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A TRACTABLE ALTERNATIVE TO COBB-DOUGLAS UTILITY FOR IMPERFECT COMPETITION

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ABSTRACT—This paper proposes a tractable alternative to Cobb-Douglas utility to resolve the problems of lack of reservation price and income effects in demand functions derived from Cobb-Douglas utility or quasilinear utility. Another advantage of this alternative is that it provides a closed-form general equilibrium in the case where some industries are monopoly while others are imperfectly competitive.

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KEY WORKS: Utility Function, Reservation Price, Income Effect, Monopoly, Oligopoly

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I. INTRODUCTION

Cobb-Douglas utility is one of most widely applied utility functions in economics. Its simple and elegant functional form enables economists to obtain closed-form solutions in a wide range of economic analysis. However, the simplicity does not come without costs. A major drawback is the lack of reservation price; that is, consumers with Cobb-Douglas utility spend a fixed proportion of their income on a commodity or a service, independent of the price of the commodity. Thus, they always purchase a positive amount of the commodity at whatever high price, although the quantity purchased tends to zero when the price tends to infinity. This obviously contradicts to commonly observed consumer behavior. An immediate result from this drawback is that the demand function drawn from a Cobb-Douglas utility function is not suitable for the analysis of monopoly, because the fixed expenditure makes a profit-maximizing monopolist set an infinitely high price and produce an infinitesimal amount of output. Applying the Cobb-Douglas utility to a multi-industry general equilibrium model, monopoly or cartelization must be excluded in all industries.

To overcome this flaw in reservation price, economists usually turn to linear demand functions in dealing with monopoly. However, a linear demand structure can only be derived from quasilinear utility (see Singh and Vives (1984)). Since a quasilinear utility function results in a demand function without income effects, linear demand functions cannot be used to study any issues relevant to consumer income and
micro-macroeconomic interaction, and consequently are not suitable for general equilibrium analysis.

After Dixit and Stiglitz’s (1977) seminal contribution, analysts become more delighted to use CES utility functions in general equilibrium analysis, especially in new growth models.\(^1\) Since the consumption of each commodity enters the utility function symmetrically, it yields constant elasticity of substitution, as the name of the utility function suggested. The CES utility brings tractable analytical form on the one hand but it excludes more sophisticated industry structure and asymmetry on the other hand. Usually, it is assumed that each commodity is supplied by one firm and all firms in the economy are identical and engage in monopolistic competition. Of course, it is possible to consider that each commodity is supplied by an industry and industry structures across industries are different. However, such a setting will result in a model hardly tractable because the closed-form equilibrium with CES utility requires all industries are symmetric.

It is clear that these most-widely used utility functions have their advantages. But they fail to provide a tractable framework for general equilibrium analysis with asymmetric industries such as a monopoly industry along with other imperfectly competitive industries. This paper suggests an alternate utility to Cobb-Douglas to resolve the problem. At a partial equilibrium level, it produces a demand structure accommodating reservation price as well as income effects. For a general equilibrium analysis, it provides a tractable framework containing different industry structures.

\section*{II. An Alternative to Cobb-Douglas Utility}

The suggested utility function is in the form\(^2\):
In this utility function, the subscript $i$ indexes the category of commodities, and $Q_i$ is the consumption quantity of commodity $i$. The number of commodities can be either finite or infinite. This utility collapse to the conventional Cobb-Douglas utility function when $a_i = 0$. On the other hand, it becomes the Stone-Geary utility function if we assume that $a_i < 0$ (see Jehle (1991), p. 211). From demand function (4) below, it is not hard to find that the indirect utility derived from (1) leads to a special case of the Gorman form (see Varian (1992), p.153). Taking partial derivative of utility (1) with respect to $Q_i$ to obtain marginal utility,

$$MU_i = ub_i(Q_i + a_i)^{-1}.$$

The marginal utility is finite when a consumer consumes an infinitesimal amount of a commodity, which contradicts to the Cobb-Douglas utility. It is the finiteness of marginal utility that makes a consumer have a finite reservation price. Setting $Q_i$ in the above equation equal to zero to obtain

$$b_i/a_i = MU_i/u,$$

which means that the ratio of two utility parameters, $b_i$ and $a_i$, is equal to the ratio of marginal utility to utility.

Assume a consumer has income $I$ so that she faces the budget constraint

$$\sum_i P_iQ_i = I,$$

where $P_i$ is the market price of commodity $i$.

