

Construction of Integrals of Higher-Order Mappings

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We find that certain higher-order mappings arise as reductions of the integrable discrete A-type KP (AKP) and B-type KP (BKP) equations. We find conservation laws for the AKP and BKP equations, then we use these conservation laws to derive integrals of the associated reduced maps.

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The search for discrete integrable systems has received considerable attention in the past decade. This has resulted in the discovery of integrable mappings of the second order, e.g., the Quispel–Roberts–Thompson (QRT) mapping,¹⁾ and discrete Painlevé equations.²⁾ Apart from second-order integrable mappings, the results for higher-order integrable mappings are few.^{3–11)} Discrete integrable systems have applications to various areas of physics, such as statistical mechanics, quantum gravity, and discrete analogues of integrable systems in classical mechanics and solid-state physics. Here, we study a novel class of higher-order integrable mappings that have bilinear forms.

As an example, we discuss the following 6th-order mapping:

$$Dx_{n+3}x_{n+2}^2x_{n+1}^3x_n^3x_{n-1}^2x_{n-2}^2x_{n-3} + Ax_{n+2}x_{n+1}^2x_n^2x_{n-1}^2x_{n-2} + Bx_{n+1}x_nx_{n-1} + C = 0. \quad (1)$$

(Here and below A, B, C, D are arbitrary parameters). How can we obtain integrals for this mapping? In ref. 3, a method for the construction of integrals was proposed and integrable third-order mappings that possess two integrals were obtained. However, this method is not applicable to higher-order mappings because this method uses some ansatz at first and requires the use of high-performance computers. If we consider 6th-order, 8th-order, and higher-order mappings, this method does not work, because current computer power is not sufficient.¹²⁾

In this Letter, we propose a systematic method of constructing integrals for a class of higher-order integrable mappings that can be transformed into a single bilinear form without the use of computers. Our proposed method is based on discrete bilinear forms related to the A-type KP (AKP) and B-type KP (BKP) soliton equations. Conservation laws for integrable partial difference equations have been studied

in refs. 13 and 14.

Before we discuss conservation laws for discrete systems, let us briefly recall conservation laws for continuous systems.¹⁵⁾ To this end, consider a (scalar) partial differential equation (PDE) $\Delta[x, u^{(i)}] = 0$. The conservation law of such a PDE is a divergence expression

$$\sum_j \frac{\partial P_j}{\partial x_j} = 0,$$

which vanishes for all solutions of the given system. It follows that there exists a function Λ (called the *characteristic* of the given conservation law) such that $\sum_j \partial P_j / \partial x_j = \Lambda \Delta$. Similarly, the conservation law of a scalar partial difference equation $\Delta[n, u_n] = 0$ is the expression $\sum_j (S_j - id)P_j = 0$, which vanishes for all solutions of the discrete system. (Here S_j is a unit shift in the n_j direction, and $\Delta[n, u_n]$ denotes a smooth function depending on n, u_n and finitely many iterates of u_n .) Again there exists a function Λ such that

$$\sum_j (S_j - id)P_j = \Lambda \Delta. \quad (2)$$

We will call Λ the characteristic of the discrete conservation law.

Here, we give a list of characteristics of the discrete AKP and BKP equations.

Discrete BKP equation

The discrete BKP (Miwa) equation¹⁶⁾ is given by

$$A\tau_{k+1,l,m}\tau_{k,l+1,m+1} + B\tau_{k,l+1,m}\tau_{k+1,l,m+1} + C\tau_{k,l,m+1}\tau_{k+1,l+1,m} + D\tau_{k,l,m}\tau_{k+1,l+1,m+1} = 0. \quad (3)$$

We have found the following 6 explicit rational characteristics for the discrete BKP equation:

