdot \quad \cdot \quad \cdot$

Intuitively, the first is more even?

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Spallation Neutron Source Accelerator

E. Bjorklund, from *The Theory of Rep-Rate Pattern Generation in the SNS Timing System*, Technical Report, Los Alamos USA (2003).
"The strategy of the SNS timing system is to distribute the timing patterns as evenly as possible over the 10-second (600 pulse) super-cycle." . . .
"The optimal pattern is not so obvious, however, when n=87"

Measures of evenness

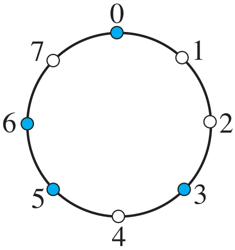
Classifying notions of evenness seems to have been one of the deeper theoretical tasks in the study of rhythms. The following article is arguably the most satisfactory culmination of the various approaches.

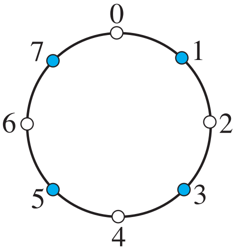
The distance geometry of music by E.D. Demaine, F. Gomez-Martin, H. Meijer, D. Rappaport, P. Taslakian, G.T. Toussaint, T. Winograd, D.R. Wood. *Computational Geometry: Theory and Applications* 42 (2009), 429–454.

Some measures of evenness

Pairwise geodesic distance sum

The geodesic distance between two points is the shortest path around the circle circumference between the points.





The sum of all pairwise geodesic distances in the left rhythm is 14, but it is 16 in the right. The right is more evenly spaced.

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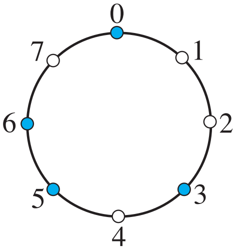

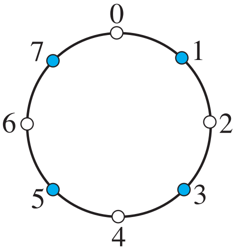

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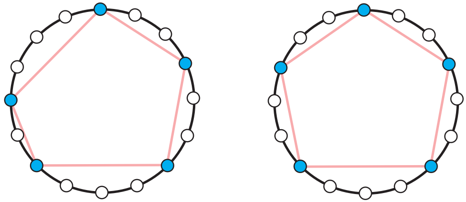


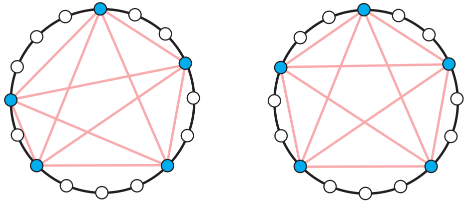
Two rhythms of 5 beats in 16.

The sum of all pairwise geodesic distances in both rhythms is 48.

Some measures of evenness

Pairwise chordal distance sum

The chordal distance between two points is the actual Euclidean distance between the points.



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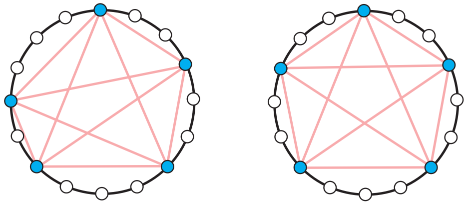

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A good result.

Theorem

- Demaine et al. show that for any $k \leq n$ there is a unique (up to cyclic shift) timespan n rhythm of k beats that maximises pairwise chordal distance sum.
- Also: several existing algorithms for producing evenly spaced rhythms actually achieve this uniquely spaced rhythm.

These "Euclidean rhythms" are very common amongst apparently complicated exotic rhythms found in world music.

A neat corollary

The reverse of a Euclidean rhythm is also a Euclidean rhythm.

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Euclidean algorithm

One of the oldest algorithmic processes (from around 300BC).

Finding greatest common divisor of n and k

- If $n = k$ then return k.
- Otherwise, subtract k from n to produce $n - k$.
- Repeat process for the smaller two of $k, n - k$.

Example: $n = 16, k = 6$.

$$\begin{aligned} 16 - 6 &= 10 \\ 10 - 6 &= 4 \\ 6 - 4 &= 2 \\ 4 - 2 &= 2 \end{aligned}$$

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Comparison between Euclid and the Bjorklund algorithm; $n = 16, k = 6$.