**Proposition 1.** With utility function (1), a consumer facing budget constraint (3) has the demand function below,
\[ Q_i = (P + I) b_i P_i^{-1} - a_i = (P_{-i} + I) b_i P_i^{-1} - (1 - b_i) a_i, \]  

where the price indexes are defined as

\[ P \equiv \sum_i a_i P_i, \quad P_{-i} \equiv \sum_{k \neq i} a_k P_k. \]  

**Proof:** Define Lagrange function

\[ \mathcal{Z} = \Pi_i (Q_i + a_i) b_i - \lambda \left( \sum_i P_i Q_i - I \right). \]

Its partial derivative with respect to \( Q_i \) provides the first-order condition that

\[ \Pi_i (Q_i + a_i) b_i (Q_i + a_i)^{-1} = \lambda P_i. \]  

Multiplying both sides of equation (6) by \( Q_i \) and summing up across \( i \) yield

\[ \Pi_i (Q_i + a_i) b_i \sum_i b_i Q_i (Q_i + a_i)^{-1} = \lambda \sum_i P_i Q_i = \lambda I. \]  

Multiplying both sides of equation (6) by \( a_i \) and summing up across \( i \) yield

\[ \Pi_i (Q_i + a_i) b_i a_i (Q_i + a_i)^{-1} = \lambda \sum_i a_i P_i = \lambda P. \]  

Substituting (7) into (6) to eliminate \( \lambda \) leads to

\[ b_i (Q_i + a_i)^{-1} = I^{-1} P \sum_i b_i Q_i (Q_i + a_i)^{-1} = I^{-1} P \left( \frac{1}{1 - \sum_k a_k b_k (Q_k + a_k)^{-1}} \right). \]  

Using (8) to eliminate \( \lambda \) in (6) we get

\[ \sum_k a_k b_k (Q_k + a_k)^{-1} = PP_{-i}^{-1} b_i (Q_i + a_i)^{-1}. \]

Substituting it into (9), (4) is immediate after routine calculation.

Given the prices of other commodities, demand function (4) shows that the consumer has a reservation price of good \( i \) that

\[ P_{r_i} = (P_{-i} + I)a_i b_i (1 - b_i)^{-1}. \]
When the market price of a commodity is higher than that, the demand for it shrinks to zero. In the determination of reservation price the taste parameter $a_i$ is critical. The smaller is $a_i$, the higher is the reservation price. When $a_i$ tends to zero, the reservation price tends to infinity, which is a characteristic of a Cobb-Douglas model. Recalling the definition of $P_i$ and $P$, the reservation can also be written as

$$P_{ri} = (P + I)a_i^{-1}b_i.$$  \hspace{1cm} (10')

Noting $P \equiv \sum a_i P_i$, i.e., $P$ is a combination of the price vector with a consumer’s preferences, the equation implies that a consumer’s reservation price is determined by her income, preferences and market prices. From (2) and (10'), we find that $P_{ri} / (P + I) = MU_{i/} u$ when $Q_i = 0$.

The own price elasticity, cross price elasticity and income elasticity of demand (4) are, respectively, equal to

$$e_i = -(P + I) b_i (P_i Q_i)^{-1} < -1, \quad e_{ik} = a_i b_i P_k (Q_i P_i)^{-1}, \quad e_{id} = b_i I (Q_i P_i)^{-1}.$$  \hspace{1cm} (10')

The absolute values of these elasticities tend to infinity when the consumption tends to zero.

The aggregate demand function can be easily obtained through integral when the distribution of consumer tastes (i.e., the distributions of $a_i$ and $b_i$) and the distribution of income are known. Here we consider homogenous consumers lonely; that is, they have identical preferences and income. Moreover, the population is normalized to unity so that the demand function (4) is also the aggregate demand function.

For a monopolistic firm that is the only suppler of commodity $i$, it maximizes profits

$$\pi = P_i Q_i - C(Q_i),$$
where $C(Q_i)$ is the cost of production. The conventional first-order condition of marginal revenue equal to marginal costs gives

$$P_i(1 + e_i^{-1}) = MC(Q_i).$$

(11)

Thus, the profit margin is equal to

$$[P_i - MC(Q_i)]P_i^{-1} = -e_i^{-1} = (P_{-i} + I)^{-1}b_i^{-1}P_iQ_i < 1.$$ 

(12)

For constant marginal cost; i.e., $MC(Q_i) ≡ c_i$, (11) provides a closed-form equilibrium.