$$\begin{aligned} \Lambda_1 &= A \left(\frac{f_1(k)\tau_{k-1,l+1,m+1}}{\tau_{k,l,m+1}\tau_{k,l+1,m+1}\tau_{k,l+1,m}} - \frac{f_1(k+1)\tau_{k+2,l,m}}{\tau_{k+1,l,m}\tau_{k+1,l+1,m}\tau_{k+1,l,m+1}} \right) + D \left(\frac{f_1(k)\tau_{k-1,l,m}}{\tau_{k,l,m}\tau_{k,l+1,m}\tau_{k,l,m+1}} - \frac{f_1(k+1)\tau_{k+2,l+1,m+1}}{\tau_{k+1,l+1,m+1}\tau_{k+1,l+1,m}\tau_{k+1,l,m+1}} \right), \\ \Lambda_2 &= B \left(\frac{f_2(l)\tau_{k+1,l-1,m+1}}{\tau_{k+1,l,m}\tau_{k+1,l+1,m}\tau_{k,l,m+1}} - \frac{f_2(l+1)\tau_{k,l+2,m}}{\tau_{k,l+1,m}\tau_{k,l+1,m+1}\tau_{k+1,l+1,m}} \right) + D \left(\frac{f_2(l)\tau_{k,l-1,m}}{\tau_{k,l,m}\tau_{k,l,m+1}\tau_{k+1,l,m}} - \frac{f_2(l+1)\tau_{k+1,l+2,m+1}}{\tau_{k+1,l+1,m+1}\tau_{k,l+1,m+1}\tau_{k+1,l+1,m}} \right), \\ \Lambda_3 &= C \left(\frac{f_3(m)\tau_{k+1,l+1,m-1}}{\tau_{k+1,l+1,m}\tau_{k,l+1,m}\tau_{k+1,l,m}} - \frac{f_3(m+1)\tau_{k,l,m+2}}{\tau_{k,l,m+1}\tau_{k,l+1,m+1}\tau_{k+1,l,m+1}} \right) + D \left(\frac{f_3(m)\tau_{k,l,m-1}}{\tau_{k+1,l,m}\tau_{k,l+1,m}\tau_{k,l,m}} - \frac{f_3(m+1)\tau_{k+1,l+1,m+2}}{\tau_{k+1,l+1,m+1}\tau_{k,l+1,m+1}\tau_{k+1,l,m+1}} \right), \end{aligned}$$

$$\begin{aligned} \Lambda_4 &= C \left(\frac{f_4(l)\tau_{k,l-1,m+1}}{\tau_{k+1,l,m+1}\tau_{k,l,m+1}\tau_{k,l,m}} - \frac{f_4(l+1)\tau_{k+1,l+2,m}}{\tau_{k+1,l+1,m}\tau_{k,l+1,m}\tau_{k+1,l+1,m+1}} \right) + A \left(\frac{f_4(l)\tau_{k+1,l-1,m}}{\tau_{k+1,l,m}\tau_{k,l,m}\tau_{k+1,l,m+1}} - \frac{f_4(l+1)\tau_{k,l+2,m+1}}{\tau_{k,l+1,m+1}\tau_{k,l+1,m}\tau_{k+1,l+1,m+1}} \right), \\ \Lambda_5 &= A \left(\frac{f_5(m)\tau_{k+1,l,m-1}}{\tau_{k+1,l+1,m}\tau_{k+1,l,m}\tau_{k,l,m}} - \frac{f_5(m+1)\tau_{k,l+1,m+2}}{\tau_{k,l+1,m+1}\tau_{k,l,m+1}\tau_{k+1,l+1,m+1}} \right) + B \left(\frac{f_5(m)\tau_{k,l+1,m-1}}{\tau_{k,l+1,m}\tau_{k,l,m}\tau_{k+1,l+1,m}} - \frac{f_5(m+1)\tau_{k+1,l,m+2}}{\tau_{k+1,l,m+1}\tau_{k,l,m+1}\tau_{k+1,l+1,m+1}} \right), \\ \Lambda_6 &= B \left(\frac{f_6(k)\tau_{k-1,l+1,m}}{\tau_{k,l+1,m+1}\tau_{k,l+1,m}\tau_{k,l,m}} - \frac{f_6(k+1)\tau_{k+2,l,m+1}}{\tau_{k+1,l,m+1}\tau_{k+1,l,m}\tau_{k+1,l+1,m+1}} \right) + C \left(\frac{f_6(k)\tau_{k-1,l,m+1}}{\tau_{k,l,m+1}\tau_{k,l,m}\tau_{k,l+1,m+1}} - \frac{f_6(k+1)\tau_{k+2,l+1,m}}{\tau_{k+1,l+1,m}\tau_{k+1,l,m}\tau_{k+1,l+1,m+1}} \right), \end{aligned}$$

where $f_1(x), \dots, f_6(x)$ are arbitrary functions. (If we choose $f_j(x)$ as constants or $(-1)^x$, we can produce 12 characteristics. Throughout this Letter, we only consider reductions in which all independent integrating factors are obtained taking $f_j(x) = \text{constant}$.) From these characteristics, we can obtain the associated conservation laws, using eq. (2). For example, P_1, P_2 , and P_3 associated to Λ_1 are