$16 - 6$	$=$	10	1,1,1,1,1,1	0,0,0,0,0,0,0,0,0,0
$10 - 6$	$=$	4	10, 10, 10, 10, 10, 10	0,0,0,0
$6 - 4$	$=$	2	100, 100, 100, 100,	10, 10
$4 - 2$	$=$	2	10010, 10010	100, 100
$2 - 2$	$=$	0	10010100 10010100	

The fact that this rhythm is simply a smaller rhythm played twice is because $\text{g.c.d}(16, 6) = 2$.

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$4 - 2$	$=$	2	10010, 10010	100, 100
$2 - 2$	$=$	0	10010100 10010100	

The fact that this rhythm is simply a smaller rhythm played twice is because $\mathrm{g.c.d}(16, 6) = 2$.

Bjorklund's algorithm

Comparison between Euclid and the Bjorklund algorithm; $n = 16, k = 6$.

$16 - 6$	$=$	10	1,1,1,1,1,1	0,0,0,0,0,0,0,0,0,0
$10 - 6$	$=$	4	10, 10, 10, 10, 10, 10	0,0,0,0
$6 - 4$	$=$	2	100, 100, 100, 100,	10, 10
$4 - 2$	$=$	2	10010, 10010	100, 100
$2 - 2$	$=$	0	10010100 10010100	

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$10 - 6$	$=$	4	10, 10, 10, 10, 10, 10	0,0,0,0
$6 - 4$	$=$	2	100, 100, 100, 100,	10, 10
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$10 - 6$	$=$	4	10, 10, 10, 10, 10, 10	0,0,0,0
$6 - 4$	$=$	2	100, 100, 100, 100,	10, 10
$4 - 2$	$=$	2	10010, 10010	100, 100
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Another approach

Spacings are also maximised by taking closest path walks in an integer lattice.

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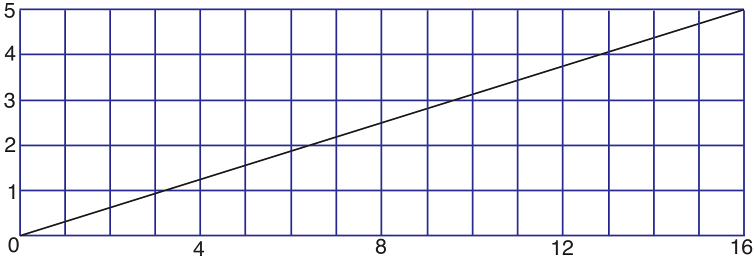

Rhythm: for 5 onsets out of 16.

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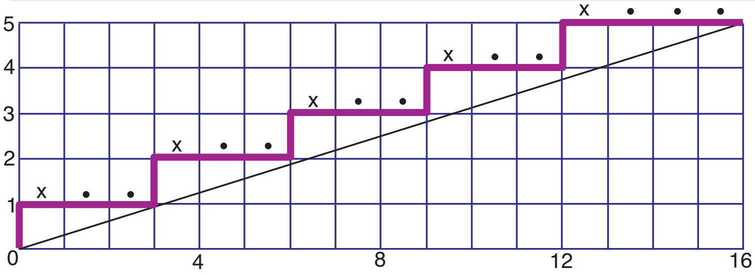

Rhythm: for 5 onsets out of 16.

Some easy cases.

2 in 3

× × ·

Hmmm.

3 in 4

× × × ·

Some easy cases.

2 in 3

× × ·

Hmmm.

3 in 4

× × × ·

Some easy cases.

2 in 5

× · · × ·

- Take 5 (Dave Brubeck quartet, 1961)

3 in 8 and 5 in 8

× · · × · · × ·

and × · × × · × × ·

- Rock and Roll Hound Dog; Elvis Presley version (1956).
- Metal Orion, by Metallica (1986).

The $3 + 3 + 2$ rhythm is very widely encountered in both world music and modern rock music derivatives.

Some easy cases.

2 in 5

× · · × ·

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3 in 8 and 5 in 8

× · · × · · × ·

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$\times \quad \cdot \quad \cdot \quad \times \quad \cdot \quad \cdot \quad \times \quad \cdot$

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The $3 + 3 + 2$ rhythm is very widely encountered in both world music and modern rock music derivatives.

Some other divisions of basic timespans

5 in 16

× · · × · · × · · × · · × · · ·

- The Bossa Nova clave rhythm Soul Bossa Nova Quincy Jones (1962).
- Bela Lugosi is Dead Bauhaus (1979). "Often considered to be the first gothic rock record released."
- Codex Radiohead (2011); piano (as $4 + 3 + 3 + 3$)

Against $2 + 2 + 2 + 2 + 2 + 2 + 2 + 2$:

× · · · × · · · × · · · × · · · × · ·

× · × · × · × · × · × · × · × · ·

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× · · × · · × · · × · · × · · ·

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× · · · × · · × · · × · · × · ·

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× · · · × · · × · · × · · × · ·

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× · · · × · · × · · × · · × · ·

× · × · × · × · × · × · × · × ·

Inevitably uneven divisions of nonstandard timespans

4 in 13

× · · × · · × · · × · · ·

- Golden Brown The Stranglers (1981).

