

k -integrals and k -Lie symmetries in discrete dynamical systems

F.A. Hagggar^a, G.B. Byrnes^a, G.R.W. Quispel^{a,*}, H.W. Capel^b

^a Department of Mathematics, La Trobe University, Bundoora 3083 Victoria, Australia

^b Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, Netherlands

Received 13 March 1996

Abstract

We generalize the concept of symplectic maps to that of k -symplectic maps: maps whose k th iterates are symplectic. Similarly, k -symmetries and k -integrals are symmetries (resp. integrals) of the k th iterate of the map. It is shown that k -symmetries and k -integrals are related by the k -symplectic structure, as in the $k = 1$ continuous case (Noether's theorem). Examples are given of k -integrals and their related k -symmetries for $k = 1, \dots, 4$.

PACS: 02.30.Ks; 02.90.+p

Keywords: Difference equations; k -symplectic maps; k -Lie symmetries; k -integrals

1. Introduction

In recent years there has been considerable effort expended to extend theorems and techniques for differential equations to their discrete analogue, difference equations [1, 4, 5, 10, 13, 15, 17, 21–25, 27]. Hamiltonian systems form an important subclass of ordinary differential equations: the corresponding discrete theory is that of symplectic maps. Such maps have been investigated by a number of authors [3, 9, 19, 28]. In particular, Maeda [16] was able to show that Noether's result linking symmetries and first integrals [20] extends to symplectic maps.

Within the last two years it has been noticed that discrete systems can admit a type of symmetry unavailable in continuous-time systems. These are k -symmetries, which are not symmetries of the map itself but of the k th iterate of the map. Discrete k -symmetries of various types have been reported [11, 12].

* Corresponding author.

In this article we introduce the concept of k -Lie symmetries (that is *continuous* k -symmetries) of discrete symplectic maps. In fact, we extend this to k -symplectic maps, whose k th iterates preserve the symplectic structure. We show that the Noether correspondence between symmetries and first integrals of Hamiltonian systems extends to a relation between k_1 -Lie symmetries and k_2 -integrals of k_3 -symplectic maps (where in general k_1, k_2 and k_3 need not be equal). While this reduces to Maeda's result in the case $k_1 = k_2 = k_3 = 1$, our proof is independent of his and rather simpler.

We also show that a k -symplectic map may imply the existence of several distinct symplectic structures for the k th iterate, so that a single k -symmetry may be used to determine several first integrals and vice versa.

In Section 4 we provide various examples for $k = 1, 2, 3$ and 4.

2. Symplectic and k -symplectic maps

A map $f : \mathbf{R}^{2n} \rightarrow \mathbf{R}^{2n}$ (or $f : \mathbf{C}^{2n} \rightarrow \mathbf{C}^{2n}$) is symplectic if it preserves a symplectic structure on \mathbf{R}^{2n} (respectively \mathbf{C}^{2n}). It is k -symplectic if the k th iterate $f^{[k]}$ preserves the symplectic structure. Throughout this section we will use \mathbf{R}^{2n} for economy: all the arguments can be extended trivially to the complex case.

Recall that a symplectic structure Ω on \mathbf{R}^{2n} satisfies the following conditions:

1. Ω is a skew-symmetric matrix, $\Omega^T = -\Omega$;
2. Ω satisfies the Jacobi identity

$$\sum_{l=1}^{2n} \left\{ \Omega_{il} \frac{\partial}{\partial x_l} \Omega_{jk} + \Omega_{jl} \frac{\partial}{\partial x_l} \Omega_{ki} + \Omega_{kl} \frac{\partial}{\partial x_l} \Omega_{ij} \right\} \equiv 0, \quad i, j, k = 1, \dots, 2n;$$

3. Ω has maximal rank.

In general, if g is a function on \mathbf{R}^{2n} we will write $g' := g \circ f$ and $g^{(k)} := g \circ f^{[k]}$, that is the image of g under the mapping and its k -iterate. Note that Ω is a matrix-valued function, so $\Omega' = \Omega \circ f$, etc.

The derivative map f_* of f is the matrix

$$f_* := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_{2n}} \\ \vdots & & \vdots \\ \frac{\partial f_{2n}}{\partial x_1} & \dots & \frac{\partial f_{2n}}{\partial x_{2n}} \end{bmatrix}.$$

Thus, if a vector field ξ on \mathbf{R}^{2n} has components $\xi_j(\partial/\partial x_j)$, it is carried by f to the vector $f_*\xi$ with components $\sum_k f_{*jk} \xi_k(\partial/\partial x_j)$.