$$\begin{aligned} P_1 &= f_1(k) \left[-A^2 \frac{\tau_{k-1,l+1,m+1}\tau_{k+1,l,m}}{\tau_{k+1,l+1,m}\tau_{k+1,l,m+1}} - D^2 \frac{\tau_{k-1,l,m}\tau_{k+1,l+1,m+1}}{\tau_{k,l+1,m}\tau_{k,l,m+1}} - AB \frac{\tau_{k-1,l+1,m}\tau_{k+1,l,m}}{\tau_{k,l,m}\tau_{k,l+1,m}} \right. \\ &\quad - BD \frac{\tau_{k-1,l+1,m}\tau_{k+1,l+1,m+1}}{\tau_{k,l+1,m+1}\tau_{k,l+1,m}} - AC \frac{\tau_{k-1,l,m+1}\tau_{k+1,l,m}}{\tau_{k,l,m}\tau_{k,l,m+1}} - CD \frac{\tau_{k-1,l,m+1}\tau_{k+1,l+1,m+1}}{\tau_{k,l+1,m+1}\tau_{k,l,m+1}} \\ &\quad \left. - AD \left(\frac{\tau_{k-1,l,m}\tau_{k,l+1,m+1}\tau_{k+1,l,m}}{\tau_{k,l,m}\tau_{k,l+1,m}\tau_{k,l,m+1}} + \frac{\tau_{k,l,m}\tau_{k-1,l+1,m+1}\tau_{k+1,l+1,m+1}}{\tau_{k,l+1,m+1}\tau_{k,l+1,m}\tau_{k,l,m+1}} \right) \right], \\ P_2 &= f_1(k) \left(AC \frac{\tau_{k+1,l,m}\tau_{k-1,l,m+1}}{\tau_{k,l,m+1}\tau_{k,l,m}} - BD \frac{\tau_{k-1,l,m}\tau_{k+1,l,m+1}}{\tau_{k,l,m+1}\tau_{k,l,m}} \right), \\ P_3 &= f_1(k) \left(AB \frac{\tau_{k+1,l,m}\tau_{k-1,l+1,m}}{\tau_{k,l,m}\tau_{k,l+1,m}} - CD \frac{\tau_{k-1,l,m}\tau_{k+1,l+1,m}}{\tau_{k,l+1,m}\tau_{k,l,m}} \right), \end{aligned}$$

where P_1, P_2 , and P_3 correspond to the k -, l -, and m -directions, respectively.

Discrete AKP equation

The discrete AKP (Hirota–Miwa) equation^{16,17)} is given by

$$A\tau_{k+1,l,m}\tau_{k,l+1,m+1} + B\tau_{k,l+1,m}\tau_{k+1,l,m+1} + C\tau_{k,l,m+1}\tau_{k+1,l+1,m} = 0. \tag{4}$$

Note that the discrete AKP equation is the special case of $D = 0$ of the discrete BKP equation. The discrete AKP equation inherits the above 6 characteristics (with $D = 0$) from the discrete BKP equation, and we have found the following additional characteristic:

$$\Lambda_7 = \frac{f_7(k+l+m)\tau_{k,l,m}}{\tau_{k+1,l,m}\tau_{k,l+1,m}\tau_{k,l,m+1}} - \frac{f_7(k+l+m+1)\tau_{k+1,l+1,m+1}}{\tau_{k,l+1,m+1}\tau_{k+1,l,m+1}\tau_{k+1,l+1,m}},$$

where $f_7(x)$ is an arbitrary function.

Reduction to Finite-Dimensional Mappings and Construction of their Integrals

First example: Consider the following 4th-order mapping:

$$Dx_{n+2}x_{n+1}^2x_n^2x_{n-1}^2x_{n-2} + Ax_{n+1}x_nx_{n-1} + B + C/x_n = 0. \tag{5}$$

Using the transformation of the dependent variable $x_n = \tau_{n+1}\tau_{n-1}/\tau_n^2$, we obtain the bilinear form

$$D\tau_{n+3}\tau_{n-3} + A\tau_{n+2}\tau_{n-2} + B\tau_{n+1}\tau_{n-1} + C\tau_n^2 = 0. \tag{6}$$