4 in 11

× · · × · · × · · × ·

- Right in Two Tool (2006).

Inevitably uneven divisions of nonstandard timespans

4 in 13

× · · × · · × · · × · · ·

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4 in 11

× · · × · · × · · × ·

- Right in Two Tool (2006).

Even more challenging

World music examples offer even more outrageous combinations.

7 in 15

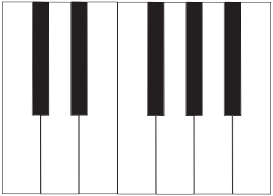
$\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot$

- Bucimis Traditional Bulgarian; here performed by Eblen Macari Trio (live 2003).

Even spacing in scales

7 note scales

Choose 7 notes, as evenly as possible from the 12 semitones.



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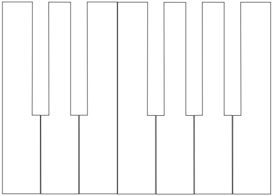

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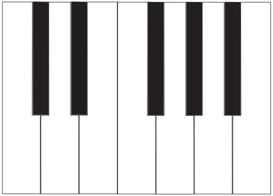

Even spacing in scales

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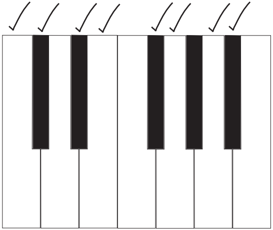
Choose 5 notes, as evenly as possible from the 12 semitones.

Pentatonic scales: precise keys depend on where you start.

Octatonic scales

Choosing 8 out of 12 notes

The octatonic scales correspond to equal spacing of 8 in 12; there are essentially two of them.



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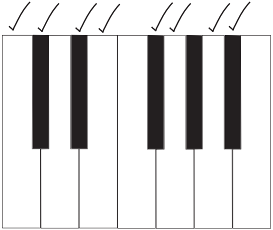

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Outline

A very familiar rhythm

365 + 365 + 365 + 366 + 365 + 365 + 365 + 366 + ...

⋯ + 365 + 365 + 365 + 366 + 365 + 365 + 365 + **365** + 365 + 365 + 365 + 366 + ...

A calendar strategy

- 365 days is a little bit short for the “real” year.
- Add an extra day if the approaching summer solstice would otherwise occur a calendar day later than usual.

This is an example of a “Sturmian word”; widely studied in the context of dynamical systems and combinatorics of infinite patterns. And closely related to the Euclidean rhythms.

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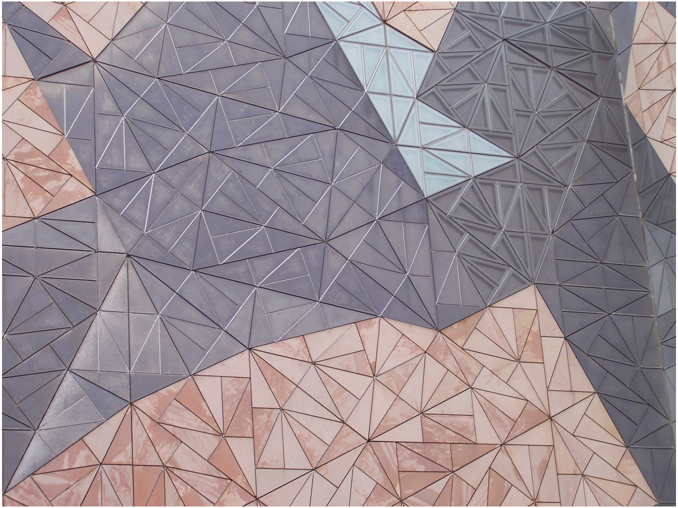
Infinite words are just 1-dimensional tilings



Desirable properties of infinite “rhythms”

1. Patterns that are heard are reheard at regular intervals.
2. There are relatively few patterns of a given length that occur.

A repeated finite rhythm has all these properties

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Example: a period 3 word

$110110110110110110110110110110\ldots$

- There are at most 3 different subwords of any finite length:
- Length one: $0, 1$
- Length two: $01, 10, 11$
- Length six: $110110, 101101, 011011$.

