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Sufficient conditions for dynamical systems to have pre-symplectic or pre-implectic structures

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Abstract

We present a number of alternative sufficient conditions for the existence of pre-symplectic or pre-implectic (Poisson) structures, for both ordinary differential (ODE) and ordinary difference (OΔE) equations. Four alternative sets of conditions are obtained for ODEs and OΔEs in n dependent variables: (1) A vector field in involution with the ODE and an integral (or two symmetries for an OΔE) imply a pre-implectic structure; (2) volume preservation and $n-2$ symmetries imply a pre-symplectic structure; (3) volume preservation and $n-2$ integrals imply a pre-implectic structure; (4) complex implectic structure implies infinitely many real implectic structures. In all but the first case the methods can give a set of distinct structures.

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1. Introduction

Hamiltonian systems are a central area of mathematics and physics. On one hand they have spawned symplectic geometry [1], Noether's theorems relating symmetries and conserved quantities (integrals) [2] and the whole panoply of integrable systems [3–6]. On the other, a Hamiltonian description of a classical system is necessary for canonical quantisation, while symplectic methods are also important in various types of stability analysis and numerical integration (symplectic integrators).

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Given the above, it is clearly desirable to know when a system of differential equations have a Hamiltonian structure and to be able to find that structure explicitly. If a regular Lagrangian is known then a Hamiltonian can be derived from it. However for systems of more than two dependent variables or more than one independent variable even the existence question has not been satisfactorily answered for Lagrangians [7,8]. For the simplest case, a system of first-order ordinary differential equations (ODEs), it is known that a local Hamiltonian description always exists but finding it is not an algorithmic procedure.

Analogous considerations apply to difference equations. Since numerical integrators are essentially difference equations, reliable long-term integration of Hamiltonian ODEs can benefit by using an approximating difference scheme which is also symplectic. Moreover there are discrete analogues of Noether's theorem, so that symmetries and integrals of symplectic difference equations are related as for Hamiltonian ODEs.

Recently there has been increased interest in what we shall call singular Hamiltonian systems. These are systems which are Hamiltonian when restricted to suitable submanifolds of phase space and are rather easier to find than full Hamiltonian systems. On the other hand they bring some complications. In particular, it becomes necessary to distinguish between symplectic and implectic structures: an implectic structure maps integrals to symmetries while a symplectic structure maps symmetries to integrals. In the maximal rank case one is the inverse of the other, so the existence of either implies the other. This no longer is the case for singular Hamiltonian systems.

Note that we call non-maximal rank symplectic structures pre-symplectic. By analogy, a non-maximal rank implectic structure will be called pre-implectic: such structures are commonly called *Poisson* in the literature, but we want to emphasise the duality between the implectic and symplectic forms. More precise definitions will be given in Section 2.

In this paper we obtain a number of sufficient conditions for the existence of symplectic, pre-symplectic, implectic and pre-implectic structures for both ODEs and ordinary difference equations (OΔEs). Section 2 also contains a brief review of most of the devices borrowed from differential geometry, in an attempt to make the paper self-contained. It also contains four distinct variations on the classical Noether theorem, covering the pre-implectic and pre-symplectic cases for both ODEs and OΔEs.

Section 3 contains the result that, given a vector field in involution (in the sense of definition 1.39 in [9]) with an ODE and an integral, there exists a rank 2 pre-implectic structure. This generalises a result obtained by Hojman in [10], where the vector field is required to be a symmetry. The result extends to OΔEs, where one requires two symmetries in involution but no integral.

Section 4 gives a construction for a pre-symplectic structure, starting from a volume (or measure) preserving ODE or OΔE in n dimensions and a set of $n-2$ volume preserving symmetries. A similar construction yields an essentially dual result in Section 5: given a volume preserving ODE or OΔE and $n-2$ integrals, we construct a pre-implectic structure.

Section 6 assumes that an analytic symplectic or implectic structure is known. From the existing structure we construct an infinite set of alternate, compatible symplectic or implectic structures in twice the dimension, via complexification.

Discussion of the results concludes the paper in Section 7.

2. Preliminaries

In this article we will usually work on a smooth manifold M of dimension n, with generic local coordinates $(x^1,\dots,x^n)$. Where it is necessary to treat time-dependent systems separately, we will replace M with $E=\mathbf{R}\times M$, having local coordinates $(t,x^1,\dots,x^n)$, or more briefly (t,x). For the definitions of basic geometric objects which occupy this section, M will usually suffice. In Section 6 we will make the additional assumption that M is a complex analytic manifold, with complex dimension n (and therefore real dimension $2n$).

We start this section with some definitions in the hope of making the article self-contained.

2.1. Vector fields, multi-vectors and differential forms

A vector field v on M has the local coordinate form

$$v=\sum_j v^j\frac{\partial}{\partial x^j}\,, \tag{1}$$

so the action $v(f)$ of v on a function $f:M\to\mathbf{R}$ is clear. The set of all vectors at $x\in M$ is the tangent space to M at x and is denoted T_xM. In some cases it will be convenient to represent v by its components $(v^1,\dots,v^n)$, leading to the traditional notation

$$v(f)=v\cdot\nabla f\,. \tag{2}$$

A 1-form on M is a linear map from vector fields to $\mathbf{R}$. Hence 1-forms are dual to vector fields and can be represented locally by

$$\theta=\sum_j\theta_j\,\mathrm{d}x^j\,. \tag{3}$$

The inner product $\langle v,\theta\rangle$ of a vector field and a 1-form is defined in coordinates by the duality relation

$$\left\langle\frac{\partial}{\partial x^j},\mathrm{d}x^k\right\rangle=\delta_j^k\,, \tag{4}$$

δ_j^k being the Kronecker delta. Thus

$$\langle v,\theta\rangle=\sum_j v^j\theta_j\,. \tag{5}$$

This can also be written as $\theta(v)$, the evaluation of θ on v.

For $1 < k \leqslant n$, a k-form is a completely skew multi-linear map from sets of k vector fields to $\mathbf{R}$. The space of k-forms on M is denoted $\Lambda^k(M)$ and has a local basis given by all the skew products of the $\mathrm{d}x^j$, taken k at a time, upto re-ordering. The dimension of $\Lambda^k(M)$ is therefore ${}^nC_k$.

The skew product is denoted by $\wedge$,

$$\mathrm{d}x^i \wedge \mathrm{d}x^j = \tfrac{1}{2}(\mathrm{d}x^i \otimes \mathrm{d}x^j - \mathrm{d}x^j \otimes \mathrm{d}x^i)\,, \tag{6}$$

in terms of the direct product $\otimes$, or if u, v are vector fields and θ, ϕ are 1-forms,

$$\theta \wedge \phi(u, v) = \tfrac{1}{2}(\theta(u)\phi(v) - \theta(v)\phi(u))\,. \tag{7}$$

The definition of $\wedge$ extends to give a $j+k$-form from the product of a j-form and a k-form by a similar anti-symmetrisation process. Explicitly, if $X_1, \ldots, X_{j+k}$ are arbitrary vector fields, $\theta \in \Lambda^j(M)$ and $\phi \in \Lambda^k(M)$, then

$$\theta\wedge\phi(X_1, \ldots, X_{j+k}) = \frac{1}{(j+k)!}\sum_{\sigma}(-1)^{|\sigma|}\theta(X_{\sigma(1)}, \ldots, X_{\sigma(j)})\phi(X_{\sigma(j+1)}, \ldots, X_{\sigma(j+k)})\,, \tag{8}$$

where the sum is over the permutations σ of the integers $1, \ldots, j+k$ and $|\sigma|$ is 0 or 1 as the permutation is even or odd.

As an example, note that a basis for $\Lambda^2(\mathbf{R}^3)$ is given by

$$\{\mathrm{d}x^1 \wedge \mathrm{d}x^2, \mathrm{d}x^2 \wedge \mathrm{d}x^3, \mathrm{d}x^3 \wedge \mathrm{d}x^1\}\,. \tag{9}$$

Then if $\theta = \theta_j\, \mathrm{d}x^j$ and $\phi = \phi_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j$,

$$\theta \wedge \phi = (\theta_1\phi_{23} + \theta_2\phi_{31} + \theta_3\phi_{12})\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\,, \tag{10}$$

as can be checked by substituting coordinate vector fields into (8).

Given $\theta \in \Lambda^k(M)$ and a vector field v, the contraction $v\rfloor\theta$ defines a $k-1$-form as follows:

$$(v\rfloor\theta)(X_1, \ldots, X_{k-1}) = \theta(v, X_1, \ldots, X_{k-1}) \tag{11}$$

for arbitrary vector fields X_j. Note that if θ is a 1-form then $v\rfloor\theta = \langle v, \theta\rangle = \theta(v)$ is a 0-form or, in other words, a function.

While k-forms and their $\wedge$-product will be familiar to most readers, the dual concepts of k-multi-vectors and the $\wedge$-product of vector fields are less well known. The definition of the wedge product is completely analogous to Eq. (8): for a pair of vector fields u, v we have that

$$u \wedge v = \tfrac{1}{2}(u \otimes v - v \otimes u) \tag{12}$$

is a 2-multi-vector or bi-vector. In general, a k-multi-vector is a completely skew multilinear map from sets of k 1-forms to $\mathbf{R}$ (or equivalently a linear map $\Lambda^k(M) \to \mathbf{R}$). For example, the space of 2-multi-vectors in $\mathbf{R}^3$ has a basis

$$\left\{\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^1}\right\}\,, \tag{13}$$

dual to the basis (9) for $\Lambda^2(\mathbf{R}^3)$.

2.2. The volume form

Since the dimension of $\Lambda^k(M)$ is ${}^nC_k$, the space of n forms on M is one dimensional. Thus all n-forms on M can be written as

$$\Omega = f\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \tag{14}$$

for some function $f : M \to \mathbf{R}$. In particular, if f is thought of as a measure on M, then Ω can be used to define the volume of a subset U of M,

$$\mathrm{vol}(U) = \int_U \Omega\,. \tag{15}$$

In this context Ω is called the volume form on M.