This bilinear form is obtained from the discrete BKP equation by applying the reduction $\tau_n \equiv \tau_{Z_1k+Z_2l+Z_3m}$, where $Z_1 = 1, Z_2 = 2, Z_3 = 3$. Using the characteristics of the discrete BKP equation, we obtain the following integrating factors for the discrete bilinear form eq. (6):

$$\Lambda_1 = -\Lambda_6, \quad \Lambda_2 = -\Lambda_4, \quad \Lambda_3 = \frac{A}{D}\Lambda_4 + \frac{B}{D}\Lambda_6,$$

$$\begin{aligned} \Lambda_4 &= C \left(\frac{\tau_{n-2}}{\tau_{n+1}\tau_n\tau_{n-3}} - \frac{\tau_{n+2}}{\tau_n\tau_{n-1}\tau_{n+3}} \right) + A \left(\frac{\tau_{n-4}}{\tau_{n-2}\tau_{n-3}\tau_{n+1}} - \frac{\tau_{n+4}}{\tau_{n+2}\tau_{n-1}\tau_{n+3}} \right), \\ \Lambda_5 &= -\frac{A}{D}\Lambda_4 - \frac{B}{D}\Lambda_6, \\ \Lambda_6 &= B \left(\frac{\tau_{n-2}}{\tau_{n+2}\tau_{n-1}\tau_{n-3}} - \frac{\tau_{n+2}}{\tau_{n+1}\tau_{n-2}\tau_{n+3}} \right) + C \left(\frac{\tau_{n-1}}{\tau_n\tau_{n-3}\tau_{n+2}} - \frac{\tau_{n+1}}{\tau_n\tau_{n-2}\tau_{n+3}} \right). \end{aligned}$$

It is confirmed using the bilinear form eq. (6) that the integrating factors $\Lambda_1, \Lambda_2, \Lambda_3$, and Λ_5 lead to the indicated linear combinations of Λ_4 and Λ_6 . Note that the two integrating factors Λ_4 and Λ_6 are independent. From the above integrating factors, we can write the integrating factors in terms of the x -variable:

$$\begin{aligned} \tilde{\Lambda}_4 &= \tau_{n-1}\tau_{n+1}\Lambda_4 = C \left(\frac{1}{x_{n-1}x_{n-2}} - \frac{1}{x_{n+2}x_{n+1}} \right) + A(x_{n-2}x_{n-3} - x_{n+3}x_{n+2}), \\ \tilde{\Lambda}_6 &= \tau_{n-1}\tau_{n+1}\Lambda_6 = B \left(\frac{1}{x_{n+1}x_nx_{n-1}x_{n-2}} - \frac{1}{x_{n+2}x_{n+1}x_nx_{n-1}} \right) + C \left(\frac{1}{x_{n+1}x_nx_{n-1}^2x_{n-2}} - \frac{1}{x_{n+2}x_{n+1}^2x_nx_{n-1}} \right). \end{aligned}$$

We then obtain the following two integrals:

$$\begin{aligned}
 Q_4 &= CDx_{n+2}x_{n+1}^2x_n^2x_{n-1} - ADx_{n+3}x_{n+2}^2x_{n+1}^2x_n^2x_{n-1}x_{n-2} \\
 &\quad - A^2(x_{n+3}x_{n+2}x_{n+1}x_nx_{n-1} + x_{n+2}x_{n+1}x_nx_{n-1}x_{n-2}) \\
 &\quad - C^2\left(\frac{1}{x_{n+2}x_{n+1}x_n} + \frac{1}{x_{n+1}x_nx_{n-1}}\right) \\
 &\quad - BC\left(\frac{1}{x_{n+2}x_{n+1}} + \frac{1}{x_{n+1}x_n} + \frac{1}{x_nx_{n-1}}\right) \\
 &\quad - AB\sum_{j=0}^4 x_{n+3-j}x_{n+2-j} - AC\sum_{j=0}^3 \frac{x_{n+3-j}x_{n+2-j}}{x_{n-j}}, \\
 Q_6 &= BDx_{n+2}x_{n+1}x_nx_{n-1} + CD(x_{n+2}x_{n+1}x_n + x_{n+1}x_nx_{n-1}) \\
 &\quad - AB\left(\frac{1}{x_{n+2}} + \frac{1}{x_{n+1}} + \frac{1}{x_n} + \frac{1}{x_{n-1}}\right) \\
 &\quad - AC\left(\frac{1}{x_{n+2}x_{n+1}} + \frac{1}{x_{n+1}x_n} + \frac{1}{x_nx_{n-1}}\right) \\
 &\quad - \frac{B^2}{x_{n+2}x_{n+1}x_nx_{n-1}} - \frac{C^2}{x_{n+2}x_{n+1}^2x_n^2x_{n-1}} \\
 &\quad - BC\left(\frac{1}{x_{n+2}x_{n+1}x_n^2x_{n-1}} + \frac{1}{x_{n+2}x_{n+1}^2x_nx_{n-1}}\right).
 \end{aligned}$$

It is not difficult to show that Q_4 and Q_6 are functionally independent.