Desirable properties of infinite “rhythms”

1. Patterns that are heard are reheard at regular intervals.
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A repeated finite rhythm has all these properties

These are very important and widely studied mathematical

- Item 1 is essentially *uniform recurrence*: associated with minimal dynamical systems.
- Item 2 relates to the *entropy* of the dynamical system.

Patterns that are heard are reheard at regular intervals

Uniform recurrence

An infinite “word” is uniformly recurrent if for every number k there is a number n_k such that any length k pattern that appears, appears somewhere within every pattern of length n_k.

Example: the Thue-Morse sequence

Fixed point of $0 \mapsto 01$, $1 \mapsto 10$.

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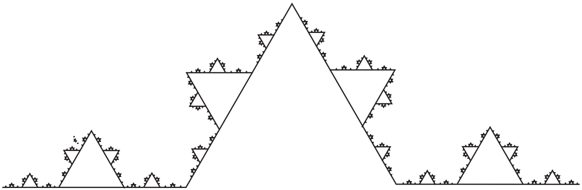
Thue-Morse sequence arises in:

number theory, combinatorics, dynamical systems, semigroup and group theory, chess and more!

Self similarity: The Koch curve



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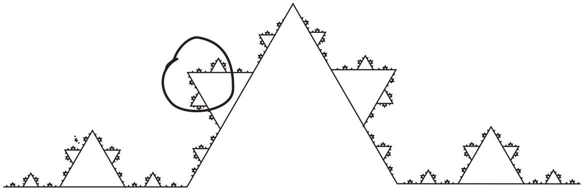

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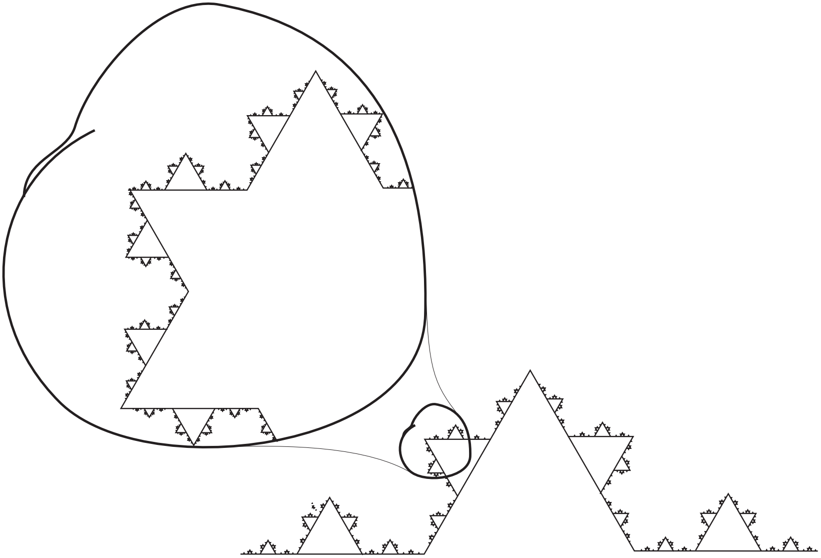

Self similarity in words: The Thue-Morse Sequence

011010011001011010010110011010011001011001101001 . . .

Self similarity in words: The Thue-Morse Sequence

$\underline{0}1\underline{1}0\underline{1}0\underline{0}1\underline{1}0\underline{0}1\underline{0}1\underline{1}0\underline{1}0\underline{0}1\underline{0}1\underline{1}0\underline{0}1\underline{1}0\underline{1}0\underline{0}1\underline{1}0\underline{0}1\underline{0}1\underline{1}0\underline{0}1\underline{1}0\underline{1}0\underline{0}1 \ldots$

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0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 1 0 0 1 0 1 1 0 . . .

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$$\overline{\underline{0}}1\underline{1}0\overline{\underline{1}}0\underline{0}1\overline{\underline{1}}0\underline{0}1\overline{\underline{0}}1\underline{1}0\overline{\underline{1}}0\underline{0}1\overline{\underline{0}}1\underline{1}0\overline{\underline{0}}1\underline{1}0\overline{\underline{1}}0\underline{0}1\overline{\underline{1}}0\underline{0}1\overline{\underline{0}}1\underline{1}0\overline{\underline{0}}1\underline{1}0\overline{\underline{1}}0\underline{0}1 \ldots$$

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$$\overline{\underline{0}} \quad \overline{\underline{1}} \quad \overline{\underline{1}} \quad \overline{\underline{0}} \quad \overline{\underline{1}} \quad \overline{\underline{0}} \quad \overline{\underline{0}} \quad \overline{\underline{1}} \quad \overline{\underline{1}} \quad \overline{\underline{0}} \quad \overline{\underline{0}} \quad \overline{\underline{1}} \quad \ldots$$

This sort of self-similarity property is common amongst words defined as fixed points of substitutions.