Some care is necessary with the transformation properties of Ω . Usually, the symplectic structure is given by a 2-form, or skew, 2-covariant tensor ω . With that choice, the Noether relationship between a symmetry ξ of a Hamiltonian system and its

corresponding integral I is given by $\nabla I = \xi^T \omega$. It will be more convenient in this article to take the matrix inverse of the 2-form, so we will have a skew, 2-contravariant tensor Ω . The Noether relation is then $\xi = (\nabla I \Omega)^T$.

With the above notation, f is symplectic if

$$f_* \Omega f_*^T = \Omega'$$

Similarly, f is k -symplectic if

$$(f^{[k]})_* \Omega (f^{[k]})_*^T = \Omega^{(k)},$$

or equivalently

$$(f_*)^{(k-1)} \dots (f_*)^{(1)} f_* \Omega f_*^T (f_*^T)^{(1)} \dots (f_*^T)^{(k-1)} = \Omega^{(k)}.$$

We say that f is strictly k -symplectic if k is the smallest positive integer for which this is true.

The gradient of a function, ∇g , will be considered to be a row vector. Note that in general $\nabla' \neq \nabla$. In fact, $\nabla' g = (\nabla g)(f_*)^{-1}$.

3. k -integrals and k -symmetries

A function $I : \mathbf{R}^{2n} \rightarrow \mathbf{R}$ is an integral (or conserved quantity) of a map f if $I' = I$. The extension of this definition to k -integrals is immediate: I is a k -integral of f if $I^{(k)} = I$. That is to say I is a k -integral of f if and only if I is an integral of $f^{[k]}$.

We say that I is a strict k -integral if it is a k -integral and in addition $I^{(j)} \neq I$ if $j < k$.

In a similar way, a vector field ξ is a symmetry of f if $f_* \xi = \xi'$ (which is equivalent to requiring that the flow of ξ maps orbits of f to orbits [4, 17]). Thus, ξ is a k -symmetry of f if and only if ξ is a symmetry of $f^{[k]}$, or $(f^{[k]})_* \xi = \xi^{(k)}$. It is a strict k -symmetry if it is not also a j -symmetry for any $j < k$.¹

Proposition 1. *If I is a (strict) k -integral and ξ a (strict) k -symmetry of f , then I' is a (strict) k -integral and $(f_*)^{-1} \xi'$ is a (strict) k -symmetry.*

The following result will be used again later, so we state it separately as a lemma.

Lemma 2. $(f^{[k]})_* (f_*)^{-1} = (f_*^{(k)})^{-1} (f^{[k]})'_*$.

Proof of Lemma 2. The LHS can be expanded as

$$(f^{[k]})_* (f_*)^{-1} = f_*^{(k-1)} \dots f_* (f_*)^{-1} = (f_*^{(k)})^{-1} (f^{[k]})'_* . \quad \square$$

¹ Note that what we call a strict k -symmetry is called a k -symmetry in Refs. [11, 12].

Proof of Proposition 1. Clearly, $(I')^{(k)} = (I^{(k)})' = I'$. Suppose then that $(I')^{(j)} = I'$ with $j < k$. If we iterate each side a further $k - 1$ times and use that fact that $I^{(k)} = I$ we obtain $I^{(k+j)} = I^{(j)} = I$, which is a contradiction.

For the symmetry case, consider $(f^{[k]})_*((f_*)^{-1}\xi')$. From the lemma we can rewrite this as $(f_*^{(k)})^{-1}[(f^{[k]})_*\xi']$. Using the fact that ξ is a k -symmetry, this becomes

$$(f_*^{(k)})^{-1}[\xi^{(k)}]' = [(f_*)^{-1}\xi']^{(k)},$$

so $(f_*)^{-1}\xi'$ is a symmetry of $f^{[k]}$.

If it is also a symmetry of $f^{[j]}$ for some $j < k$, then $(f^{[j]})_*((f_*)^{-1}\xi') = [(f_*)^{-1}\xi']^{(j)}$. Using the calculation above with j in place of k , we find that this is equivalent to $(f_*^{[j]})_*\xi = \xi^{(j)}$, contradicting the assumption that ξ is a strict k -symmetry. \square

The following theorem is equivalent to results derived by Maeda [16].