In the special case of $D = 0$, the fourth-order mapping eq. (5) reduces to the second-order mapping

$$Ax_{n+1}x_nx_{n-1} + B + C/x_n = 0, \tag{7}$$

which is a special case of the QRT mapping.¹⁾ Using the transformation of the dependent variable $x_n = \tau_{n+1}\tau_{n-1}/\tau_n^2$, we obtain the bilinear form

$$A\tau_{n+2}\tau_{n-2} + B\tau_{n+1}\tau_{n-1} + C\tau_n^2 = 0. \tag{8}$$

This bilinear form is obtained from the discrete AKP equation by applying the reduction $\tau_n \equiv \tau_{Z_1k+Z_2l+Z_3m}$, where $Z_1 = 1, Z_2 = 2, Z_3 = 3$. Using the characteristics of the discrete AKP equation, we obtain the following integrating factors for the discrete bilinear form eq. (8):

$$\begin{aligned}
 \Lambda_1 &= \left(\frac{\tau_{n+1}}{\tau_n\tau_{n+2}\tau_{n-1}} - \frac{\tau_{n-1}}{\tau_{n-2}\tau_n\tau_{n+1}}\right), \quad \Lambda_2 = -\Lambda_1, \\
 \Lambda_3 &= \frac{C}{A}\Lambda_1, \quad \Lambda_4 = B\Lambda_1, \quad \Lambda_5 = -\frac{C^2}{A}\Lambda_1, \\
 \Lambda_6 &= -A\Lambda_1, \quad \Lambda_7 = \frac{C}{A}\Lambda_1.
 \end{aligned}$$

There is only one independent integrating factor, Λ_1 . From Λ_1 , we can write the integrating factor in terms of the x -variable: $\tilde{\Lambda}_1 = \tau_{n+1}\tau_{n-1}\Lambda_1 = 1/x_{n+1} - 1/x_{n-1}$. We then obtain the following integral:

$$Q_1 = -Ax_{n+1}x_n + B\left(\frac{1}{x_{n+1}} + \frac{1}{x_n}\right) + \frac{C}{x_{n+1}x_n}.$$

Second example: As a second example, let us discuss the following 6th-order mapping:

$$\begin{aligned}
 Dx_{n+3}x_{n+2}^2x_{n+1}^3x_n^3x_{n-1}^2x_{n-2}x_{n-3} \\
 + Ax_{n+2}x_{n+1}^2x_n^2x_{n-1}^2x_{n-2} + Bx_{n+1}x_nx_{n-1} + C = 0.
 \end{aligned} \tag{9}$$

Using the transformation of the dependent variable $x_n = \tau_{n+1}\tau_{n-1}/\tau_n^2$, we obtain the bilinear form

$$A\tau_{n+3}\tau_{n-3} + B\tau_{n+2}\tau_{n-2} + C\tau_{n+1}\tau_{n-1} + D\tau_{n+4}\tau_{n-4} = 0. \tag{10}$$

This bilinear form is obtained from the discrete BKP equation by applying the reduction $\tau_n \equiv \tau_{Z_1k+Z_2l+Z_3m}$ where $Z_1 = 1, Z_2 = 2, Z_3 = 5$ or $Z_1 = 1, Z_2 = 3, Z_3 = 4$. Using the characteristics of the discrete BKP equation, we obtain the following integrating factors for the discrete bilinear form eq. (10):