Paradiddle of parradiddle of paraddidle of. . .

Left paradiddle: *LRRLRLLR*

Right paradiddle: *RLLRLRRL*

A left paradiddle of parradiddles

$$\underbrace{LRRLRLLR}_{\text{Left}}\;\underbrace{RLLRLRRL}_{\text{Right}}\;\underbrace{RLLRLRRL}_{\text{Right}}\;\underbrace{LRRLRLLR}_{\text{Left}}\;\underbrace{RLLRLRRL}_{\text{Right}}\;\text{etc}$$

The Thue-Morse sequence is just the paradiddle of paradiddle of paradiddle of. . .

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Thue-Morse

Layered Thue-Morse rhythm

0								1							
0				1				1				0			
0		1		1		0		1		0		0		1	
0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0

Etcetera!

Sturmian words

Complexity

- In an eventually periodic infinite word, the number of length k subwords is at most the period length.
- Otherwise though, the number of subwords of length k in an infinite pattern is at least $k + 1$.

Words achieving this minimal nonbounded growth rate are called *Sturmian words*.

Sturmian words

Sturmian words are precisely those obtained by taking a line of *irrational slope* and approximating it by a walk in the integer lattice! They are also uniformly recurrent.

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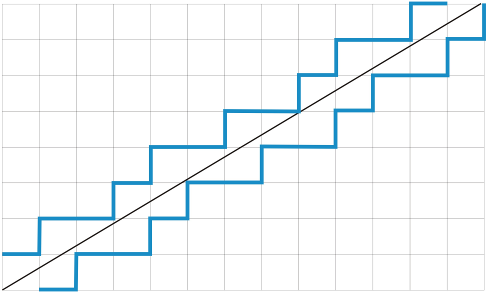
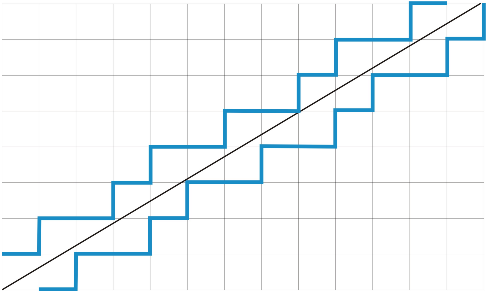
A line of golden ratio slope



Golden Ratio Rhythm

A line of golden ratio slope



Golden Ratio Rhythm

Paperfolding rhythm

Paperfolding word; aka Dragon Curve word

A remarkable uniformly recurrent fractal word can be obtained by folding paper.

- Keep folding to the right until you get bored.
- Unfold, and record the pattern of valleys and peaks amongst the creases.

1101100111001001110110001100100111011001100...

The Dragon's Rhythm sounds pretty good too!

The Dragon's Rhythm (Here over 4/4 time.)

This curve is the fixed point of the substitution $11 \mapsto 1101$, $10 \mapsto 1100$, $01 \mapsto 1001$, $00 \mapsto 1000$.

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A remarkable uniformly recurrent fractal word can be obtained by folding paper.

- Keep folding to the right until you get bored.
- Unfold, and record the pattern of valleys and peaks amongst the creases.

11011001110010011101100011001001110110011100...

The Dragon's Rhythm sounds pretty good too!

The Dragon's Rhythm (Here over 4/4 time.)

This curve is the fixed point of the substitution $11 \mapsto 1101$, $10 \mapsto 1100$, $01 \mapsto 1001$, $00 \mapsto 1000$.

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Some references

Aside from the two articles mentioned above, the following resources are excellent sources of information on infinite words.

- M. Lothaire, *Algebraic Combinatorics on Words*, Encyclopedia of Mathematics and its Applications, 90, Cambridge University Press 2002.
- Jean-Paul Allouche and Jeffrey Shallit, *Automatic Sequences*, Cambridge University Press 2003.
- Jean-Paul Allouche and Jeffrey Shallit The Ubiquitous Prouhet-Thue-Morse Sequence, in C. Ding. T. Helleseth, and H. Niederreiter, eds., *Sequences and Their Applications*: Proceedings of SETA '98, Springer-Verlag, 1999, pp. 1–16. (Search the web.)
- Both Wikipedia and Eric Weisstein's MathWorld (a Wolfram resource) also have a huge amount of information!