**Proposition 2.** If $MC(Q_i) ≡ c_i$ and $P_{Ri} > c_i$, there is a unique monopoly equilibrium

$$Q_i = [(P_{-i} + I)(1 - b_i)a_ib_i^{-1}]^{1/2} - (1 - b_i)a_i,$$

(13)

$$P_i = [(P_{-i} + I)(1 - b_i)^{-1}a_i^{-1}b_i]^{1/2}.$$ 

(14)

**Proof:** Substituting $MR(Q_i) = (P_{-i} + I)(1 - b_i)a_i[Q_i + (1 - b_i)a_i]^{-2}$ into (11), we immediately obtain (13). Noting $MR(0) = P_{Ri} > c_i$, $\lim_{Q_i \to \infty} MR(Q_i) = 0$ and the monotonicity of $MR(Q_i)$, we find $Q_i$ is the only solution to (11). Substituting (13) into (4) and solving for $P_i$ yield (14).

Equations (12)-(14) show that as the marginal cost declines, not only equilibrium profits rise but also equilibrium revenue and profit margin increase. The monopoly output and price determined by (13) and (14) include the effects of aggregate income and price of other commodities on them. In a model of Cobb-Douglas utility or quasilinear utility, it is impossible to accommodate these effects simultaneously. Proposition 2 can be considered more dynamically. Suppose an entrepreneur brings a brand new product into a market through certain breakthroughs in product innovation.
However, as far as the marginal production cost is higher than the reservation price, the market will not accept the product. Only when the cost is lower than the reservation price, would the innovation be economically successful. This characteristic contradicts a demand structure with infinitely high reservation price, where any new product can succeed in marketplace no matter how high the production cost is.

III. A FURTHER EXTENSION

Of course, the model in the previous section is not necessarily limited to monopoly. Consider a situation that each commodity is supplied by an industry and each industry accommodates more than one firm. Assuming all firms produce a homogenous product, it is straightforward to extend the previous model to oligopoly or other imperfect competition forms. A more interesting case is that each firm produces one and only differentiated variety of a commodity. Each variety enters the utility function through a CES composition that

\[
Q_i = \left( \frac{\theta_i}{q_i} \sum_{j=1}^{n_i} q_{ij}^\theta \right)^{1/(1-\theta_i)}, \quad 0 < \theta_i < 1,
\]

where \( n_i \) is the number of varieties in commodity \( i \), \( q_{ij} \) is the consumption quantity of variety \( j \) and \( 1/(\theta_i - 1) \) is the elasticity of substitution between any pair of varieties in the same industry. Thus, (1) and (15) consist of a utility function of a range of commodities and each commodity has a number of close substitutes. Given the aggregate expenditure, \( I \), and the price of each product, \( p_{ij} \), the budget constraint becomes

\[
\sum_{i,j} p_{ij} q_{ij} = I.
\]

**Proposition 3.** A representative consumer with utility (1) and (15) and budget constraint (16) has a demand function
\[ q_{ij} = (P_i p_{ij}^{-1})^{\theta_i/(\theta_i - 1)} n_{ij}^{-1} Q_i, \]  
(17)
where industry price index \( P_i \) is defined as
\[ P_i \equiv \left( n_{ij}^{-1} \sum_{j=1}^{n_i} p_{ij}^{\theta_i/(\theta_i - 1)} \right)^{\theta_i - 1}/\theta_i, \]  
(18)
and industry demand \( Q_i \) is given by (4).

**Proof:** Define Lagrange function
\[ \mathcal{L}_i = \Pi \left[ \left( n_{ij}^{\theta_i - 1} \sum_{j=1}^{n_i} q_{ij}^{\theta_i} \right)^{\theta_i} + a_i \right] - \lambda_i \left( \sum_{i,j} p_{ij} q_{ij} - I \right). \]
Its partial derivative with respect to \( q_{ij} \) provides the first-order condition
\[ \Pi_j (Q_i + a_i)^{\theta_i} b_i (Q_i + a_i)^{-1} \left( n_{ij}^{\theta_i - 1} \sum_{j=1}^{n_i} q_{ij}^{\theta_i} \right)^{\theta_i - 1} n_{ij}^{\theta_i - 1} q_{ij}^{\theta_i - 1} = \lambda_i p_{ij}. \]  
(19)
Recalling (15), multiplying both sides of the equation by \( q_{ij} \) and summing up across \( i \) and \( j \) yield
\[ \Pi_j (Q_i + a_i)^{\theta_i} b_i (Q_i + a_i)^{-1} \left( n_{ij}^{\theta_i - 1} \sum_{j=1}^{n_i} q_{ij}^{\theta_i} \right)^{\theta_i - 1} n_{ij}^{\theta_i - 1} q_{ij}^{\theta_i - 1} = \lambda_i I. \]  
(20)
Taking a \( \theta_i/(\theta_i - 1) \)-root of both sides of (19) and summing up across \( j \), then taking a \((\theta_i - 1)/\theta_i \)-root and using (18) to obtain
\[ \Pi_j (Q_i + a_i)^{\theta_i} b_i (Q_i + a_i)^{-1} = \lambda_i p_{ij}. \]  
(21)
It is clear that (20) and (21) are identical to (6) and (7) so that they determine an industry demand function (4). Substituting (21) into (19) to eliminate \( \lambda_i \), we obtain (17).