Theorem 3. *Suppose that f is a symplectic map with respect to Ω and that I is an integral. Then $\xi := (\nabla I \Omega)^T$ is a symmetry of f such that $\mathcal{L}_\xi \Omega = 0$.² Conversely, if ξ is a symmetry of f which satisfies $\mathcal{L}_\xi \Omega = 0$, then there is a (possibly time-dependent) integral $I : \mathbf{R}^{2n} \rightarrow \mathbf{R}$ such that $\xi = (\nabla I \Omega)^T$.*

Proof. Consider $\xi := (\nabla I \Omega)^T$, so that $\xi' = (\nabla' I' \Omega')^T$. Since f is assumed to be symplectic we have $\Omega' = f_* \Omega f_*^T$. Together with the transformation property of ∇ we have

$$\xi' = (\nabla' I' (f_*)^{-1} f_* \Omega f_*^T)^T = f_* (\nabla I \Omega)^T = f_* \xi.$$

Thus ξ is a symmetry. Since Ω satisfies the Jacobi identity, the condition $\mathcal{L}_\xi \Omega = 0$ is equivalent to requiring that

$$\frac{\partial}{\partial x_i} (\omega_{kj} \xi_k) - \frac{\partial}{\partial x_j} (\omega_{ki} \xi_k) = 0, \quad (1)$$

where $\omega := \Omega^{-1}$. Note that in three dimensions, this is just $\nabla \times (\xi^T \omega) = 0$. That this is satisfied follows directly from the fact that $\xi^T \omega = \nabla I$.

For the converse we have $\mathcal{L}_\xi \Omega = 0$ and therefore

$$\frac{\partial}{\partial x_i} (\omega_{kj} \xi_k) - \frac{\partial}{\partial x_j} (\omega_{ki} \xi_k) = 0.$$

It is a standard result (the 1-dimensional case of the Poincaré Lemma [6, p. 224]) that this implies the existence of a function I such that $\xi^T \omega = \nabla I$, whence $\xi = (\nabla I \Omega)^T$. Now since ξ is a symmetry of f ($f_* \xi = \xi'$),

$$(\nabla I)' = (\xi^T \omega)' = (f_* \xi)^T (f_*)^{-T} \omega (f_*)^{-1} = \xi^T \omega (f_*)^{-1} = \nabla' I,$$

² Here \mathcal{L}_ξ denotes the Lie derivative with respect to ξ [6, p. 147]. For a symplectic structure Ω , the condition $\mathcal{L}_\xi \Omega = 0$ reduces to equation (1).

where $A^{-T} := (A^{-1})^T$. Therefore, $\nabla(I - I') = 0$, so $I' = I + c$, leading to the time-dependent integral $\hat{I}(n) = I - nc$. \square

Note. The relation between symmetries and integrals is exactly as in the continuous case [2, 14]. Therefore, the well known results for the continuous case combined with the theorem suffice to show that the algebra of symmetries ξ such that $\mathcal{L}_\xi \Omega = 0$ (sometimes called Noether symmetries) with the Lie bracket, is isomorphic to the algebra of first integrals, modulo constants, with the Poisson bracket. The result proved by Maeda is somewhat different to the theorem above, but is equivalent to it.

Corollary 4. *If I is a k_1 -integral of a k_2 -symplectic map f , then $\xi := (\nabla I \Omega)^T$ is a k_3 -symmetry of f and $\mathcal{L}_\xi \Omega = 0$, where k_3 is the least common multiple of k_1, k_2 . Conversely, if ξ is a k_1 -symmetry of f satisfying $\mathcal{L}_\xi \Omega = 0$ and f is k_2 -symplectic then there is a (possibly time-dependent) k_3 -integral I such that $\xi = (\nabla I \Omega)^T$.*

Proof. Replace f by $f^{[k_3]}$ in the theorem. \square

Note. It may occur that the resulting k_3 -symmetry is also a j -symmetry, where j is some divisor of k_3 . In that case the k_3 -symmetry is not strict. The same applies to integrals.

An interesting question appears at this point. Given a k -integral I , we know from Proposition 1 that $I^{(1)}, I^{(2)}, \dots, I^{(k-1)}$ are also distinct k -integrals (although they need not be independent). We know from Theorem 3 and its corollary that there is a symmetry ξ corresponding to I . Using Proposition 1 again, we have that $\tilde{\xi} := (f_*)^{-1} \xi'$ is a symmetry of $f^{[k]}$. The question is whether $\tilde{\xi}$ is the symmetry associated to I' .