There is a tradition in the physics literature to say that a map is volume preserving if it preserves Ω (in a sense to be defined later) with $f \equiv 1$, or measure preserving if Ω is more general. The distinction is slight, since f depends on the Jacobian of any coordinate transform: if $(y^1, \ldots, y^n)$ are alternative coordinates,

$$\Omega = f\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \frac{\partial(x^1, \ldots, x^n)}{\partial(y^1, \ldots, y^n)} f\,\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n\,. \tag{16}$$

2.3. The Lie bracket and Lie derivative

The space of vector fields on M has a Lie algebra structure, with product given by the Lie bracket $[\cdot,\cdot]$. If $u = u^j\partial/\partial x^j$ and $v = v^j\partial/\partial x^j$, then

$$[u,v] = \left(u^i\frac{\partial v^j}{\partial x^i} - v^i\frac{\partial u^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}\,. \tag{17}$$

The Lie bracket defines a derivation, since it satisifes the Leibnitz condition $[u, fv] = u(f)v + f[u,v]$, so it is also written in the form of a linear differential operator,

$$[u,v] =: \mathscr{L}_u v\,. \tag{18}$$

The definition is extended to scalar functions by setting $\mathscr{L}_u f = u(f)$, then the action on differential forms and multi-vectors is determined by the Leibnitz condition. If $X_1,\ldots,X_k$ are arbitrary vector fields and $\theta \in \Lambda^k(M)$,

$$\begin{aligned} v(\theta(X_1,\ldots,X_k)) &= (\mathscr{L}_v\theta)(X_1,\ldots,X_k) \\ &\quad + \theta(\mathscr{L}_v X_1, X_2, \ldots, X_k) + \cdots + \theta(X_1,\ldots,X_{k-1},\mathscr{L}_v X_k)\,, \end{aligned} \tag{19}$$

and the expression for $\mathscr{L}_v\theta$ is obtained by re-arranging the above equality. Similarly if τ is a k-multi-vector and $\alpha_1,\ldots,\alpha_k$ are arbitrary 1-forms,

$$\begin{aligned} v(\tau(\alpha_1,\ldots,\alpha_k)) &= (\mathscr{L}_v\tau)(\alpha_1,\ldots,\alpha_k) \\ &\quad + \tau(\mathscr{L}_v\alpha_1,\alpha_2,\ldots,\alpha_k) + \cdots + \tau(\alpha_1,\ldots,\alpha_{k-1},\mathscr{L}_v\alpha_k)\,. \end{aligned} \tag{20}$$

In particular if $\tau = \xi \wedge \eta$ is a bi-vector, then

$$\mathscr{L}_v\tau = (\mathscr{L}_v\xi) \wedge \eta + \xi \wedge (\mathscr{L}_v\eta)\,. \tag{21}$$

We say that an object σ (vector field, multi-vector or differential form) is preserved by a vector field v if $\mathscr{L}_v\sigma = 0$. In particular, a vector field v is volume preserving if $\mathscr{L}_v\Omega = 0$.

2.4. Push-forward and pull-back

Suppose M and N are smooth manifolds and ϕ is a diffeomorphism. Clearly a smooth curve in M maps to a smooth curve in N, so by taking tangents to the curve we have a map of vectors $\phi_*\colon T_xM \to T_{\phi(x)}N$. This map is sometimes called the push-forward or the derivative map of ϕ. If v has coordinate expression $v = v^j\partial/\partial x^j$ and $y = \phi(x)$,

$$\phi_* v = \left(v^k \frac{\partial \phi^j}{\partial x^k} \right) \frac{\partial}{\partial y^j} \,. \tag{22}$$

Similarly, if $\alpha \in \Lambda^k(N)$, the duality of vector fields and differential forms induces a map $\phi^* \Lambda^k(N) \to \Lambda^k(M)$, called the pull-back:

$$(\phi^*\alpha)_x(X_1,\ldots,X_k) := \alpha_{\phi(x)}(\phi_*X_1,\ldots,\phi_*X_k) \,. \tag{23}$$

Here the subscript gives the point at which the form is evaluated. For a 1-form $\alpha = \alpha_j\,\mathrm{d}y^j$,

$$\phi^*\alpha = \alpha_k \frac{\partial \phi^k}{\partial x^j}\,\mathrm{d}x^j \,. \tag{24}$$

A k-form α is said to be preserved by a map ϕ if $\phi^*\alpha = \alpha$: in particular, ϕ is volume-preserving if it preserves the volume form Ω, $\phi^*\Omega = \Omega$.

2.5. The exterior derivative

Given the Lie derivative, the other important linear differential operator on differential forms is the exterior derivative

$$\mathrm{d} : \Lambda^k(M) \to \Lambda^{k+1}(M), \quad k = 0, 1, \ldots, n \,, \tag{25}$$

with $\Lambda^k(M)$ understood to contain no non-zero elements if $k > n$. If $\theta \in \Lambda^k(M)$ and $X_0,\ldots,X_k$ are arbitrary vector fields, then

$$\begin{aligned} \mathrm{d}\theta(X_0,\ldots,X_k) = & \sum_{0\leqslant j\leqslant n} (-1)^j X_j(\theta(X_0,\ldots,\hat{X}^j,\ldots,X_k)) \\ & + \sum_{0\leqslant i<j\leqslant k} (-1)^{i+j}\theta([X_i,X_j],\; X_0,\ldots,\hat{X}^i,\ldots,\hat{X}^j,\ldots,X_k) \,. \end{aligned} \tag{26}$$

Here $\hat{X}^j$ denotes the omission of X^j from the sequence. As examples, when $k = 0$

$$\mathrm{d}f = \frac{\partial f}{\partial x^j}\,\mathrm{d}x^j \,. \tag{27}$$

When $k = 1$ we can set $\theta = \theta_j\,\mathrm{d}x^j$ and

$$\mathrm{d}\,\theta = \frac{1}{2}\left(\frac{\partial \theta_j}{\partial x^i} - \frac{\partial \theta_i}{\partial x^j} \right) \mathrm{d}x^i \wedge \mathrm{d}x^j \,. \tag{28}$$

Coordinate expressions for $k \geqslant 2$ can be derived by substituting coordinate vectors for $X_0, \ldots, X_k$ in (26).

An important fact that can be readily checked is that $\mathrm{d}^2 \equiv 0$. Conversely, if $\theta \in \Lambda^k(M)$, $k > 0$ and $\mathrm{d}\theta = 0$, then there is a locally defined $\alpha \in \Lambda^{k-1}(M)$ such that $\theta = \mathrm{d}\alpha$. If in addition M is such that the cohomology class $H^k(M)$ is trivial, then α exists globally on M.

Finally, it can be shown that the Lie derivative can be expressed in terms of the exterior derivative,

$$\mathscr{L}_v \theta = v \rfloor \mathrm{d}\theta + \mathrm{d}(v \rfloor \theta) . \tag{29}$$

Using this expression it is easy to show that the exterior and Lie derivatives commute, $\mathscr{L}_v \, \mathrm{d}\theta = \mathrm{d}(\mathscr{L}_v \theta)$. It can also be shown that d commutes with the pull-back,

$$\mathrm{d}(\phi^* \alpha) = \phi^*(\mathrm{d}\alpha) \tag{30}$$

for all smooth maps ϕ and differential forms α.

2.6. ODEs, symmetries and integrals

If a vector field Γ on M has a local coordinate expression

$$\Gamma := f^j \frac{\partial}{\partial x^j} \tag{31}$$

(with the summation convention assumed), then the flow of Γ on M solves the system of m ODEs

$$\dot{x}^j = f^j(x) . \tag{32}$$

If instead we consider a product manifold $\mathbf{R} \times M$ with coordinates $(t, x) \equiv (t, x^1, \ldots, x^m)$ and a vector field with coordinate expression

$$\Gamma := \frac{\partial}{\partial t} + f^j \frac{\partial}{\partial x^j} , \tag{33}$$

then the flow of Γ satisfies the time-dependent system of m ODEs

$$\dot{x}^j = f^j(t, x) . \tag{34}$$

Second- and higher-order ODEs may also be considered as vector fields on appropriate tangent or jet spaces of M [11], leading to a geometrical version of the standard expansion to a larger system of first-order ODEs.

Conversely, a given system of m first-order ODEs (which may originate in a higher-order system) can always be expressed as a vector field on $\mathbf{R}^m$ (or $\mathbf{R} \times \mathbf{R}^m$ in the time-dependent case). In the following, we will use E to denote either M or $\mathbf{R} \times M$ as appropriate.

A Lie symmetry of a system of ODEs is a continuous transformation which maps solutions to solutions (see Olver [9] for a more precise definition). Given a representation of the ODEs as a vector field Γ on E, the vector field ξ on E generates a symmetry of Γ if and only if $[\Gamma, \xi] = 0$.

Such symmetry generators form a Lie algebra as a consequence of the Jacobi identity for the Lie bracket. Suppose that ξ and η each generate a symmetry, so that $[\Gamma,\xi]=[\Gamma,\eta]=0$. Then the Jacobi identity can be re-arranged to give

$$[\Gamma,[\xi,\eta]]=[[\Gamma,\xi],\eta]-[[\Gamma,\eta],\xi]=0\,, \tag{35}$$

so $[\xi,\eta]$ also generates a symmetry.

If we relax the definition of a symmetry to require only that solutions are mapped to re-parametrisations of solutions we obtain projective symmetries. The condition for ξ to generate a projective symmetry is

$$[\Gamma,\xi]=\lambda\Gamma, \quad \lambda\in C^{\infty}(E)\,. \tag{36}$$

Generators of projective symmetries also form a Lie algebra, using an argument analogous to that above.

Note that for any $g\in C^{\infty}(E)$, $g\Gamma$ generates a projective symmetry of Γ. In fact symmetries generated this way are called trivial [9], since they map each solution to a re-parametrisation of itself. Thus if $[\Gamma,\xi]=\lambda\Gamma$ and $g\in C^{\infty}(E)$ satisfies $\Gamma(g)=\lambda$, then $\bar{\xi}:=\xi-g\Gamma$ generates a (non-projective) symmetry.

Alternately, by setting $\tilde{\xi}:=\xi-\langle\xi,\mathrm{d}t\rangle\Gamma$ we obtain an evolutionary symmetry $\tilde{\xi}$, that is $\langle\xi,\mathrm{d}t\rangle=0$ [9].

An integral of the ODE is simply a function $I:E\to\mathbf{R}$ which is constant along the flow of Γ. Thus $f:E\to\mathbf{R}$ is an integral of Γ iff $\Gamma(f)=0$.

2.7. OΔEs, symmetries and integrals

Let $\phi:M\to M$ be a smooth map. Then in local coordinates ϕ determines a system of first-order ordinary difference equations (OΔE),

$$(x^j)'=\phi^j(x)\,. \tag{37}$$

If we work on $E=\mathbf{R}\times M$ and restrict to maps $\phi:E\to E$ which have the form

$$(t+1,x')=\phi(t,x)\,, \tag{38}$$

then we have a system of time-dependent OΔEs.

As in the ODE case, a Lie symmetry of a system of OΔEs is a continuous transformation mapping solutions to solutions. It can be shown [12] that a vector field ξ generates a symmetry iff

$$\phi_*\xi=\xi\circ\phi \tag{39}$$

and

$$\frac{\partial}{\partial x^j}\langle\xi,\mathrm{d}t\rangle=0, \quad j=1,\ldots,n\,. \tag{40}$$

The second condition requires ξ to be a projectible vector field: all symmetries of OΔEs are projectible [12], which is not the case for ODEs.

2.8. Implectic and pre-implectic structures

Recall that an implectic structure on E is a skew symmetric, 2-contravariant tensor of maximal rank which satisfies the Jacobi condition:

$$J^{ij}_{,k}J^{kl} + J^{jl}_{,k}J^{ki} + J^{li}_{,k}J^{kj} = 0\,, \tag{41}$$

the comma denoting the differentiation and the summation convention is employed over repeated indices. The rank of J, that is the number of non-zero eigenvalues, is always even and so may be less than the dimension of E. A pre-implectic structure, often called a Poisson structure, satisfies all but the maximal rank condition.

An implectic or pre-implectic structure can be used to define a Lie product on the functions on E,

$$\{g,h\} := J^{ij}g_{,i}h_{,j}\,, \tag{42}$$

called the Poisson bracket.

An alternative viewpoint is that J is a bi-vector field,

$$J = J^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j} = a^{ij}\xi_i\wedge\xi_j\,. \tag{43}$$

Note that the coefficient matrix with respect to the coordinate vector fields is taken to be J^{ij}, whereas for arbitrary, possibly non-commuting vector fields ξ_j, η_j it is written as a^{ij}.