An interesting feature of demand function (17) is that the demand for a variety is positive as far as the industry demand is positive. However, the industry demand has a reservation price of (10). By the definition, the industry price index is determined by
the price of each individual variety. If variety prices are very high, so is the industry price index and in turn the demand for each individual variety becomes zero.

Demand function (17) provides a flexible structure for either partial or general equilibrium analysis. We start with partial equilibrium of a particular industry \( i \), where the commodity prices of other industries and aggregate income are exogenous variables. Firms within the industry compete with each other to maximize their own profits

\[
\pi_j = p_j q_j - C_j(q_j) - f_j,
\]

where \( C_j(q_j) \) is the variable costs of firm \( j \) in industry \( i \) and \( f_j \) is the fixed costs. Although it is hard to figure out equilibrium in a closed-form with a general functional form of variable costs, a symmetric equilibrium with identical constant marginal cost across all firms within an industry can be plotted out in a closed-form.\(^5\)

**Proposition 4.** When firms in an industry have a common constant marginal cost \( c_j < P_{Ri} \), there exists a unique symmetric partial equilibrium

\[
P_i = \frac{1}{\xi_{i2}} (-\xi_{i2} - (\xi_{i2}^2 - 4\xi_{i1}\xi_{i3}))^{1/2} \xi_{i1}^{-1}, \quad \text{when } \xi_{i1} \neq 0, \quad (22)
\]

\[
P_i = -\xi_{i2}^{-1} \xi_{i1}, \quad \text{when } \xi_{i1} = 0, \quad (23)
\]

where

\[
\xi_{i1} \equiv (1 - b_i)a_i(\theta_i - n_i^{-1}),
\]

\[
\xi_{i2} \equiv [(P_{-i} + I)b_i\theta_i + (1 - b_i)a_ic_j](n_i^{-1} - 1),
\]

\[
\xi_{i3} \equiv (P_{-i} + I)b_ic_j(1 - \theta_i n_i^{-1}).
\]

**Proof:** From (4) and (17), we have

\[
\pi_j = n_i^{-1}(p_j - c_i)P_j^{-1/\theta_i}[(P_{-i} + I)b_iP_i^{\theta_i/\theta_i(1-\theta_i)} - (1 - b_i)a_iP_j^{\theta_i/(1-\theta_i)}] - f_i.
\]
Recalling (18), \( \frac{\partial \pi_y}{\partial p_y} = 0 \rightarrow \)

\[
q_y - n_i^{-1}(p_y - c_i)(1 - \Theta_i)^{-1} p_y^{-(1 - \Theta_i)^{-1}}[(P_i + I)b_i P_i^{0,/(1 - \Theta_i)}] + n_i^{-1}(p_y - c_i) p_y^{-(1 - \Theta_i)^{-1}}[(P_i + I)b_i P_i^{0,/(1 - \Theta_i)} - (1 - b_i) a_i P_i^{1/(1 - \Theta_i)}] = 0
\]

In symmetric equilibrium, \( p_y = P_i, \ q_y = q_i = n_i^{-1}Q_i = n_i^{-1}[(P_i + I)b_i P_i^{-1} - (1 - b_i) a_i] \).

Substituting them into the last equation and multiplying both sides of the equation by \((1 - \Theta_i)n_i P_i^{-2}\) yield

\[
\xi_{i1} P_i^{-2} + \xi_{i2} P_i + \xi_{i3} = 0. \tag{24}
\]

The solutions to the equation are

\[
P_i = \frac{1}{2}[-\xi_{i2} + (\xi_{i2}^2 - 4\xi_{i1}\xi_{i3})^{1/2}]\xi_{i1}^{-1} \text{ and } P_i = \frac{1}{2}[-\xi_{i2} - (\xi_{i2}^2 - 4\xi_{i1}\xi_{i3})^{1/2}]\xi_{i1}^{-1} \tag{25}
\]

Note, \( \xi_{i2} \leq 0, \ \xi_{i3} > 0 \) but the sign of \( \xi_{i1} \) is ambiguous. If \( \xi_{i1} > 0 \), the profit increases when \( p_y \) is small since \( \lim_{p_y \to 0} \frac{\partial \pi_y}{\partial p_y} \to +\infty \) so that the second solution in (25) determines the equilibrium price. If \( \xi_{i1} < 0 \), only the second solution is positive. If \( \xi_{i1} = 0 \), (24) leads to equilibrium price \( P_i = -\xi_{i2}^{-1}\xi_{i3} \).