The answer is yes if we impose the condition that f be symplectic.

Proposition 5. *Let f be a symplectic map with a k -integral I and corresponding k -symmetry ξ . Then the k -symmetry corresponding to I' is $(f_*)^{-1} \xi$.*

Proof. Consider

$$(\nabla I \Omega)' = \nabla' I' \Omega' = \nabla I' (f_*)^{-1} f_* \Omega f_*^T = (\nabla I' \Omega) f_*^T .$$

Therefore, the symmetry $\tilde{\xi} = (\nabla I' \Omega)^T$ corresponding to I' satisfies

$$f_* \tilde{\xi} = (\nabla I \Omega)'^T = \xi'$$

or $\tilde{\xi} = (f_*)^{-1} \xi'$. \square

In the more general case where f is strictly k -symplectic, this need no longer be true.

Example 6. Consider the involution

$$f(x, y, u, v) = (v, y, u, x) . \tag{2}$$

As $f^{[2]}$ is the identity f is trivially 2-symplectic for any symplectic structure on \mathbf{R}^4 and any function $\mathbf{R}^4 \rightarrow \mathbf{R}$ is a 2-integral of f . So take the symplectic structure

$$\Omega = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and the 2-integral $I(x, y, u, v) := x$.

The symmetry corresponding to I is $-\partial/\partial u$. Now $I'(x, y, u, v) = v$, with corresponding symmetry $\partial/\partial y$. On the other hand, $(f_*)^{-1}\partial/\partial u' = \partial/\partial u$, so the answer is no.

The reason that this fails if f is only k -symplectic is that if $f_*\Omega f_*^T \neq \Omega'$ the relation between vector fields and functions is changed. In this example, the symplectic structure transforms to

$$\tilde{\Omega} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

after one iteration, i.e. $\tilde{\Omega} = (f_*)^{-1}\Omega'(f_*)^{-T}$. Note that $\tilde{\Omega}$ does pair I' with $\partial/\partial u$.

The next proposition demonstrates that the new correspondence is still a symplectic structure.

Proposition 7. *If f is k -symplectic with respect to Ω , then $\tilde{\Omega} := (f_*)^{-1}\Omega'(f_*)^{-T}$ is also a symplectic structure and f is k -symplectic with respect to $\tilde{\Omega}$.*

Proof. If f is invertible then so is f_* , hence $(f_*)^{-1}\Omega'(f_*)^{-T}$ is skew and maximal rank.

Let $\{\cdot, \cdot\}_\Omega$ be the Poisson bracket determined by Ω . The Jacobi condition on Ω is then

$$\{\{h_1, h_2\}_\Omega, h_3\}_\Omega + (\text{cyclic permutations}) = 0, \quad \forall h_j \in C^\infty(\mathbf{R}^{2n}).$$

Now we claim that

$$\{\{h_1, h_2\}_\Omega\}' = \{\{h'_1, h'_2\}_{\tilde{\Omega}}\}.$$

To see this, recall that the LHS is equal to $(\nabla h_1 \Omega (\nabla h_2)^T)'$, so using the transformation properties of ∇ we obtain

$$\{\{h_1, h_2\}_\Omega\}' = \nabla h'_1 (f_*)^{-1} \Omega' (f_*)^{-T} (\nabla h'_2)^T = \{\{h'_1, h'_2\}_{\tilde{\Omega}}\}.$$

It follows that

$$\{\{h'_1, h'_2\}_{\tilde{\Omega}}, h'_3\}_{\tilde{\Omega}} + (\text{cyclic permutations}) = 0, \quad \forall h_j \in C^\infty(\mathbf{R}^{2n}).$$

Since for every function h on \mathbf{R}^{2n} there is a function g such that $h = g'$, this demonstrates that $\tilde{\Omega}$ satisfies the Jacobi identity.