The action of J on pairs of function is then given by the Poisson bracket

$$\{g,h\} = \tfrac{1}{2}a^{ij}(\langle\xi_i, \mathrm{d}g\rangle\langle\xi_j, \mathrm{d}h\rangle - \langle\xi_j, \mathrm{d}h\rangle\langle\xi_i, \mathrm{d}g\rangle)\,. \tag{44}$$

It can be seen that J also defines a map from the space $\Lambda^1(E)$ to the dual space of vector fields,

$$J : \theta \mapsto \tfrac{1}{2}a^{ij}(\langle\xi_i,\theta\rangle\xi_j - \langle\xi_j,\theta\rangle\xi_i), \quad \forall\theta\in\Lambda^1(E)\,. \tag{45}$$

Note that if $v = J(\mathrm{d}f)$, then

$$\{f,g\} = \langle v, \mathrm{d}g\rangle = v(g), \quad \forall g : E \to \mathbf{R}\,. \tag{46}$$

The Jacobi identity ensures that if θ is closed ($\mathrm{d}\theta = 0$ or equivalently there is a local function g such that $\theta = \mathrm{d}g$) then $v := J(\theta)$ satisfies $\mathscr{L}_v J = 0$. A consequence of this is that J is a Lie algebra homomorphism, so that if $v := J(\mathrm{d}g)$ and $w := J(\mathrm{d}h)$, then $[v,w] = J(\mathrm{d}\{g,h\})$.

2.9. Symplectic and pre-symplectic structures

Recall that a symplectic structure on E is a 2-form ω of maximal rank satisfying the Jacobi condition

$$\mathrm{d}\omega = 0\,. \tag{47}$$

A pre-symplectic structure satisfies all but the maximal rank condition. In coordinates,

$$\omega = \omega_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j\,. \tag{48}$$

Note that in some circumstances it is appropriate to consider so-called exact-symplectic structures, where ω satisfies the (globally) stronger condition $\omega = \mathrm{d}\sigma$ for some 1-form σ. We will not need this condition anywhere in this paper.

If v is a vector field, then $v\rfloor\omega$ is a one form, so a (pre)-symplectic structure provides a map dual to the implectic structure,

$$\omega : v \mapsto v\rfloor\omega \tag{49}$$

or in coordinates

$$\omega : v^j\frac{\partial}{\partial x^j} \mapsto v^i\omega_{ij}\,\mathrm{d}x^j\,. \tag{50}$$

Here, the Jacobi condition implies that if $\mathscr{L}_v\omega = 0$ then $\mathrm{d}(v\rfloor\omega) = 0$, so there is a local function f such that $v\rfloor\omega = \mathrm{d}f$. Also, if $v\rfloor\omega = \mathrm{d}f$ and u is another vector field then

$$\omega(v,u) = u(f)\,. \tag{51}$$

In fact if ω is a symplectic structure (and is therefore invertible on M or the constant time hypersurfaces M_τ of E), then the inverse map defines an implectic structure. Conversely, the inverse of an implectic structure J is symplectic.

2.10. Hamiltonian systems and the Noether theorems

In addition to the conditions for J to be (pre-)implectic, we say that J is (pre-) implectic for the autonomous ODE defined by Γ if $\mathscr{L}_\Gamma J = 0$ and Γ is in the image of J. Similarly a (pre-)symplectic structure ω is (pre-)symplectic for autonomous Γ if $\mathscr{L}_\Gamma\omega = 0$ and $\Gamma\rfloor\omega \neq 0$. If J is inverse to ω then it is easy to show that

$$\mathscr{L}_\Gamma J = 0 \;\;\Leftrightarrow\;\; \mathscr{L}_\Gamma\omega = 0\,, \tag{52}$$

so in this case J is implectic for Γ if and only if ω is symplectic for Γ.

The most familiar case of a Hamiltonian system arises when a system of second-order autonomous ODEs can be derived from a non-singular Lagrangian variational problem. Both the Hamiltonian function H and the implectic structure are obtained via the Legendre transform.

When the second-order equations are time dependent, the manifold $E = \mathbf{R} \times M$ is necessarily of odd dimension. The coordinate t for the first factor of $E = \mathbf{R} \times M$ is then chosen to be a Casimir of J, $J(\mathrm{d}t) = 0$. In the pre-symplectic case, the additional condition is $(\partial/\partial t)\rfloor\omega = 0$.

In general, if Γ defines a system of non-autonomous ODEs on $E = \mathbf{R} \times M$, then J is said to be (pre-)implectic for Γ if J is (pre-)implectic, $\mathscr{L}_\Gamma J = 0$ and $\Gamma - \partial/\partial t$ is in the image of J. The corresponding conditions for the (pre-)symplectic structure ω is that $\mathscr{L}_\Gamma\omega = 0$ and

$$\Gamma\rfloor\omega = 0 \;\;\Leftrightarrow\;\; \Gamma - \frac{\partial}{\partial t} = 0\,. \tag{53}$$

From the previously stated results it follows that if ω is pre-symplectic for Γ, there is a locally defined function H, the Hamiltonian for Γ, such that

$$\Gamma \rfloor \omega = \mathrm{d}H\,. \tag{54}$$

We say that the system of ODEs defined by Γ is a singular Hamiltonian system. If ω is symplectic, we say Γ is Hamiltonian.

Similarly, if J is pre-implectic for autonomous Γ, then there is a locally defined Hamiltonian H such that

$$\Gamma = J(\mathrm{d}H) \tag{55}$$

in the autonomous case, or

$$\Gamma = \frac{\partial}{\partial t} + J(\mathrm{d}H) \tag{56}$$

in the non-autonomous case.

The vector field corresponding to the first-order Hamilton equations may have arisen as a first-order formulation of a second- or higher-order system, or may have originated as a first-order system. In particular, if the system arose as a second-order system it need not have any equivalent Lagrangian formulation [8].

2.10.1. Symplectic and implectic maps

In the case of difference equations, there is no Hamiltonian, only the analogue of the $\mathscr{L}_\Gamma\omega = \mathscr{L}_\Gamma J = 0$ condition: The pre-symplectic form ω is pre-symplectic for the map ϕ if $\phi^*\omega = \omega$. Similarly, the pre-implectic structure J is pre-implectic for ϕ if $\phi_* J = J \circ \phi$. In some circumstances, particularly in the study of KAM-type results for maps, it is necessary to impose the additional condition that the map be globally canonical [13]. This requires the map to preserve exact-symplectic structures. We have no use for this stronger condition in this paper.

2.10.2. The Noether theorems

We now give the four Noether-type theorems relevant in this paper. The first two are the standard Noether theorems of Hamiltonian mechanics, usually combined as one in the symplectic or implectic case and equivalent to the first of two theorems given by Noether in [2].

Theorem 1. *If the system of ODEs defined by Γ has a pre-implectic structure J and an integral I, then $v := J(\mathrm{d}I)$ is an evolutionary symmetry of Γ and $\mathscr{L}_v J = 0$.*

Proof. As I is an integral of Γ we have $\Gamma(I) = 0$, so $\mathscr{L}_\Gamma(\mathrm{d}I) = 0$. Now

$$[\Gamma, v] = \mathscr{L}_\Gamma(J(\mathrm{d}I)) = (\mathscr{L}_\Gamma J)(\mathrm{d}I) + J(\mathscr{L}_\Gamma\, \mathrm{d}I) = 0 \tag{57}$$

and so v is a symmetry. It is evolutionary because $J(\mathrm{d}t) = 0$. To see that it satisfies $\mathscr{L}_v J = 0$, first recall that for an arbitrary function h, $v(h) = \{I, h\}$. Now with arbitrary function f and g,

$$(\mathscr{L}_v J)(\mathrm{d}f, \mathrm{d}g) = v(J(\mathrm{d}f, \mathrm{d}g)) - J(\mathrm{d}(v(f)), \mathrm{d}g) - J(\mathrm{d}f, \mathrm{d}(v(g))) \tag{58}$$

$$= \{I, \{f, g\}\} - \{\{I, f\}, g\} - \{f, \{I, g\}\} \tag{59}$$

$$= 0\,, \tag{60}$$

the last equality holding by the Jacobi condition for J. □

Theorem 2. *If v is a symmetry of Γ and $\mathscr{L}_v\omega = 0$, where ω is pre-symplectic for Γ, then there is a function f such that $v\rfloor\omega = \mathrm{d}f$ and a constant $c \in \mathbf{R}$ such that $f - ct$ is an integral of Γ.*

Proof. As $\mathscr{L}_v\omega = 0$, there is a function f such that $v\rfloor\omega = \mathrm{d}f$. Taking the Lie derivative of this equation we have

$$\mathscr{L}_\Gamma(v\rfloor\omega) = (\mathscr{L}_\Gamma v)\rfloor\omega + v\rfloor(\mathscr{L}_\Gamma\omega) = 0 = \mathscr{L}_\Gamma\,\mathrm{d}f = \mathrm{d}(\Gamma(f))\,. \tag{61}$$

Hence for some $c \in \mathbf{R}$, $\Gamma(f) = c$ and we can define a time-dependent integral I of Γ by setting

$$I(t,x) := f(t,x) - ct\,. \qquad \Box \tag{62}$$

The next two are as above, but re-cast for the mapping case.

Theorem 3. *If a map ϕ has a pre-implectic structure J and an integral I, then $v := J(\mathrm{d}I)$ is a symmetry of ϕ and $\mathscr{L}_v J = 0$.*

Proof. As J is pre-implectic for ϕ we have

$$J_{\phi(x)}(\mathrm{d}I_{\phi(x)}, \mathrm{d}g_{\phi(x)}) = J_x(\phi^*\,\mathrm{d}I_x, \phi^*\,\mathrm{d}g_x)\,. \tag{63}$$

Here the subscript is the point at which the vector field, differential form, etc. is evaluated and g is an arbitrary function. Since I is an integral, $\phi^*\,\mathrm{d}I = \mathrm{d}I$ and so

$$J_{\phi(x)}(\mathrm{d}I_{\phi(x)}, \mathrm{d}g_{\phi(x)}) = J_x(\mathrm{d}I_x, \phi^*\,\mathrm{d}g_x)\,, \tag{64}$$

or

$$\langle v_{\phi(x)}, \mathrm{d}g_{\phi(x)}\rangle = \langle \phi_* v_x, \mathrm{d}g_{\phi(x)}\rangle\,. \tag{65}$$

This being true for arbitrary functions g, $v \circ \phi = \phi_* v$ as required.