$$\begin{aligned}
 \Lambda_1 &= -\Lambda_6, \quad \Lambda_2 = \Lambda_4, \quad \Lambda_3 = -\Lambda_5, \\
 \Lambda_4 &= C\left(\frac{\tau_{n-1}}{\tau_{n+2}\tau_{n+1}\tau_{n-4}} - \frac{\tau_{n+1}}{\tau_{n-1}\tau_{n-2}\tau_{n+4}}\right) \\
 &\quad + A\left(\frac{\tau_{n-5}}{\tau_{n-3}\tau_{n-4}\tau_{n+2}} - \frac{\tau_{n+5}}{\tau_{n+3}\tau_{n-2}\tau_{n+4}}\right), \\
 \Lambda_5 &= A\left(\frac{\tau_{n-8}}{\tau_{n-1}\tau_{n-3}\tau_{n-4}} - \frac{\tau_{n+8}}{\tau_{n+3}\tau_{n+1}\tau_{n+4}}\right) \\
 &\quad + B\left(\frac{\tau_{n-7}}{\tau_{n-2}\tau_{n-4}\tau_{n-1}} - \frac{\tau_{n+7}}{\tau_{n+2}\tau_{n+1}\tau_{n+4}}\right), \\
 \Lambda_6 &= B\left(\frac{\tau_{n-3}}{\tau_{n+3}\tau_{n-2}\tau_{n-4}} - \frac{\tau_{n+3}}{\tau_{n+2}\tau_{n-3}\tau_{n+4}}\right) \\
 &\quad + C\left(\frac{\tau_n}{\tau_{n+1}\tau_{n-4}\tau_{n+3}} - \frac{\tau_n}{\tau_{n-1}\tau_{n-3}\tau_{n+4}}\right).
 \end{aligned}$$

It is confirmed using the bilinear form eq. (10) that integrating factors Λ_1, Λ_2 , and Λ_3 lead to Λ_6, Λ_4 , and Λ_5 respectively. Note that the 3 integrating factors Λ_4, Λ_5 , and Λ_6 are independent. From the above integrating factors, we can write the integrating factors in terms of the x -variable:

$$\begin{aligned}
 \tilde{\Lambda}_4 &= \tau_{n+1}\tau_{n-1}\Lambda_4 \\
 &= \frac{C}{x_{n+1}x_n^2x_{n-1}}\left(\frac{1}{x_{n-1}^2x_n^2x_{n-2}x_{n-3}} - \frac{1}{x_{n+3}x_{n+2}^2x_{n+1}^2}\right) + \frac{A}{x_{n+1}x_nx_{n-1}}(x_{n-3}x_{n-4} - x_{n+4}x_{n+3}), \\
 \tilde{\Lambda}_5 &= \tau_{n+1}\tau_{n-1}\Lambda_5 \\
 &= Ax_n(x_{n-1}^2x_{n-2}^3x_{n-3}^4x_{n-4}^3x_{n-5}^2x_{n-6}x_{n-7} - x_{n+7}x_{n+6}^2x_{n+5}^3x_{n+4}^4x_{n+3}^3x_{n+2}^2x_{n+1}^2) \\
 &\quad + Bx_n(x_{n-1}^2x_{n-2}^3x_{n-3}^3x_{n-4}^2x_{n-5}x_{n-6} - x_{n+6}x_{n+5}^2x_{n+4}^3x_{n+3}^3x_{n+2}^2x_{n+1}^2), \\
 \tilde{\Lambda}_6 &= \tau_{n+1}\tau_{n-1}\Lambda_6 \\
 &= \frac{B}{x_{n+2}x_{n+1}^2x_n^2x_{n-1}x_{n-2}}\left(\frac{1}{x_{n-3}} - \frac{1}{x_{n+3}}\right) + \frac{C}{x_{n+2}x_{n+1}^2x_n^2x_{n-1}x_{n-2}}\left(\frac{1}{x_{n-1}x_{n-2}x_{n-3}} - \frac{1}{x_{n+3}x_{n+2}x_{n+1}}\right).
 \end{aligned}$$

We then obtain the following three integrals. [Note that eq. (9) can be used to eliminate, e.g., x_{n+4} and x_{n-3} from Q_4 , and similarly for Q_5 and Q_6 .]