To see that f is k -symplectic wrt $\tilde{\Omega}$, we have to show

$$(f^{[k]})_* \tilde{\Omega} (f^{[k]})_*^T = \tilde{\Omega}^{(k)}.$$

Using Lemma 2 and the definition of $\tilde{\Omega}$, the LHS becomes

$$(f_*^{(k)})^{-1} (f_*^{[k]})'_* \Omega' (f_*^{[k]})'^T (f_*^{(k)})^{-T} = (f_*^{(k)})^{-1} \Omega^{(k+1)} (f_*^{(k)})^{-T} = \tilde{\Omega}^{(k)}. \quad \square$$

Thus, if a map f is k -symplectic but not symplectic, f effectively maps between different symplectic structures. While the k -integrals and k -symmetries are the same for each symplectic structure, the homomorphism between them changes, as shown in the following proposition.

Proposition 8. *If I is a k_1 -integral and $\xi := (\nabla I \Omega)^T$ is a k_2 -symmetry of a map f which is k_3 -symplectic with respect to Ω , then the k_1 -integral I' and the k_2 -symmetry $\tilde{\xi} := (f_*)^{-1} \xi'$ are related by $\tilde{\xi} = (\nabla I' \tilde{\Omega})^T$ for arbitrary $k_1, k_2, k_3 \in \mathbf{N}$, where $\tilde{\Omega}$ is defined as in Proposition 7.*

Proof.

$$\begin{aligned} (f_*)^{-1} \tilde{\xi}' &= (f_*)^{-1} (\nabla I \Omega)^T = (f_*)^{-1} (\nabla I' \Omega')^T \\ &= (\nabla I' (f_*)^{-1} \Omega' (f_*)^{-T})^T = (\nabla I' \tilde{\Omega})^T. \quad \square \end{aligned}$$

It should be noted that the combination of the above propositions can be used to find additional k -symmetries by the following two step procedure.

1. Simply calculate the k -integrals $I^{(1)}, \dots, I^{(k-1)}$.
2. From each of these we can derive further k -symmetries (which may or may not be new) by using the alternative symplectic structures $\tilde{\Omega}, \tilde{\tilde{\Omega}}, \dots$

Example 6 above demonstrates this. The function $I(x, y, u, v) = u$ is an integral of the map (2), so step 1 gives nothing new. However, the corresponding symmetry generators using Ω and $\tilde{\Omega}$ are distinct, $-\partial/\partial x$ and $-\partial/\partial v$, respectively.

Let $\Omega^{\{j\}}$ be defined inductively by

$$\Omega^{\{1\}} := \tilde{\Omega}; \quad \Omega^{\{j+1\}} := \tilde{\tilde{\Omega}}^{\{j\}}, \quad j = 1, 2, \dots$$

The above discussion is summarized in the theorem below, which follows from Propositions 1, 7 and Theorem 3.

Theorem 9. *Let I be a k -integral of a k -symplectic map f . Then the vector fields $\xi_{i,j} := \nabla I^{(i)} \Omega^{\{j\}}$ are k -symmetries of f .*

Note that there remains an open question: while Theorem 3 guarantees that $\mathcal{L}_{\xi_{i,j}} \Omega^{\{j\}} = 0$, it is not clear if $\mathcal{L}_{\xi_{i,j}} \Omega^{\{k\}}$ is zero for arbitrary k . It can be shown that $\mathcal{L}_{\xi_{i,j}} \Omega^{\{k\}} = 0$ in the case where a non-trivial linear combination $a\Omega^{\{j\}} + b\Omega^{\{k\}}$ is a symplectic structure, so that $\Omega^{\{j\}}$ and $\Omega^{\{k\}}$ are compatible in the sense of [8, 18].

The importance of this is that since $\xi_{i,j}$ is then a Noether symmetry with respect to $\Omega^{\{k\}}$, there is a (possibly new) k -integral \hat{I} such that

$$\xi_{i,j}^T (\Omega^{\{k\}})^{-1} = \nabla \hat{I}.$$

In other words, $\Omega^{\{j\}}(\Omega^{\{k\}})^{-1}$ is a recursion operator. For a general discussion of recursion operators in finite-dimensional Hamiltonian mechanics, see [7]. Recursion operators are best known for their role in completely integrable, infinite-dimensional Hamiltonian dynamics. For example, they can be used to generate the hierarchy of conserved quantities for the KdV equation, see [8, 18].

Referring again to Example 6, we find that Ω and $\tilde{\Omega}$ are compatible. Thus, we have that $\mathcal{L}_{\partial/\partial x}\tilde{\Omega} = 0$ and $\mathcal{L}_{\partial/\partial v}\tilde{\Omega} = 0$, leading to the new integral y in each case.