That $\mathscr{L}_v J = 0$ is proved in exactly the same manner as in Theorem 1. □

Theorem 4. *If a map ϕ has a pre-symplectic structure ω and a symmetry v such that $\mathscr{L}_v\omega = 0$, then there is a function f satisfying $v\rfloor\omega = \mathrm{d}f$ and a constant $c \in \mathbf{R}$ such that $f - ct$ is an integral of ϕ.*

Proof. Once again there is a function f such that $v\rfloor\omega = \mathrm{d}f$. Now if X is an arbitrary vector field.

$$(\phi^* \mathrm{d}f)_x(X_x) = (v_{\phi(x)}\rfloor\omega_{\phi(x)})(\phi_* X_x) \tag{66}$$

$$= \omega_{\phi(x)}(\phi_* v_x, \phi_* X_x) \tag{67}$$

$$= (\phi^* \omega)_x(v_x, X_x)\,, \tag{68}$$

since $\phi_* v = v \circ \phi$ as it is a symmetry. As $\phi^*\omega = \omega$ we then have

$$\phi^* \mathrm{d}f = v\rfloor\omega = \mathrm{d}f\ . \tag{69}$$

Thus $\mathrm{d}(f \circ \phi) = \mathrm{d}f$ and so $f \circ \phi - f = c \in \mathbf{R}$. If we now define

$$I(t,x) := f(t,x) - ct\ , \tag{70}$$

we have $I \circ \phi = I$. □

In the case where J and ω are invertible, these theorems define an algebra isomorphism between first integrals modulo constants and evolutionary symmetries v satisfying $\mathscr{L}_v J = \mathscr{L}_v \omega = 0$, called Noether symmetries.

In the maximal-rank, time-dependent case the result still holds but with the constants replaced by functions of t (Theorem 6.33 in [9]). The Liouville–Arnold theorem [1] is also usually stated only for the case of invertible J, but can be proved for the maximal rank case [14].

2.11. Some comments on compatibility

A question normally associated with multiple symplectic or implectic structures is "are they compatible?" Unfortunately, in the non-maximal rank cases we discuss in this paper, it is not clear what compatibility means.

Typically one wishes to know if the recursion operator $R := J_2^{-1} J_1$ maps integrals to integrals. Alternately R can be defined from symplectic forms as $R^{-1} := \omega_1 \omega_2^{-1}$ and it is desired that R^{-1} maps Noether symmetries with respect to ω_1 to Noether symmetries with respect to ω_2. Clearly, neither of these definitions is acceptable if the J_i and ω_i are not of maximal rank. The time-dependent situation of maximal rank, non-invertible structures can be dealt with by working on constant-time hypersurfaces.

In the case where J_1 and J_2 have maximal rank, compatibility is assured if $J_1 + J_2$ (or some other linear combination) satisfies the Jacobi condition [9]. Since this condition can be checked regardless of rank, it is tempting to use it as a definition but it is not clear what this would mean, if anything. In the symplectic case the usual test is that $\omega_1 \omega_2^{-1} \omega_1$ must satisfy the Jacobi condition, $\mathrm{d}(\omega_1 \omega_2^{-1} \omega_1) = 0$. This requires at least ω_2 to be invertible, so there is no corresponding easy definition in the symplectic case. It is possible to work directly from the Lie derivative condition (recall that a symmetry v is a Noether symmetry if $\mathscr{L}_v \omega = 0$), but the resulting condition is much stronger than adopting the $J_1 + J_2$ condition in the implectic case. We hope to examine the question of compatibility and non-maximal rank structures in a future publication.

While there will be no further discussion of compatibility in this paper, we will still check if linear combinations $aJ_1 + bJ_2$ satisfy the Jacobi condition, simply to determine if the resulting new structures are also implectic. The corresponding test in the symplectic case is trivial, since the Jacobi condition for symplectic structures is trivial.

3. One symmetry and one integral imply a pre-implectic structure

3.1. The ODE case

Consider an n-dimensional manifold N, on which is defined a pair of vector fields ξ and η which are pointwise linearly independent. Their span K_x at $x \in N$ is a subspace of the tangent space T_xN. We are interested in the case where there exist two-dimensional submanifolds S through each point $x \in N$, such that $T_xS_{(x)} = K_x \subset T_xN \ \forall x \in N$. Here $S_{(x)}$ denotes the unique submanifold passing through x.

It is a consequence of Frobenius's theorem (see for example [15]) that such submanifolds S exist if and only if

$$[\xi, \eta] = a\xi + b\eta \,, \tag{71}$$

with a, b smooth functions on N. Vector fields which satisfy this condition are said to form an *integrable system*.

Suppose ξ, η form an integrable system and let S be a two-dimensional submanifold to which they are tangent. Then the restrictions $\xi|_S, \eta|_S$ are vector fields on S, forming a basis for T_xS for all $x \in S$. It follows that the skew product (or bi-vector) $J_S := A_S\xi|_S \wedge \eta|_S$, with A_S an arbitrary function $A_S : S \to \mathbf{R}$, is a non-singular map $T^*S \to TS$. Consequently there is an inverse map ω_S which is a maximal rank 2-form on S. Since all 3-forms are identically zero on a 2-manifold, the Jacobi condition $\mathrm{d}\omega = 0$ is satisfied trivially and ω is a symplectic structure on S. The inverse of a symplectic structure being implectic, J_S is an implectic structure on S.

Moreover since ξ and η are smooth vector fields on E, if we choose A_S to be the restriction to S of a smooth function $A : E \to \mathbf{R}$ then we can define $J := A\xi \wedge \eta$ which satisfies

$$J_S = J|_S \,. \tag{72}$$

Note also that if we choose an arbitrary implectic structure J' on E, then its restriction to S differs from J_S only by a scalar multiplier.

The preceding facts allow us to prove some results on the existence of singular Hamiltonian structures for systems of ODEs.

Proposition 5. *Assume a system of autonomous ODEs represented by a vector field Γ on M, with a second vector field η, point-wise linearly independent of Γ, which satisfies $[\Gamma,\eta] = a\Gamma + b\eta$ for some smooth $a,b : E \to \mathbf{R}$ and an integral of motion $H : M \to \mathbf{R}$ such that $\eta(H) \neq 0$ on M. Then there is a foliation of M by 2 dimensional submanifolds S on which Γ restricts to a Hamiltonian system and Γ is a singular Hamiltonian system on M.*

Conversely, if Γ represents a singular Hamiltonian system then in a neighbourhood of every point of M there is a smooth vector field ξ which forms an integrable system with Γ.

Proof. Γ and η form an integrable system, so there is a foliation by submanifolds S. If $J := (1/\eta(H))\Gamma \wedge \eta$, then the restriction $J|_S$ is implectic on each S from the discussion above. Moreover, since H is an integral of motion, $\Gamma(H) = 0$ and thus $J(\mathrm{d}H) = \Gamma$.

The additional condition for Γ to be a singular Hamiltonian is that Γ is in the image of J, which is obvious from the definition.

The converse is just Darboux's theorem (see Olver Theorem 6.22 in [9]). □

The time-dependent version of Proposition 5 can be proved by working on each constant time hyper-surface.

Proposition 6. *Assume a system of time-dependent ODEs represented by a vector field Γ on $E = \mathbf{R} \times M$, together with an evolutionary vector field η, pointwise independent of $\tilde{\Gamma} := \Gamma - \partial/\partial t$, and an integral of motion $H : M \to \mathbf{R}$ such that $\eta(H) \neq 0$ on E. Then there is a foliation of each constant time slice $M_\tau := \{(t,x) \in E : t = \tau\}$ by 2 dimensional submanifolds S such that Γ restricts to a Hamiltonian system on each $\mathbf{R} \times S$ and Γ is a singular Hamiltonian system on E.*

Conversely, if Γ represents a singular Hamiltonian system then in a neighbourhood of every point of E there is a smooth evolutionary vector field ξ which forms an integrable system with $\tilde{\Gamma}$.

Note that an autonomous system can of course be treated as time dependent, allowing η and/or H to depend on t.

Comparison with the definition of a projective symmetry yields the following corollary to Proposition 5.

Corollary 7. *If an autonomous ODE Γ has a nowhere trivial, time-independent projective symmetry generator η and an integral H such that $\eta(H) \neq 0$, then Γ is a singular Hamiltonian system with respect to $J := (1/\eta(H))\Gamma \wedge \eta$.*

This result appears in Hojman [10] for the special case of symmetries rather than projective symmetries, although he does not show the relationship between symmetries and integrable submanifolds.

3.1.1. The mapping case

There is a simple extension of these results to the case of differentiable mappings.

Proposition 8. *Suppose that the independent vector fields ξ and η are symmetries of a differentiable map ϕ and form an integrable system. Then $J := \xi \wedge \eta$ is a pre-implectic structure for ϕ.*

Proof. From the previous discussion we know that the integrability condition implies that J satisfies the Jacobi condition, so we need only show that J is preserved by ϕ, $\phi_* J \phi_*^{\mathrm{T}} = J \circ \phi$. However since ξ and η are symmetries, $\phi_* \xi = \xi \circ \phi$ and $\phi_* \eta = \eta \circ \phi$. Now

$$\phi_*(\xi \wedge \eta)\phi_*^{\mathrm{T}} = (\phi_* \xi) \wedge (\phi_* \eta) = (\xi \circ \phi) \wedge (\eta \circ \phi) = J \circ \phi\,. \qquad \square \tag{73}$$

Note that an extra symmetry is required to replace the one provided by the ODE itself, but it is not necessary to know an integral of the map.

4. Volume preservation and $n-2$ symmetries imply pre-symplectic structure

4.1. ODE case

Let M be an n-dimensional manifold with generic local coordinates $x^1, \ldots, x^n$, and a volume form $\Omega = f\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, where f is a function $f : M \to \mathbf{R}$. A system of n first-order ODE is defined on M by a vector field Γ. The ODE is said to be volume preserving if Γ preserves Ω, that is the Lie derivative $\mathscr{L}_\Gamma \Omega = 0$.

Proposition 9. *If Γ is a volume preserving system of ODE on M and there is a system of volume preserving symmetries $X_1, \ldots, X_{n-2}$ which satisfy, for $j < k$,*

$$[X_j, X_k] = \sum_{i=1}^{j=1} a^i_{jk} X_i \tag{74}$$

then the 2-form $\omega := X_{n-2} \rfloor \cdots X_1 \rfloor \Omega$ is pre-symplectic for Γ. That is to say ω is preserved by Γ, $\mathscr{L}_\Gamma \omega = 0$, and satisfies the Jacobi identity $\mathrm{d}\omega = 0$.

Proof. Define inductively $\Omega_{j+1} := X_{j+1} \rfloor \Omega_j$, $j = 1, \ldots, n-3$ and $\Omega_1 := X_1 \rfloor \Omega$. We will show that setting $\omega := \Omega_{n-2}$ satisfies the theorem.

Assume for some $k > 0$ that $\mathrm{d}\Omega_j = 0\ \forall j \leqslant k$. As Ω is an n-form, $\mathrm{d}\Omega = 0$ automatically. Note also that $\mathscr{L}_{X_j} \Omega = 0$ implies $\mathrm{d}(X_j \rfloor \Omega) = 0$ and in particular $\mathrm{d}\Omega_1 = 0$.