It is interesting to check two special cases for the equilibrium determined by (22). For the monopoly case, \( n_i = 1 \), there are

\[
\xi_{i1} = (1 - b_i)a_i(\Theta_i - 1), \quad \xi_{i2} = 0, \quad \xi_{i3} = (P_i + I)b_i c_i(1 - \Theta_i).
\]

Thus, equation (22) collapse to (14). On the other hand, monopolistic competition is usually presumed to have many competitors in an industry so that \( n_i \gg 1 \). Neglecting terms of order \( n_i^{-1} \) in the definitions of \( \xi_{i1}, \xi_{i2} \) and \( \xi_{i3} \), we have

\[
\xi_{i1} = (1 - b_i)a_i\Theta_i, \quad \xi_{i2} = -(P_i + I)b_i \Theta_i + (1 - b_i) a_i c_i, \quad \xi_{i3} = (P_i + I)b_i c_i\]

Substituting them into (22) yields
In Proposition 4, the number of firms in an industry is exogenously given. To endogenize it and industry structure, we require a zero profit condition determining the number of firms in equilibrium,

\[ n_i = (1 - \theta_i)P_i Q_i / f_i = (1 - \theta_i)[(P_{-i} + I) b_i - (1 - b_i)a_i P_i ]/f_i . \]  

(27)

In this condition, \( P_i \) is given by (22) or (23).

Turning to general equilibrium, the aggregate income can be normalized to unity; i.e., \( I \equiv 1 \) (see chapter 2 of Grossman and Helpman (1991)) but all prices are endogenous variables. Since the price index of industry \( i \) in (22) or (23) is a function of price indexes of other industries, we need to pool equation (22) or (23) of each industry together and simultaneously solve them to obtain general equilibrium. Because (22) and (23) are nonlinear, it is impossible to calculate the closed-form equilibrium for a general setting of the model. However, for many interesting specifications, the model is analytically tractable. For instance, the model can be applied to study the interaction between a giant monopolist like Microsoft and the whole economy. In this case, we can consider one industry is monopoly and the others are monopolistically competitive. The general equilibrium then is given in the proposition below.

**Proposition 5.** If industry 1 is monopoly and other industries are monopolistically competitive, there exists a unique quasi-symmetric general equilibrium

\[ P_1 = [(\sum_{i \neq 1} a_i P_i + 1)(1 - b_1)^{-1} a_1^{-1} b_1 c_1 ]^{1/2}, \]  

(28)

\[ P_i = c_i / \theta_i , \quad i \neq 1 \]  

(29)

and equilibrium output for each industry can be obtained by substituting (28) and (29) into (4).
Proof: (29) is immediate from (26). (28) is derived from (14) by substituting $I = 1$ and

\[ P_{-1} = \sum_{i \neq 1} a_i P_i. \]

IV. CONCLUDING REMARKS

This paper proposes a tractable alternative to Cobb-Douglas utility. The demand function derived from this utility function can resolve the problems of missing reservation price or income effects in other commonly applied demand functions. It has a particularly advantage in the study of the interaction between an industry and the whole economy. The model presented in Proposition 5 is only a simple example illustrating such an advantage in general equilibrium analysis. The great potential of the utility function remains to be explored in further research. For instance, the model of Proposition 5 can be easily extended to investigate the dynamics of economic development, where a new monopolist emerges through the invention of a drastically new product. When the technology is matured and/or the patent expires, more firms enter the industry and it becomes monopolistically competitive. Then another new technology breakthrough creates a newer industry… The economy develops through the creation of new industry and the evolution of industries. In comparison with existing growth models, such a model can accommodate more sophisticated microeconomic specification and provide more room for the analysis of economic evolution.
REFERENCES


2 The assumption that $\sum b_i = 1$ is not necessary but it can simplify notations.

3 For the case of infinite categories, it is better to consider that $i$ is distributed on $[0, 1]$ and use the utility form $u = \int_0^1 b_i \log(q_i + a_i)di$.

4 The author would like to thank an anonymous referee for indicating this.

5 Symmetric equilibrium means that firms in an industry charge the same price for their products. However, the prices across industries can be different.

6 Quasi-symmetry is in the sense that firms with an industry set the same equilibrium price, but the prices index for each industry can differ.