4. Examples

Here we give some examples of k -integrals and k -symmetries for $k = 1, \dots, 4$. Examples with higher values of k can easily be constructed in higher dimensions.

4.1. The case $k = 1$

The case $k = 1$ is of course that of integrals of symplectic maps. For the case of maps in the plane, an 18-parameter family of integrable (and reversible) maps is given in [26].

Define the quartic polynomials f_j and g_j as components of

$$f(x) = (f_1(x), f_2(x), f_3(x)) = (A_0 X) \times (A_1 X),$$

$$g(x) = (g_1(x), g_2(x), g_3(x)) = (A_0^T X) \times (A_1^T X),$$

where

$$A_i := \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \delta_i & \varepsilon_i & \zeta_i \\ \kappa_i & \lambda_i & \mu_i \end{pmatrix}, \quad i = 0, 1; \quad X := \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}.$$

The entries in A_i are 18 arbitrary parameters.

If we now define the map $M(x, y) = (x', y')$ by

$$x' := \frac{f_1(y) - x f_2(y)}{f_2(y) - x f_3(y)}, \quad y' := \frac{g_1(x') - y g_2(x')}{g_2(x') - y g_3(x')},$$

it can be shown that M is symplectic with respect to

$$\Omega := \rho_1^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where

$$\rho_i(x, y) := \alpha_i x^2 y^2 + \beta_i x^2 y + \gamma_i x^2 + \delta_i x y^2 + \varepsilon_i x y + \xi_i x + \kappa_i y^2 + \lambda_i y + \mu_i,$$

$$i = 0, 1.$$

Moreover, the function

$$I(x, y) := \frac{\rho_0(x, y)}{\rho_1(x, y)}$$

$$= \frac{\alpha_0 x^2 y^2 + \beta_0 x^2 y + \gamma_0 x^2 + \delta_0 x y^2 + \varepsilon_0 x y + \xi_0 x + \kappa_0 y^2 + \lambda_0 y + \mu_0}{\alpha_1 x^2 y^2 + \beta_1 x^2 y + \gamma_1 x^2 + \delta_1 x y^2 + \varepsilon_1 x y + \xi_1 x + \kappa_1 y^2 + \lambda_1 y + \mu_1}$$

is an integral of M .

The corresponding symmetry is generated by the vector field $\xi := (\nabla I \Omega)^T$, or

$$\xi = \frac{1}{\rho_1} \left(\frac{\partial I}{\partial y} \frac{\partial}{\partial x} - \frac{\partial I}{\partial x} \frac{\partial}{\partial y} \right).$$

As an example with two degrees of freedom we take a system investigated by Bruschi et al. [3], in the form used by Suris to produce integrable maps with $2N$ degrees of freedom [28].

Define $M(x, y, u, v) := (x', y', u', v')$ as follows (with a an arbitrary constant):

$$x' := -y + \frac{2ax}{1 + axu},$$

$$y' := x,$$

$$u' := -v + \frac{2au}{1 + axu},$$

$$v' := u.$$

The map M is symplectic with respect to

$$\Omega := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

and has integrals

$$I_1(x, y, u, v) := xv - yu,$$

$$I_2(x, y, u, v) := xu + yv - a(xv + yu) + axyv.$$

It is then easy to find the corresponding symmetries

$$\begin{aligned}\xi_1(x, y, u, v) &:= -x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}, \\ \xi_2(x, y, u, v) &:= (-y + ax(1 - yu)) \frac{\partial}{\partial x} + (x - ay(1 - xv)) \frac{\partial}{\partial y} \\ &\quad + (-v + au(1 - xv)) \frac{\partial}{\partial u} + (u - av(1 - yu)) \frac{\partial}{\partial v}.\end{aligned}$$

4.2. The case $k = 2$

We can construct a symplectic 2-dimensional map with a 2-integral as follows. First take an arbitrary bi-quadratic in the dependent variables x, y :

$$\tilde{I}(x, y) = \tilde{A}x^2y^2 + \tilde{B}x^2y + \tilde{C}xy^2 + \tilde{D}xy + \tilde{E}x^2 + \tilde{F}y^2 + \tilde{G}x + \tilde{H}y.$$

Then choose a linear involution

$$L(x, y) := (y, x),$$

(i.e. $L^2 = \text{Id}$).