Now consider

$$\mathscr{L}_{X_{k+1}} \Omega_k = \mathscr{L}_{X_{k+1}}(X_k \rfloor \Omega_{k-1}) \tag{75}$$

$$= [X_{k+1}, X_k] \rfloor \Omega_{k-1} + X_k \rfloor \mathscr{L}_{X_{k+1}} \Omega_{k-1} \tag{76}$$

$$= \left(-\sum_{i=1}^{k-1} a^i_{k,\,k+1} X_i \right) \rfloor \Omega_{k-1} + X_k \rfloor \mathscr{L}_{X_{k+1}} \Omega_{k-1} \tag{77}$$

$$= X_k \rfloor \mathscr{L}_{X_{k+1}} \Omega_{k-1} \,. \tag{78}$$

However

$$\mathscr{L}_{X_{k+1}} \Omega_{k-1} = [X_{k+1}, X_{k-1}] \rfloor \Omega_{k-2} + X_{k-1} \rfloor \mathscr{L}_{X_{k+1}} \Omega_{k-2} \tag{79}$$

$$= X_{k-1} \rfloor \mathscr{L}_{X_{k+1}} \Omega_{k-2} \,. \tag{80}$$

Continuing in this way we obtain

$$\mathscr{L}_{X_{k+1}} \Omega_k = X_k \rfloor \cdots X_1 \rfloor (\mathscr{L}_{X_{k+1}} \Omega) = 0 \,. \tag{81}$$

Now

$$\mathscr{L}_{X_{k+1}} \Omega_k = X_{k+1} \rfloor \mathrm{d}\Omega_k + \mathrm{d}(X_{k+1} \rfloor \Omega_k) \tag{82}$$

$$= \mathrm{d}\Omega_{k+1} \tag{83}$$

so $\mathrm{d}\Omega_j = 0$ for $j = 1, \ldots, n-2$.

Thus ω is a closed 2-form and therefore a pre-symplectic form on M. It remains to show that ω is preserved by Γ, that is $\mathscr{L}_\Gamma \omega = 0$. For arbitrary k,

$$\mathscr{L}_\Gamma \Omega_k = [\Gamma, X_k] \rfloor \Omega_{k-1} + X_k \rfloor \mathscr{L}_\Gamma \Omega_{k-1} \,,$$

for which we obtain

$$\mathscr{L}_\Gamma \Omega_k = X_k \rfloor \cdots X_1 \rfloor \mathscr{L}_\Gamma \Omega = 0 \,. \tag{84}$$

Consequently $\mathrm{d}(\Gamma \rfloor \omega) = 0$ and there is a local Hamiltonian H such that

$$\Gamma \rfloor \Omega_{n-2} = \mathrm{d}H \,. \qquad \square \tag{85}$$

4.2. Mapping case

An analogous result holds for volume preserving maps.

Proposition 10. *If ϕ is a volume preserving map, $\phi^* \Omega = \Omega$, with volume preserving symmetries $X_1, \ldots, X_{n-2}$ satisfying the same integrability conditions as in Proposition 9, then ω is a pre-symplectic form for ϕ.*

Proof. We construct ω as in Proposition 9, so we have already that ω satisfies the Jacobi condition $\mathrm{d}\omega = 0$. It remains to prove that ω is preserved by the map, $\phi^* \omega = \omega$. To do this we first prove a lemma. $\square$

Lemma 11. *Let σ be a $k+1$-form preserved by ϕ, $\phi^* \sigma = \sigma$. Then if X is a symmetry vector field of $\phi, \phi^*(X \rfloor \sigma) = (X \rfloor \sigma)$.*

Proof. A vector field X is a symmetry of ϕ iff $\phi_* X = X \circ \phi$. Now if we evaluate $X \rfloor \sigma$ on arbitrary vector fields $Y_1, \ldots, Y_k$ at the point $x \in M$, we have

$$\phi^*(X \rfloor \sigma)_x(Y_1|_x, \ldots, Y_k|_x) = \sigma_{\phi(x)}(X|_{\phi(x)}, \phi_* Y_1|_x, \ldots, \phi_* Y_k|_x) \tag{86}$$

$$= \sigma_{\phi(x)}(\phi_* X|_x, \phi_* Y_1|_x, \ldots, \phi_* Y_k|_x) \tag{87}$$

$$= (X \rfloor \phi^* \sigma)_x(Y_1|_x, \ldots, Y_k|_x)\,. \tag{88}$$

Hence $\phi^*(X \rfloor \sigma) = X \rfloor \phi^* \sigma$. □

Proof of Proposition 10 (*Conclusion*). The proof is now completed by induction. If we set $\Omega_0 := \Omega$, then $\phi^* \Omega_0 = \Omega_0$. Using the lemma with $\omega = \Omega_k$ and $X = X_{k+1}$, $k = 0, \ldots, n-3$, $\phi^* \Omega_k = \Omega_k$ implies that $\phi^* \Omega_{k+1} = \Omega_{k+1}$ and so with $k = n-3$ we obtain $\phi^* \omega = \omega$. □

4.3. Multiple pre-symplectic structures

Note that a re-ordering of the symmetries $X_1, \ldots, X_{n-2}$ does not affect the resulting pre-symplectic form Ω_{n-2} beyond a possible change of sign. More generally, any set of $n-2$ symmetries constructed from linear combinations of $X_1, \ldots, X_{n-2}$ will give the same Ω_{n-2} except for multiplication by the Jacobian of the transformation of the symmetries.

However, if $n-1$ symmetries are known, infinitely many independent sets of $n-2$ symmetries can be constructed from them, parameterised by an $n-1$ sphere, each yielding an independent pre-symplectic form. Moreover, any linear combination of them also satisfies the Jacobi condition, as it is a linear condition in the (pre)-symplectic case.

5. Volume preservation and $n-2$ integrals imply a pre-implectic structure

This section contains results essentially dual to those in Section 4. Whereas in that section symmetries and a conserved volume form were used to construct a pre-symplectic form for either an ODE or a mapping, here we will use integrals and an n-multi-vector dual to the conserved volume form to construct a pre-implectic structure.

Proposition 12. *Let τ be a non-zero n-multi-vector on an n-dimensional manifold M and let $I_1, \ldots, I_{n-2}$ be independent functions on M, so that $\mathrm{d}I_1 \wedge \cdots \wedge \mathrm{d}I_{n-2} \neq 0$. Then the skew 2-contravariant tensor $J := \tau \rfloor \mathrm{d}I_1 \cdots \rfloor \mathrm{d}I_{n-2}$ satisfies the Jacobi condition.*

Proof. Note that the common kernel of $\mathrm{d}I_1, \ldots, \mathrm{d}I_{n-2}$ is everywhere 2-dimensional and contains J. Thus in a neighbourhood of any point $x \in M$, there exist independent vector fields ξ and η and a function g such that $J = g \xi \wedge \eta$. Moreover, the kernel defines an integrable foliation (the common level surfaces of $I_1, \ldots, I_{n-2}$) so ξ and η form an integrable system, $[\xi,\eta] = a\xi + b\eta$ for some local functions a, b. From the discussion in Section 2.8 it follows that J satisfies the Jacobi condition. □

5.1. The ODE case

If the system of ODEs is defined by a vector field Γ, then it remains to show that with a suitable choice of τ and I_j, $\mathscr{L}_\Gamma J = 0$.

Proposition 13. *Let Ω be a volume form satisfying $\mathscr{L}_\Gamma \Omega = 0$ and define τ to be the n-multi-vector such that $\tau \rfloor \Omega = 1$. Then if $I_1, \dots, I_{n-2}$ are integrals of Γ with $\mathrm{d}I_1 \wedge \cdots \wedge \mathrm{d}I_{n-2} \neq 0$, then $J := \tau \rfloor \mathrm{d}I_1 \cdots \rfloor \mathrm{d}I_{n-2}$ is a pre-implectic form for Γ.*

Proof. We have that

$$0 = \mathscr{L}_\Gamma(\tau \rfloor \Omega) = \mathscr{L}_\Gamma \tau \rfloor \Omega + \tau \rfloor \mathscr{L}_\Gamma \Omega = \mathscr{L}_\Gamma \tau \rfloor \Omega \tag{89}$$

and since the kernel of Ω is trivial, we have $\mathscr{L}_\Gamma \tau = 0$. We now set $\tau_0 := \tau$ and proceed by induction. If it is assumed that $\mathscr{L}_\Gamma \tau_k = 0$, $k < n-2$, then

$$\mathscr{L}_\Gamma \tau_{k+1} = \mathscr{L}_\Gamma(\tau_k \rfloor \mathrm{d}I_{k+1}) = \mathscr{L}_\Gamma \tau_k \rfloor \mathrm{d}I_{k+1} + \tau_k \rfloor \mathscr{L}_\Gamma \, \mathrm{d}I_{k+1} = 0 \,. \tag{90}$$

Combined with Proposition 12, this gives the result with $J := \tau_{n-2}$. □

Unlike the dual case discussed in Section 4, we do not automatically obtain a Hamiltonian for Γ. However we do have the following result.

Proposition 14. *Let H be any integral of Γ independent of $I_1, \dots, I_{n-2}$. Then there is a function f such that $\Gamma = fJ\nabla H$, so that H is a Hamiltonian for Γ with respect to the pre-implectic structure fJ.*

Proof. Recall that for pre-implectic structures of rank 2, the Jacobi condition is conformally invariant. Thus fJ is pre-implectic.

Now let ξ be any local vector field independent of Γ and contained in the common kernel of $\mathrm{d}I_1, \dots, \mathrm{d}I_{n-2}$. Then since the kernel is spanned by Γ and ξ, there is a local function g such that $J = g\Gamma \wedge \xi$. The common kernel of $\mathrm{d}H, \mathrm{d}I_1, \dots, \mathrm{d}I_{n-2}$ is spanned by Γ, so we can define $f := 1/(g\xi(H))$.

Checking, we have

$$fJ\nabla H = \frac{1}{g\xi(H)} g\xi(H)\Gamma = \Gamma \,. \qquad \square \tag{91}$$

5.2. The mapping case

In the mapping case, recall that the pre-implectic structure is required to satisfy $\phi_* \tau_{n-2} = \tau_{n-2} \circ \phi$ in addition to the Jacobi condition.

Proposition 15. *Let $\phi : M \to M$ be a smooth mapping. Let Ω be a volume form satisfying $\phi^* \Omega = 0$ and define τ to be the n-multi-vector such that $\tau \rfloor \Omega = 1$. Then if*

$I_1,\dots,I_{n-2}$ are integrals of ϕ with $\mathrm{d}I_1 \wedge \cdots \wedge \mathrm{d}I_{n-2} \neq 0$, then $J := \tau \rfloor \mathrm{d}I_1 \cdots \rfloor \mathrm{d}I_{n-2}$ is a pre-implectic form for ϕ.

Proof. The Jacobi condition holds for J by Proposition 12. Since $\tau \rfloor \Omega = 1$ and $\phi^* \Omega = \Omega$,

$$\langle \phi^* \Omega, \tau \rangle|_x = \langle \Omega_{\phi(x)}, \phi_* \tau_x \rangle = \langle \Omega_x, \tau_x \rangle = \langle \Omega_{\phi(x)}, \tau_{\phi(x)} \rangle . \tag{92}$$

The trivial kernel of Ω then implies that $\phi_* \tau = \tau \circ \phi$. We now set $\tau_0 := \tau$ and proceed by induction: assume that for some $k < n-2, \phi_* \tau_k = \tau_k \circ \phi$. Then since $\phi^* \mathrm{d}I_j = \mathrm{d}I_j$,

$$\begin{aligned} \tau_{k+1} \circ \phi &= (\tau_k \circ \phi) \rfloor \mathrm{d}I_{k+1} \circ \phi = (\phi_* \tau_k) \rfloor (\phi_*)^{-1} \mathrm{d}I_{k+1} \\ &= \phi_* (\tau_k \rfloor \mathrm{d}I_{k+1}) = \phi_* \tau_{k+1} . \end{aligned} \tag{93}$$

The result is obtained recognising that $J = \tau_{n-2}$. □

5.3. *Multiple pre-implectic structures*

As with the dual case discussed in Section 4, the order in which the integrals are taken only affects the sign of the resulting pre-implectic structure J. Again, taking linear combinations of the $n-2$ integrals multiplies J by the Jacobian of the transformation. In fact, this is true even of arbitrary functions of $I_1,\dots,I_{n-2}$. If $J_i = f_i(I_1,\dots,I_{n-2})$, then the exterior derivatives $\mathrm{d}J_i$ satisfy

$$\mathrm{d}J_i = \sum_{k=1}^{n-2} f_{i,k}\, \mathrm{d}I_k , \quad i = 1,\dots,n-2 \tag{94}$$

and the derivatives $f_{i,k}$ are themselves integrals (either $\Gamma(f_{i,k}) = 0$ or $f_{i,k} \circ \phi = f_{i,k}$ as appropriate). Since J depends linearly on the $\mathrm{d}I_i$ rather than the I_i, it transforms by multiplication by $\det(f_{i,k})$.