$$\begin{aligned}
 Q_4 &= CD(x_{n+3}x_{n+2}^2x_{n+1}^2x_n + x_{n+2}x_{n+1}^2x_n^2x_{n-1} + x_{n+1}x_n^2x_{n-1}^2x_{n-2}) \\
 &\quad - ADx_{n+4}x_{n+3}^2x_{n+2}^2x_{n+1}^2x_n^2x_{n-1}^2x_{n-2}x_{n-3} - A^2x_{n+3}x_{n+2}x_{n+1}x_nx_{n-1}x_{n-2}(x_{n+4} + x_{n-3}) \\
 &\quad - BC \sum_{j=0}^2 \frac{1}{x_{n+3-j}x_{n+2-j}^2x_{n+1-j}^2x_{n-j}} - AB \sum_{j=0}^6 x_{n+4-j}x_{n+3-j} \\
 &\quad - C^2 \left(\frac{1}{x_{n+3}x_{n+2}^2x_{n+1}^3x_n^2x_{n-1}} + \frac{1}{x_{n+2}x_{n+1}^2x_n^3x_{n-1}^2x_{n-2}} \right) - AC \sum_{j=0}^3 \frac{x_{n+4-j}x_{n+3-j}}{x_{n+1-j}x_{n-j}x_{n-1-j}}, \\
 Q_5 &= AD \sum_{j=0}^3 x_{n+7-j}x_{n+6-j}^2x_{n+5-j}^3x_{n+4-j}^4x_{n+3-j}^5x_{n+2-j}^5x_{n+1-j}^5x_{n-j}^4x_{n-1-j}^3x_{n-2-j}^2x_{n-3-j} \\
 &\quad + BD \sum_{j=0}^2 x_{n+6-j}x_{n+5-j}^2x_{n+4-j}^3x_{n+3-j}^4x_{n+2-j}^4x_{n+1-j}^5x_{n-j}^4x_{n-1-j}^3x_{n-2-j}^2x_{n-3-j} \\
 &\quad + A^2 \sum_{j=0}^4 x_{n+7-j}x_{n+6-j}^2x_{n+5-j}^3x_{n+4-j}^4x_{n+3-j}^4x_{n+2-j}^4x_{n+1-j}^3x_{n-j}^2x_{n-1-j}^2x_{n-2-j} \\
 &\quad + AB \sum_{j=0}^4 x_{n+6-j}x_{n+5-j}^2x_{n+4-j}^3x_{n+3-j}^4x_{n+2-j}^4x_{n+1-j}^3x_{n-j}^2x_{n-1-j}^2x_{n-2-j} \\
 &\quad + AB \sum_{j=0}^4 x_{n+7-j}x_{n+6-j}^2x_{n+5-j}^3x_{n+4-j}^4x_{n+3-j}^4x_{n+2-j}^3x_{n+1-j}^3x_{n-j}^2x_{n-1-j} \\
 &\quad + B^2 \sum_{j=0}^5 x_{n+6-j}x_{n+5-j}^2x_{n+4-j}^3x_{n+3-j}^3x_{n+2-j}^3x_{n+1-j}^2x_{n-j}^2x_{n-1-j} \\
 &\quad + BC \sum_{j=0}^5 x_{n+6-j}x_{n+5-j}^2x_{n+4-j}^3x_{n+3-j}^3x_{n+2-j}^2x_{n+1-j}^2x_{n-j} \\
 &\quad + AC \sum_{j=0}^6 x_{n+7-j}x_{n+6-j}^2x_{n+5-j}^3x_{n+4-j}^4x_{n+3-j}^4x_{n+2-j}^3x_{n+1-j}^2x_{n-j}, \\
 Q_6 &= BDx_{n+3}x_{n+2}x_{n+1}x_nx_{n-1}x_{n-2} + CD \sum_{j=0}^3 x_{n+3-j}x_{n+2-j}x_{n+1-j} \\
 &\quad - AB \sum_{j=0}^5 \frac{1}{x_{n+3-j}} - AC \sum_{j=0}^2 \frac{1}{x_{n+3-j}x_{n+2-j}x_{n+1-j}x_{n-j}} \\
 &\quad - \frac{B^2}{x_{n+3}x_{n+2}x_{n+1}x_nx_{n-1}x_{n-2}} - \frac{C^2}{x_{n+3}x_{n+2}^2x_{n+1}^3x_n^3x_{n-1}^2x_{n-2}} \\
 &\quad - \frac{BC}{x_{n+3}x_{n+2}x_{n+1}^2x_n^2x_{n-1}^2x_{n-2}} - \frac{BC}{x_{n+3}x_{n+2}^2x_{n+1}^2x_n^2x_{n-1}x_{n-2}}.
 \end{aligned}$$

Using a Mathematical software package such as Mathematica, one can show that Q_4 , Q_5 , and Q_6 are functionally independent.

We consider the 4th-order mapping

$$Ax_{n+2}x_{n+1}^2x_n^2x_{n-1}^2x_{n-2} + Bx_{n+1}x_nx_{n-1} + C = 0. \quad (11)$$

This mapping is the special case of $D = 0$ of the 6th-order mapping eq. (9). Applying $x_n = \tau_{n+1}\tau_{n-1}/\tau_n^2$, we obtain the bilinear form

$$A\tau_{n+3}\tau_{n-3} + B\tau_{n+2}\tau_{n-2} + C\tau_{n+1}\tau_{n-1} = 0. \quad (12)$$