From \tilde{I} we construct an I such that $I(L(x, y)) = \lambda$ with $\lambda^2 = 1$,

$$I := \tilde{I}(x, y) + \lambda \tilde{I}(L(x, y)). \quad (3)$$

With the choice $\lambda = -1$,

$$I(x, y) = (\tilde{B} - \tilde{C})(x^2y - y^2x) + (\tilde{E} - \tilde{F})(x^2 - y^2) + (\tilde{G} - \tilde{H})(x - y).$$

Now we search for a new mapping M which leaves I invariant. To simplify the task, we assume that M has the form $M(x, y) = (x'(x, y), y)$. Using new constants $B := \tilde{B} - \tilde{C}$, $E := \tilde{E} - \tilde{F}$ and $G = \tilde{G} - \tilde{H}$, the invariance condition is

$$\begin{aligned}0 = I(x', y) - I(x, y) &= B(x'^2y - x^2y - x'y^2 + xy^2) + E(x'^2 - x^2) + G(x' - x) \\ &= (x' - x)\{By(x' + x - y) + E(x' + x) + G\}.\end{aligned}$$

So provided that M is not trivial, we have

$$\begin{aligned}x' &= -x + \left(\frac{By^2 - G}{By + E}\right), \\ y' &= y.\end{aligned}$$

The desired map f is then defined to be the composite $f := L \circ M$,

$$f(x, y) = \left(y, -x + \left(\frac{By^2 - G}{By + E}\right)\right).$$

Since $I(L(x, y)) = -I(x, y)$ and $I(M(x, y)) = I(x, y)$, we have

$$I(f(x, y)) = -I(x, y),$$

$$I(f^{[2]}(x, y)) = I(x, y),$$

so I is a 2-integral of f as required.

Checking the Poisson bracket for the canonical symplectic form

$$\Omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

demonstrates that

$$\{x, y\} = \{x', y'\} = 1$$

with $(x', y') := f(x, y)$.

The corresponding 2-symmetry ξ then satisfies

$$\xi_i = \sum_j \Omega_{ij} \frac{\partial I}{\partial x_j},$$

so the 2-symmetry is generated by

$$[B(2xy - x^2) + 2yE + G] \frac{\partial}{\partial x} + [B(2xy - y^2) + 2xE + G] \frac{\partial}{\partial y}.$$

4.3. The case $k = 3$

The simplest way to obtain an example of a 3-integral is to take the real form of a 2-dimensional complex system, then set the λ in Eq. (3) to be a 3rd root of unity.

Starting from the quadratic

$$I(z, w) := z^2 w^2 + Bz + Cw$$

and the class of mappings $L(z, w) := (w, \nu z)$, we require that

$$I(L(z, w)) = \lambda I(z, w), \quad \lambda = \nu^2$$

(note that it is not necessary that $L^{[3]} = \text{Id}$). This yields the conditions

$$\lambda^3 = 1, \quad C = B/\lambda.$$

Thus,

$$I(z, w) := z^2 w^2 + Bz + \lambda^2 Bw.$$

Taking a second map of the form $M(z, w) := (z'(z, w), w)$ and solving the invariance condition $I(z', w) = I(z, w)$ as in Section 4.2, the condition is

$$(z' - z)(w^2(z' + z) + B) = 0$$

which can be solved for z' :

$$z' = -z - \frac{B}{w^2}.$$

Combining L and M into $f := L \circ M$ as before, we have

$$f(z, w) := \left(w, -\lambda^2 \left(z + \frac{B}{w^2} \right) \right).$$

To obtain the real form we set

$$\lambda = \frac{-1 + i\sqrt{3}}{2}, \quad z = x + iy, \quad w = u + iv,$$

and assume, for convenience, that the constant $B \in \mathbf{R}$. This gives

$$\begin{aligned} \hat{f}(x, y, u, v) = & \left(u, v, \frac{1}{2}(x - y\sqrt{3}) + \frac{B}{2(u^2 + v^2)^2}(u^2 - v^2 + 2uv\sqrt{3}), \right. \\ & \left. \frac{1}{2}(y + x\sqrt{3}) + \frac{B}{2(u^2 + v^2)^2}[(u^2 - v^2)\sqrt{3} - 2uv] \right). \end{aligned}$$

It can be checked that this map is 3-symplectic with respect to either of

$$\Omega_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \Omega_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