If more than $n-2$ integrals are known, then as in Section 4 it is possible to choose inequivalent subsets of $n-2$ independent integrals and so derive independent pre-implectic structures J. Their linear combinations are also pre-implectic.

Corollary 16. *Suppose that $I_1,\dots,I_{n-2}$ and $J_1,\dots,J_{n-2}$ are two sets of independent integrals. If σ and $\tilde{\sigma}$ are the corresponding pre-implectic structures ($\sigma := \tau \rfloor \mathrm{d}I_1 \cdots \rfloor \mathrm{d}I_{n-2}$; $\tilde{\sigma} := \tau \rfloor \mathrm{d}J_1 \cdots \rfloor \mathrm{d}J_{n-2}$), then for any constants a, b, $a\sigma + b\tilde{\sigma}$ is also pre-implectic.*

Proof. Assume *w.l.o.g.* that $I_2 = J_2,\dots,I_{n-2} = J_{n-2}$. If I_1 and J_1 are integrals, then so is $aI_1 + bJ_1$. The linearity of the inner product yields

$$a\sigma + b\tilde{\sigma} = \tau \rfloor (a\, \mathrm{d}I_1 + b\, \mathrm{d}J_1) \rfloor \mathrm{d}I_2 \cdots \rfloor \mathrm{d}I_{n-2} , \tag{95}$$

which proves the result. □

5.4. *Mapping in 3-D*

To illustrate the preceding theory, we specialise in this section to volume-preserving maps in 3 dimensions, with one known integral. We work on a 3 dimensional manifold

M with generic local coordinates (u, v, w), with a smooth map ϕ such that

$$(u', v', w') = \phi(u, v, w) . \tag{96}$$

Consider a volume form $\Omega = f\, \mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}w$. The dual multi-vector is

$$\tau := m \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial w} , \tag{97}$$

where for convenience we have set $m := 1/f$. It can be verified that m transforms as a measure under a map ϕ:

$$\tau = \det(\phi_*) m \frac{\partial}{\partial u'} \wedge \frac{\partial}{\partial v'} \wedge \frac{\partial}{\partial w'} , \tag{98}$$

so the measure preservation condition has the form $m \circ \phi = \det(\phi_*) m$. Now if we have an integral I, the following holds:

$$J = \tau \rfloor \mathrm{d}I = m \left(\frac{\partial I}{\partial u} \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial w} - \frac{\partial I}{\partial v} \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial w} + \frac{\partial I}{\partial w} \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial w} \right) , \tag{99}$$

or we have J as a matrix with respect to the (u, v, w) coordinates:

$$J = m \begin{pmatrix} 0 & \frac{\partial I}{\partial w} & -\frac{\partial I}{\partial v} \\ -\frac{\partial I}{\partial w} & 0 & \frac{\partial I}{\partial u} \\ \frac{\partial I}{\partial v} & -\frac{\partial I}{\partial u} & 0 \end{pmatrix} . \tag{100}$$

That J satisfies the Jacobi condition and is preserved by ϕ,

$$J \circ \phi = \phi_* J \phi_*^{\mathrm{T}} \tag{101}$$

follows from the general results proved above. However, it is instructive to prove them directly in the 3 dimensional case. First, the transformation properties of forms and vector fields yields the following useful identity (valid in n dimensions): if $\alpha_1, \ldots, \alpha_{n-2}$ are one forms whose coordinates are acted on by the matrix R, then

$$R(\tau \rfloor \alpha_1 \cdots \rfloor \alpha_{n-2}) R^{\mathrm{T}} = \det(R) \tau \rfloor (\alpha_1 R^{-1}) \cdots \rfloor (\alpha_{n-2} R^{-1}) . \tag{102}$$

In dimension 3, this simplifies to

$$R(\tau \rfloor \alpha) R^{\mathrm{T}} = \det(R) \tau \rfloor (\alpha R^{-1}) . \tag{103}$$

Thus in familiar 3 dimensional notation with R the Jacobi matrix ϕ_*, this becomes

$$\phi_* J \phi_*^{\mathrm{T}} = \det(\phi_*) \tau \cdot (\nabla I \phi_*^{-1}) \tag{104}$$

$$= (\tau \circ \phi) \cdot \nabla I \phi_*^{-1} . \tag{105}$$

This partly resolves on open question raised in [16]: are the alternate k-symplectic structures generated by a k-symplectic map compatible with the original structure? Many examples of such maps are constructed via complexification, with the alternate k-symplectic structures falling within the space of compatible structures generated by the method of Section 6.

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Since $(\nabla I)\circ\phi = (\nabla I)\phi_*^{-1}$, the RHS of Eq. (105) can be written as

$$(\tau \cdot \nabla I)\circ\phi = J \circ \phi \,. \tag{106}$$

This verifies condition (101).

To directly verify the Jacobi condition, we first note that if $J = (1/m)\tau\rfloor\alpha$, then the 3-dimensional Jacobi condition can be written as

$$(\nabla \times \alpha)\cdot\alpha = 0 \,. \tag{107}$$

Substituting $\alpha = m\nabla I$ we obtain $(\nabla m \times \nabla I)\cdot\nabla I$ for the LHS, which is clearly zero.

It should be noted that all the above results generalise immediately to k-integrals and k-pre-implectic structures. Recall that I is a k-integral for ϕ if it is an integral for $\phi^{(k)}$, the kth iterate of ϕ. Similarly J is a k-(pre-)symplectic structure for ϕ if it is (pre-)implectic for $\phi^{(k)}$ (see [16] for details).

5.5. Examples

The following trace map can be found in [17,18]. The map ϕ is defined in coordinates by

$$\phi(u,v,w) = (v, w, avw - u) \tag{108}$$

(here and below a is a constant) and preserves the integral

$$I(u,v,w) = u^2 + v^2 + w^2 - auvw \,. \tag{109}$$

The determinant $\det(\phi_*) = -1$, so although ϕ does not conserve volume, the iterate $\phi^{(2)}$ does: ϕ is 2-volume-conserving, by extension of the nomenclature of [16]. The preceding theory then implies that the pre-implectic structure

$$J = \begin{pmatrix} 0 & 2w - auv & -2v + auw \\ -2w + auv & 0 & 2u - avw \\ 2v - auw & -2u + avw & 0 \end{pmatrix} \tag{110}$$

is preserved upto sign by ϕ, and preserved by $\phi^{(2)}$, so that J is 2-pre-implectic for ϕ. In fact

$$\phi_* J \phi_*^{\mathrm{T}} = -J \circ \phi \,. \tag{111}$$

Another example is the discrete Sine-Gordon equation, see [19]. Here ϕ is defined by

$$\phi(u,v,w) = (u', v', w') = \left(v, w, \frac{1 - avw}{u(vw - a)}\right) \,, \tag{112}$$

and preserves the integral

$$I(u,v,w) = a\left(\frac{w}{u} + \frac{u}{w}\right) - uv - \frac{1}{uv} - vw - \frac{1}{vw} \,. \tag{113}$$

The determinant of the mapping is given by $\det(\phi_*) = -u'v'w'/(uvw)$, so if we set $m(u,v,w) := uvw$ the map is anti-measure-preserving, hence 2-measure-preserving. So once again we obtain a 2-pre-implectic structure J for ϕ by setting

$$J = \begin{pmatrix} 0 & a\left(\frac{1}{u} - \frac{u}{w^2}\right) - v + \frac{1}{vw^2} & u - \frac{1}{uv^2} + w - \frac{1}{v^2}w \\ a\left(-\frac{1}{u} + \frac{u}{w^2}\right) + v - \frac{1}{vw^2} & 0 & a\left(-\frac{w}{u^2} + \frac{1}{w}\right) - v + \frac{1}{u^2 v} \\ -u + \frac{1}{uv^2} - w + \frac{1}{v^2}w & a\left(\frac{w}{u^2} - \frac{1}{w}\right) + v - \frac{1}{u^2 v} & 0 \end{pmatrix} . \tag{114}$$

Finally, as an example of a map with two integrals having a "bi-Hamiltonian" structure (more accurately a pair of independent pre-implectic structures), we take a volume-preserving map given by Fordy et al. [20]: $\phi(u,v,w) = (u',v',w')$ with

$$u' = -u + \frac{2w}{2v - 2u^2 - a} , \tag{115}$$

$$v' = -v + u^2 - u'^2 , \tag{116}$$

$$w' = w . \tag{117}$$

The two conserved quantities are

$$I_1(u,v,w) = w ; \tag{118}$$

$$I_2(u,v,w) = -\frac{1}{2}v^2 + \frac{1}{2}u^4 + \frac{a}{2}u^2 + uw , \tag{119}$$

to which correspond the two pre-implectic structures

$$J_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & u & v \\ -u & 0 & 2u^3 + au + w \\ -v & -2u^3 - au - w & 0 \end{pmatrix} . \tag{120}$$

6. Complex structures imply infinitely many real structures

6.1. Implectic and pre-implectic case

In this section we assume that an implectic or pre-implectic structure J is given and that it is complex analytic. The only point at which we assume J to have maximal rank is in Proposition 18, where it is shown that the derived real structures are implectic if the original complex structure is. All other results hold equally for pre-implectic J.

In the ODE case we assume that a complex analytic Hamiltonian function H is also available and in the mapping case that the map ϕ is analytic. We work on an n-dimensional complex manifold M with generic coordinates $(z^1,\dots,z^n)$ and write $z^j =$

$x^j + iy^j$, the x^j, y^j being coordinates for the underlying $2n$-dimensional real manifold. The complex structure will be denoted $\mathscr{I}, \mathscr{I}^2 = -I$, so that if a complex vector field v has the real form w, then iv has the real form $\mathscr{I}w$. In this section coordinates will be ordered so that $\mathscr{I}$ has the matrix representation

$$\mathscr{I} = \begin{pmatrix} & I \\ -I & \end{pmatrix} . \tag{121}$$

The real and imaginary parts of functions and maps (in the given coordinates) will be written as $\tilde{f} := \Re(f), \hat{f} = \Im(f)$. The following real forms of J will be important:

$$K_1 := \begin{pmatrix} \tilde{J} & \hat{J} \\ \hat{J} & -\tilde{J} \end{pmatrix}, \qquad K_2 := \begin{pmatrix} -\hat{J} & \tilde{J} \\ \tilde{J} & \hat{J} \end{pmatrix} . \tag{122}$$

Note that $-\mathscr{I}K_1 = K_1\mathscr{I} = K_2$. Also a caution: it is tempting to look at the matrix representation of J and treat it as an element of the linear group $\mathrm{GL}(n, \mathbf{C})$. There is a standard real representation of $\mathrm{GL}(n, \mathbf{C})$ in $\mathrm{GL}(2n, \mathbf{R})$, which would give the spurious real form

$$J_C := \begin{pmatrix} \tilde{J} & \hat{J} \\ -\hat{J} & \tilde{J} \end{pmatrix} . \tag{123}$$

This is *not* an implectic structure. The method fails because J acts as an inner product on vectors at each point, not an automorphism. Therefore, it is not the representation of an element of $\mathrm{GL}(n, \mathbf{C})$.

