

Explicit, volume preserving splitting methods for divergence-free polynomial vector fields

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In this talk we address the problem of the numerical integration of divergence-free vector fields by volume-preserving methods. It is well known that devising methods which preserve volume is quite a hard task. To-date the general techniques consist in splitting the given vector field into the sum of two-dimensional volume-preserving systems and solve those by symplectic methods (Feng Kang) or by solving for $n - 1$ variables and then correct for the last one in order to obtain a volume-preserving method (Shang Zai-jiu, 1994, Quispel 1995). However, these methods are generally implicit and expensive. Explicit methods exist for particular problems, for instance trigonometric ones.

We address the case of polynomial vector fields.

The main idea is to split the given divergence free vector field into the sum of pieces that

- Can be integrated exactly easily, or
- Can be integrated in a volume preserving manner by simple explicit methods (a.k.a. Forward Euler) and whose adjoint is also explicit.

Then, the basic split terms can be combined to obtain higher order integrator, either by Yoshida's technique, or by other symmetric composition methods.

Several new methods are presented for linear and quadratic problems. Roughly, these can be divided into two classes: i) methods that distinguish the diagonal part (all the terms in equation i that include the variable x_i , for $i = 1, 2 \dots, n$), and ii) methods that do not distinguish the diagonal part. The diagonal part is generally more difficult to treat as its coefficient are interconnected as a result of volume-preservation, however it is computationally less expensive, as the off-diagonal part requires computations of a order of n higher.

Among the methods for the diagonal part, we mention the splitting in d shears, where d is the degree of the polynomial vector field, and exponentiation. As for the off-diagonal part, we consider splitting in strictly lower triangular systems (as these can be integrated in a volume-preserving manner by any Runge–Kutta method) by permutations, as well as splitting in n natural shears, which are integrated exactly by a step of Forward Euler. As for methods that do not distinguish the diagonal part, we consider a splitting in $n + d$ shears. The splitting in shears has been

successfully used in the context of Hamiltonian integration in particle accelerator physics.