This bilinear form is obtained from the discrete AKP equation by applying the reduction $\tau_n \equiv \tau_{Z_1k+Z_2l+Z_3m}$, where $Z_1 = 1, Z_2 = 2, Z_3 = 5$ or $Z_1 = 1, Z_2 = 3, Z_3 = 4$. Using the characteristics of the discrete AKP equation, we obtain the

following integrating factors for the discrete bilinear form eq. (12):

$$\begin{aligned}
 \Lambda_1 &= \frac{\tau_{n+2}}{\tau_{n+1}\tau_{n+3}\tau_{n-2}} - \frac{\tau_{n-2}}{\tau_{n-3}\tau_{n-1}\tau_{n+2}}, \\
 \Lambda_2 &= \frac{\tau_n}{\tau_{n-3}\tau_{n+2}\tau_{n+1}} - \frac{\tau_n}{\tau_{n-2}\tau_{n+3}\tau_{n-1}}, \\
 \Lambda_3 &= -\frac{C^2}{A^2}\Lambda_1 - \frac{B^2C}{A^3}\Lambda_2, \quad \Lambda_4 = -B\Lambda_2, \\
 \Lambda_5 &= \frac{C^3}{A^2}\Lambda_1 + \frac{B^2C^2}{A^3}\Lambda_2, \quad \Lambda_6 = -A\Lambda_1, \quad \Lambda_7 = -\frac{C}{A}\Lambda_2.
 \end{aligned}$$

It is confirmed using the bilinear form eq. (12) that the two integrating factors Λ_1 and Λ_2 are independent. From the above integrating factors, we can write the integrating factors in terms of the x -variable:

$$\tilde{\Lambda}_1 = \tau_{n+1}\tau_{n-1}\Lambda_1 = \frac{1}{x_{n+1}x_nx_{n-1}} \left(\frac{1}{x_{n+2}} - \frac{1}{x_{n-2}} \right),$$

$$\tilde{\Lambda}_2 = \tau_{n+1}\tau_{n-1}\Lambda_2 = \frac{1}{x_{n+1}x_n^2x_{n-1}} \left(\frac{1}{x_{n-1}x_{n-2}} - \frac{1}{x_{n+2}x_{n+1}} \right).$$

We then obtain the following two integrals:

$$Q_1 = -Ax_{n+2}x_{n+1}x_nx_{n-1} + B \left(\frac{1}{x_{n+2}} + \frac{1}{x_{n+1}} + \frac{1}{x_n} + \frac{1}{x_{n-1}} \right) + \frac{C}{x_{n+2}x_{n+1}x_nx_{n-1}},$$

$$Q_2 = A(x_{n+2}x_{n+1} + x_{n+1}x_n + x_nx_{n-1}) - B \left(\frac{1}{x_{n+2}x_{n+1}x_n} + \frac{1}{x_{n+1}x_nx_{n-1}} \right) - \frac{C}{x_{n+2}x_{n+1}^2x_n^2x_{n-1}}.$$

We note that eq. (11) preserves the symplectic structure

$$\begin{pmatrix} 0 & x_{n-2}x_{n-1} & -x_{n-2}x_n & x_{n-2}x_{n+1} \\ -x_{n-1}x_{n-2} & 0 & x_{n-1}x_n & -x_{n-1}x_{n+1} \\ x_nx_{n-2} & -x_nx_{n-1} & 0 & x_nx_{n+1} \\ -x_{n+1}x_{n-2} & x_{n+1}x_{n-1} & -x_{n+1}x_n & 0 \end{pmatrix},$$

and that the two integrals Q_1 and Q_2 are in involution w.r.t. this structure, giving an independent confirmation of the integrability of eq. (11).

We have studied a class of integrable mappings that can be transformed into a single bilinear form. We have proposed a method of constructing integrals of these higher-order integrable maps. The key to the construction is the conservation laws of the discrete bilinear forms of the associated AKP and BKP equations. Note that, generalizing the examples in this Letter, we can construct a family of higher-order mappings from the discrete AKP and BKP equations, by applying the reduction $\tau_n \equiv \tau_{Z_1k+Z_2l+Z_3m}$ for any Z_1, Z_2 , and Z_3 .

In general, soliton equations in the 1+1-dimension have infinitely many conservation laws. One may expect the existence of infinitely many conservation laws for discrete AKP and BKP equations. However, finding infinitely many conservation laws in the form of eq. (2) is difficult in the case of 2+1-dimensional discrete soliton equations. Finding

infinitely many conservation laws for discrete AKP and BKP equations is still an open problem.

We hope to discuss details of our methods and higher-order mappings in the class given here in a forthcoming paper.

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