Proposition 17. *If the complex analytic skew matrix J satisfies the Jacobi condition, then so does each of the real forms K_1 and K_2 and so does any constant linear combination $aK_1 + bK_2$.*

Proof. The Jacobi condition can be written in components as

$$\sum_{a,b,c \text{ cyclic}} J^{ab}_d J^{dc} = 0 , \tag{124}$$

or more briefly as

$$(J \cdot \nabla J)^{[abc]} = 0 , \tag{125}$$

where enclosing the superscripts in square brackets is a traditional notation for skew-symmetrisation.

Making use of the Cauchy–Riemann conditions

$$\nabla_x \tilde{J} = \nabla_y \hat{J}, \quad \nabla_x \hat{J} = -\nabla_y \tilde{J} \tag{126}$$

and taking real and imaginary components we obtain

$$\Re((J \cdot \nabla J)^{[abc]}) = (\tilde{J} \cdot \nabla_x \tilde{J} - \hat{J} \cdot \nabla_x \hat{J})^{[abc]} = 0 , \tag{127}$$

$$\Im((J \cdot \nabla J)^{[abc]}) = (\tilde{J} \cdot \nabla_x \hat{J} + \hat{J} \cdot \nabla_x \tilde{J})^{[abc]} = 0 . \tag{128}$$

If we then define

$$\nabla_C f := \begin{pmatrix} \nabla_x f \\ \nabla_y f \end{pmatrix}, \tag{129}$$

the real forms K_1 and K_2 must satisfy the four conditions

$$(K_i \cdot \nabla_C K_j)^{[\gamma\delta\varepsilon]} = 0, \quad i,j = 1,2, \;\; \gamma,\delta,\varepsilon = 1,\ldots,2n \tag{130}$$

if the family $\alpha K_1 + \beta K_2$ is to satisfy the Jacobi identity. We now introduce the indices $a,b,c = 1,\ldots,n$ and $\bar{a},\bar{b},\bar{c} = n+1,\ldots,2n$. Each of the conditions (130) then splits into four further conditions:

(1) $(K_i \cdot \nabla_C K_j)^{[abc]} = 0$;
(2) $(K_i \cdot \nabla_C K_j)^{[\bar{a}bc]} = 0$;
(3) $(K_i \cdot \nabla_C K_j)^{[\bar{a}\bar{b}c]} = 0$;
(4) $(K_i \cdot \nabla_C K_j)^{[\bar{a}\bar{b}\bar{c}]} = 0$.

Cases 1 and 4 are the simplest to handle. For case 1 with $i = j = 1$ we obtain

$$(\tilde{J} \cdot \nabla_x \tilde{J} + \hat{J} \cdot \nabla_y \tilde{J})^{[abc]} = (\tilde{J} \cdot \nabla_x \tilde{J} - \hat{J} \cdot \nabla_x \hat{J})^{[abc]} = 0\,. \tag{131}$$

and with $i = j = 2$

$$(-\hat{J} \cdot \nabla_x \hat{J} + \tilde{J} \cdot \nabla_y \hat{J})^{[abc]} = (-\hat{J} \cdot \nabla_x \hat{J} + \tilde{J} \cdot \nabla_x \tilde{J})^{[abc]} = 0\,. \tag{132}$$

Similarly the mixed terms $i = 1, j = 2$ and $i = 2, j = 1$ both give

$$-(\tilde{J} \cdot \nabla_x \hat{J} + \hat{J} \cdot \nabla_x \tilde{J})^{[abc]} = 0\,. \tag{133}$$

In case 4, the two diagonal terms $i = j = 1$ and $i = j = 2$ each give (upto sign)

$$(\tilde{J} \cdot \nabla_x \hat{J} + \hat{J} \cdot \nabla_x \tilde{J})^{[abc]} = 0 \tag{134}$$

while the cross terms each give

$$-(\tilde{J} \cdot \nabla_x \tilde{J} - \hat{J} \cdot \nabla_x \hat{J})^{[abc]} = 0\,. \tag{135}$$

The other two cases mix terms. In case 2,

$$(K_i \cdot \nabla_C K_j)^{[\bar{a}bc]} = (K_i \cdot \nabla_C K_j)^{\bar{a}bc} + (K_i \cdot \nabla_C K_j)^{c\bar{a}b} + (K_i \cdot \nabla_C K_j)^{bc\bar{a}}\,, \tag{136}$$

so setting $i = j = 1$ we obtain

$$\begin{aligned}(\hat{J} \cdot \nabla_x \tilde{J} + \tilde{J} \cdot \nabla_x \hat{J})^{abc} &+ (\tilde{J} \cdot \nabla_x \hat{J} + \hat{J} \cdot \nabla_x \tilde{J})^{cab} + (\tilde{J} \cdot \nabla_x \hat{J} + \hat{J} \cdot \nabla_x \tilde{J})^{bca} \\ &= (\tilde{J} \cdot \nabla_x \hat{J} + \hat{J} \cdot \nabla_x \tilde{J})^{[abc]} = 0\,.\end{aligned} \tag{137}$$

The same result is obtained for case 2 with $i = j = 2$ and for $i \neq j$ in case 3. The remaining four subcases, case 2 with $i \neq j$ and case 3 with $i = j$, each yield, upto sign,

$$(\tilde{J} \cdot \nabla_x \tilde{J} - \hat{J} \cdot \nabla_x \hat{J})^{[abc]} = 0\,. \tag{138}$$

It then follows that if a,b are constants,

$$((aK_1 + bK_2) \cdot \nabla_C (aK_1 + bK_2))^{[\gamma\delta\varepsilon]} = 0\,, \tag{139}$$

so $aK_1 + bK_2$ is a pre-implectic structure for any choice of constants a,b provided that J is an analytic pre-implectic structure. $\square$

Proposition 18. *If J is an invertible analytic implectic structure, then any linear combination $aK_1 + bK_2$ (where K_1 and K_2 are as defined in Eqs.* (122)) *is implectic.*

Proof. Together with Proposition 17, it suffices to prove that $aK_1 + bK_2$ is maximal rank unless $a = b = 0$. As M has real dimension $2n$, this means $aK_1 + bK_2$ must be invertible. Suppose it is not, so there is a one-form $\bar{\alpha}$, the complex conjugate of α, such that

$$(aK_1 + bK_2)\alpha = 0\,. \tag{140}$$

This is equivalent to the matrix equation

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \tilde{J}\tilde{\alpha} - \hat{J}\hat{\alpha} \\ \hat{J}\tilde{\alpha} + \tilde{J}\hat{\alpha} \end{pmatrix} = 0\,, \tag{141}$$

so unless the determinant $a^2 + b^2 = 0$ we have $J\alpha = 0$ and so J has at least one zero eigenvalue. □

6.1.1. ODE case

Proposition 19. *If we are given an n-dimensional Hamiltonian system*

$$\dot{z} = v = J\nabla H\,, \tag{142}$$

where H is an analytic function of the coordinates $(z^1, \ldots, z^n)$ and J is an arbitrary analytic implectic structure, then for all $\alpha, \beta \in \mathbf{R}$ such that $\alpha^2 + \beta^2 = 1$,

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = (\alpha K_1 + \beta K_2)\nabla_C(\alpha\tilde{H} + \beta\hat{H})\,. \tag{143}$$

Proof. We have the Cauchy–Riemann conditions

$$\frac{\partial\tilde{H}}{\partial x^j} = \frac{\partial\hat{H}}{\partial y^j}, \quad \frac{\partial\hat{H}}{\partial x^j} = -\frac{\partial\tilde{H}}{\partial y^j}\,, \tag{144}$$

from which it follows that

$$\Re\left(\frac{\partial H}{\partial z^j}\right) = \frac{\partial\tilde{H}}{\partial x^j} = \frac{\partial\hat{H}}{\partial y^j}\,, \tag{145}$$

$$\Im\left(\frac{\partial H}{\partial z^j}\right) = \frac{\partial\hat{H}}{\partial x^j} = -\frac{\partial\tilde{H}}{\partial y^j}\,. \tag{146}$$

We then have

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = \begin{pmatrix} \tilde{J} & \hat{J} \\ \hat{J} & -\tilde{J} \end{pmatrix} \nabla_C\tilde{H} = \begin{pmatrix} -\hat{J} & \tilde{J} \\ \tilde{J} & -\hat{J} \end{pmatrix} \nabla_C\hat{H} \tag{147}$$

$$= a\begin{pmatrix} \tilde{J} & \hat{J} \\ \hat{J} & -\tilde{J} \end{pmatrix} \nabla_C\tilde{H} + b\begin{pmatrix} -\hat{J} & \tilde{J} \\ \tilde{J} & -\hat{J} \end{pmatrix} \nabla_C\hat{H}, \quad a + b = 1\,. \tag{148}$$

Also, if we calculate the cross terms

$$\begin{pmatrix} \tilde{J} & \hat{J} \\ \hat{J} & -\tilde{J} \end{pmatrix} \nabla_C \tilde{H} + \begin{pmatrix} -\hat{J} & \tilde{J} \\ \tilde{J} & -\hat{J} \end{pmatrix} \nabla_C \tilde{H} \tag{149}$$

we obtain (using the Cauchy–Riemann equations again)

$$\begin{pmatrix} \tilde{J} & \hat{J} \\ \hat{J} & -\tilde{J} \end{pmatrix} \nabla_C \tilde{H} + \begin{pmatrix} -\hat{J} & \tilde{J} \\ \tilde{J} & -\hat{J} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla_C \hat{H} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{150}$$

Hence if we take $\alpha^2 + \beta^2 = 1$ we have

$$\left\{ \alpha \begin{pmatrix} \tilde{J} & \hat{J} \\ \hat{J} & -\tilde{J} \end{pmatrix} + \beta \begin{pmatrix} -\hat{J} & \tilde{J} \\ \tilde{J} & -\hat{J} \end{pmatrix} \right\} \nabla_C (\alpha \tilde{H} + \beta \hat{H}) \tag{151}$$

$$= \alpha^2 \begin{pmatrix} \tilde{J} & \hat{J} \\ \hat{J} & -\tilde{J} \end{pmatrix} \nabla_C \tilde{H} + \beta^2 \begin{pmatrix} -\hat{J} & \tilde{J} \\ \tilde{J} & -\hat{J} \end{pmatrix} \nabla_C \hat{H} \tag{152}$$

$$+ \alpha\beta \left\{ \alpha \begin{pmatrix} \tilde{J} & \hat{J} \\ \hat{J} & -\tilde{J} \end{pmatrix} \nabla_C \hat{H} + \begin{pmatrix} -\hat{J} & \tilde{J} \\ \tilde{J} & -\hat{J} \end{pmatrix} \nabla_C \tilde{H} \right\} \tag{153}$$

$$= (\alpha^2 + \beta^2) \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} . \tag{154}$$

Thus we have a set of Hamiltonian structures indexed by the unit circle: that the Jacobi condition holds for all elements follows from Proposition 17. □

6.1.2. Mapping case

In the case of a complex analytic implectic mapping $\phi : M \to M$, there is no Hamiltonian. The Jacobi condition holds as in the ODE case and it is necessary only to check that the expression $\phi_* J \phi_*^{\mathrm{T}}$ retains its form when complexified.

The partial Jacobi matrix of $\tilde{\phi}$ with respect to the (real) x coordinates is written $\tilde{\phi}_x$, while $\tilde{\phi}_y$, $\hat{\phi}_x$ and $\hat{\phi}_y$ have the obvious meanings. The Cauchy–Riemann equations can then be written as

$$\tilde{\phi}_x = \hat{\phi}_y, \quad \tilde{\phi}_y = -\hat{\phi}_x . \tag{155}$$

Proposition 20. *If the map ϕ preserves the implectic structure $J, \phi_* J \phi_*^{\mathrm{T}} = J \circ \phi$, and both J and ϕ are complex analytic, then with K_j defined as in the previous section $aK_1 + bK_2$ is preserved by the real form of ϕ for any choice of $a, b \in \mathbf{R}$.*

Proof. The real and imaginary components of the condition $\phi_* J \phi_*^{\mathrm{T}} = J \circ \phi$ expand to

$$J \tilde{\circ} \phi = \tilde{\phi}_x \tilde{J} \tilde{\phi}_x^{\mathrm{T}} - \hat{\phi}_x \tilde{J} \hat{\phi}_x^{\mathrm{T}} - \hat{\phi}_x \hat{J} \tilde{\phi}_x^{\mathrm{T}} - \tilde{\phi}_x \hat{J} \hat{\phi}_x^{\mathrm{T}} , \tag{156}$$

$$J \hat{\circ} \phi = \tilde{\phi}_x \tilde{J} \hat{\phi}_x^{\mathrm{T}} + \tilde{\phi}_x \hat{J} \tilde{\phi}_x^{\mathrm{T}} + \hat{\phi}_x \tilde{J} \tilde{\phi}_x^{\mathrm{T}} - \hat{\phi}_x \hat{J} \hat{\phi}_x^{\mathrm{T}} . \tag{157}$$

The real form of the Jacobi matrix of ϕ, making use of the Cauchy–Riemann equations, is

$$\Phi_* := \begin{pmatrix} \tilde{\phi}_x & -\hat{\phi}_x \\ \hat{\phi}_x & \tilde{\phi}_x \end{pmatrix} , \tag{158}$$

so we need to check $\Phi_* K_j \Phi_*^{\mathrm{T}} = K_j \circ \phi$, $j = 1, 2$. The result follows from calculating $\Phi_* K_j \Phi_*^{\mathrm{T}}$ and comparing with the equations above. □

A special case of this result and some examples can be found in [16].

6.2. Symplectic and pre-symplectic case

In the case where a complex implectic structure J is known, so too is a complex symplectic structure $\omega = J^{-1}$. It quickly follows that the real symplectic forms corresponding to K_1 and K_2 are their inverses

$$\sigma^1 := \begin{pmatrix} \tilde{\omega} & -\hat{\omega} \\ -\hat{\omega} & -\tilde{\omega} \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} \hat{\omega} & \tilde{\omega} \\ \tilde{\omega} & -\hat{\omega} \end{pmatrix} . \tag{159}$$

The Jacobi condition in the symplectic case has coordinate expression

$$\sigma^j_{[\alpha,\beta,\gamma]} = 0 , \tag{160}$$

the square brackets denoting skew symmetrisation as before. The method of checking that the Jacobi condition for ω implies that it holds for σ^j is as for the implectic case and need not be repeated here. The maximal rank condition is trivial: if ω is invertible then the inverse of σ^j is explicitly given by K_j in terms of the components of $J := \omega^{-1}$. This is summarised in the following proposition.

Proposition 21. *If ω is an invertible analytic symplectic structure, the σ^1 and σ^2, defined as above, are also sympletic.*

If the maximal rank assumption is relaxed, then σ^1 and σ^2 can still be defined as in Eq. (159) and the Jacobi condition is satisfied. Moreover, since the Jacobi condition is linear in the sympletic case, any linear combination $a\sigma^1 + b\sigma^2$, $a, b \in \mathbf{R}$ is also a pre-symplectic structure.

6.3. Symmetries and integrals

If we know a symmetry or an integral for a complex map or system of ODEs then it is natural to expect to construct a pair of symmetries or integrals for the real form of the map or ODEs. In this section we show this to be true and give the corresponding commutation or involution properties, together with the Noether relations between symmetries and integrals.

If I is a complex analytic integral of Γ or ϕ, then its real and imaginary parts $\tilde{I}$ and $\hat{I}$ are also integrals, as can be checked by a simple computation.

Recall that if v is the real form of a complex vector field then it must leave the complex structure $\mathscr{I}$ invariant, since $\mathscr{I}$ is the real form of a constant complex scalar: $\mathscr{L}_v\mathscr{I}=0$. Similarly if ϕ is the real form of a complex map, $\phi_*\mathscr{I}\phi_*^{-1}=\mathscr{I}$. Consequently, if $[\Gamma,X]=0$ then

$$\mathscr{L}_\Gamma(\mathscr{I}X)=(\mathscr{L}_\Gamma\mathscr{I})X+\mathscr{I}[\Gamma,X]=0 \tag{161}$$

and so $\mathscr{I}X$ is a symmetry of Γ whenever X is. If X is a symmetry of the map ϕ, $\phi_*X=X\circ\phi$, so

$$\phi_*(\mathscr{I}X)=\phi_*\mathscr{I}\phi_*^{-1}\phi_*X=\mathscr{I}\phi_*X=\mathscr{I}X\circ\phi \tag{162}$$

and $\mathscr{I}X$ is also a symmetry of ϕ.

The effect of interchanging K_1 and K_2 in the Noether relations is now clear.

Proposition 22. *Let $\tilde{I}$ and $\hat{I}$ arise as the real and imaginary parts of a complex analytic integral. If ξ and η are the corresponding Noether symmetries via the pre-implectic structure K_1 :*

$$\xi=K_1\nabla_C\tilde{I};\quad \eta=K_1\nabla_C\hat{I}\,, \tag{163}$$

then the relations are interchanged by replacing K_1 by K_2 :

$$\eta=K_2\nabla_C\tilde{I};\quad \xi=-K_2\nabla_C\hat{I}\,. \tag{164}$$

Proof. First note that the Cauchy–Riemann conditions imply $\nabla_C\ (if)\ =\mathscr{I}\nabla_C f$ and we have trivially that $(\widetilde{iI}=-\hat{I}$, $(\widehat{iI})=\tilde{I}$. Thus

$$\mathscr{I}\nabla_C\tilde{I}=\nabla_C\hat{I};\quad \mathscr{I}\nabla_C\hat{I}=-\nabla_C\tilde{I}\,. \tag{165}$$

It follows immediately that

$$K_2\nabla_C\tilde{I}=K_1\mathscr{I}\nabla_C\tilde{I}=K_1\nabla_C\hat{I}=\eta\,, \tag{166}$$

$$K_2\nabla_C\hat{I}=K_1\mathscr{I}\nabla_C\hat{I}=-K_1\nabla_C\tilde{I}=-\xi\,.\qquad\Box \tag{167}$$

Note also that the two integrals derived from a single complex integral are in involution with respect to any linear combination of K_1 and K_2. This follows from the fact that

$$\{\tilde{I},\hat{I}\}_{K_j}=\nabla_C\tilde{I}K_j\nabla_C\hat{I}=\nabla_C\tilde{I}K_j\mathscr{I}\nabla_C\tilde{I}\,. \tag{168}$$

Since $K_1\mathscr{I}=K_2$ and $K_2\mathscr{I}=-K_1$ and both are skew, the right-hand side of Eq. (168) must vanish.

In fact we can do better than this, as shown by the following proposition.

Proposition 23. *Let I_1 and I_2 be any two complex analytic integrals, in involution with respect to the implectic structure J. Then the real and imaginary parts of I_1 and I_2 are all in involution with respect to any of the implectic structures aK_1+bK_2.*

Proof. Juggling with Cauchy–Riemann conditions yields the following expressions for the real and imaginary parts of the Poisson bracket of I_1 and I_2:

$$\Re\,(\nabla I_1 J \nabla I_2) = \nabla_C \tilde{I}_1 K_1 \nabla_C \tilde{I}_2 \tag{169}$$

$$= \nabla_C \tilde{I}_1 K_2 \nabla_C \hat{I}_2 \tag{170}$$

$$= \nabla_C \hat{I}_1 K_2 \nabla_C \tilde{I}_2 \tag{171}$$

$$= \nabla_C \hat{I}_1 K_1 \nabla_C \hat{I}_2\,, \tag{172}$$

$$\Im\,(\nabla I_1 J \nabla I_2) = \nabla_C \tilde{I}_1 K_1 \nabla_C \tilde{I}_2 \tag{173}$$

$$= \nabla_C \tilde{I}_1 K_1 \nabla_C \hat{I}_2 \tag{174}$$

$$= \nabla_C \hat{I}_1 K_1 \nabla_C \tilde{I}_2 \tag{175}$$

$$= \nabla_C \hat{I}_1 K_2 \nabla_C \hat{I}_2\,. \tag{176}$$

Now the right-hand side of these equations are simply the eight possible Poisson brackets of the real and imaginary parts of I_1 and I_2 with respect to the implectic structures K_1 and K_2. The result is completed by linearity. □

7. Conclusion

We have provided several distinct sets of sufficient conditions for constructing pre-symplectic and pre-implectic structures for given ODEs and OΔEs (maps). These include knowledge of symmetries or integrals, volume preservation and complex structure.

The conditions for pre-symplectic structures differ from those for pre-implectic structures, as these are not equivalent as in the maximal rank case. Specifically, if the given n-dimensional ODE or map is volume preserving then knowledge of $n-2$ integrals is sufficient to construct a pre-implectic structure. To construct a pre-symplectic structure we need instead $n-2$ volume preserving symmetries. Note that finding symmetries may be easier than finding integrals, due to the existence of various computer packages dedicated to the task. See for example [21] or the review [22].

The method of Section 3 stands apart, requiring one integral and one symmetry (in fact a weaker condition suffices) for the ODE case, or two symmetries for a system of OΔEs. This is sufficient to construct a pre-implectic structure for a system of arbitrary dimension.

Section 6 differs by virtue of producing maximal rank implectic structures by complexifying existing ones. Although the results are explicitly concerned with implectic structures, the results apply equally to the symplectic case by taking inverses. Hence if we have an analytic Hamiltonian ODE or symplectic/implectic map in dimension n, we can produce an infinite set of mutually compatible symplectic or implectic structures in dimension $2n$.

This partly resolves on open question raised in [16]: are the alternate k-symplectic structures generated by a k-symplectic map compatible with the original structure? Many examples of such maps are constructed via complexification, with the alternate k-symplectic structures falling within the space of compatible structures generated by the method of Section 6.

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