In fact, $\hat{f}_* \Omega_j \hat{f}_*^T = A \Omega'_j$, where $A = \frac{1}{2}(-1 + J\sqrt{3})$ is the real form of λ with J the complex structure

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad J^2 = -1.$$

These two symplectic structures satisfy $\Omega_2 = J\Omega_1$ and they can be interpreted as arising from the real and imaginary parts of the symplectic structure on \mathbf{C}^2 : if the symplectic structure is

$$\Omega = (a + ib) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad a, b \in \mathbf{R},$$

then the corresponding symplectic structure on \mathbf{R}^4 is $a\Omega_1 + b\Omega_2$. Note that in this example

$$\tilde{\Omega}_1 = -\frac{1}{2}(1 + J\sqrt{3})\Omega_1 = -\frac{1}{2}\Omega_1 - \frac{\sqrt{3}}{2}\Omega_2,$$

so had we known only one symplectic structure, call it Ω , then the other could have been found by calculating $\tilde{\Omega}$ as defined in Proposition 7.

The original complex 3-integral I can be decomposed into real and imaginary parts, providing a pair of 3-integrals for \hat{f} :

$$I_1(x, y, u, v) = (x^2 - y^2)(u^2 - v^2) - 4xyuv + \frac{B}{2}(2x - u + v\sqrt{3}),$$

$$I_2(x, y, u, v) = 2xy(u^2 - v^2) + 2uv(x^2 - y^2) + \frac{B}{2}(2y - v - u\sqrt{3}).$$

Moreover, provided we use a symplectic structure such that $J\Omega$ is skew, the Cauchy–Riemann equations guarantee that integrals arising in this way are in involution, $\{I_1, I_2\} = 0$ as in this case.

The symmetries corresponding to I_1 and I_2 (using Ω_1) are

$$\xi_1(x, y, u, v) = (-2v(x^2 - y^2) - 4xyu + B\sqrt{3}/2)\frac{\partial}{\partial x} + (2u(x^2 - y^2) - 4xyv - B/2)\frac{\partial}{\partial y}$$

$$+ (2y(u^2 - v^2) + 4xuv)\frac{\partial}{\partial u} - (2x(u^2 - v^2) - 4yuv + B)\frac{\partial}{\partial v},$$

$$\xi_2(x, y, u, v) = (2u(x^2 - y^2) - 4xyv - B/2)\frac{\partial}{\partial x} + (2v(x^2 - y^2) + 4xyu - B\sqrt{3}/2)\frac{\partial}{\partial y}$$

$$- (2x(u^2 - v^2) - 4yuv + B)\frac{\partial}{\partial u} - (2y(u^2 - v^2) + 4xuv)\frac{\partial}{\partial v}.$$

Note that here

$$\tilde{\Omega}_1\Omega_1^{-1} = -\frac{1}{2}(1 + \sqrt{3}\Omega_2\Omega_1^{-1}).$$

If we calculate $\nabla I_j\tilde{\Omega}_1\Omega_1^{-1}$ as in step 2 of the procedure above Theorem 9, we obtain a linear combination of I_1 and I_2 , so in this case the second step provides no new 3-integrals.

4.4. The case $k = 4$

As in the $k = 3$ case, we proceed by working with a symplectic map on \mathbf{C}^2 , this time choosing λ to be a fourth root of unity.

Starting from the quadratic

$$I(z, w) := Az^2w + Czw^2 + Bz + Dw$$

and the 4th order mapping $L(z, w) := (w, -z)$, we then require that

$$I(L(z, w)) = \lambda I(z, w),$$

or explicitly

$$C = \lambda^{-1}A, \quad D = \lambda B, \quad \lambda^2 = -1.$$

Thus,

$$I(z, w) := A(z^2w - \lambda zw^2) + B(z + \lambda w).$$

We then seek another map $M(z, w) := (z'(z, w), w)$ satisfying $I(z', w) = I(z, w)$ as before. The invariance condition is

$$A[(z' + z)w - \lambda w^2] + B = 0,$$

which can be solved for z' :

$$z' = -z + \lambda w - \frac{B}{Aw}.$$

Combining L and M into $f := L \circ M$ as before, we have

$$f(z, w) := \left(w, z - \lambda w + \frac{B}{Aw} \right).$$

Setting $\lambda = i$, $z = x + iy$, $w = u + iv$ and assuming the constants A and B to be real, we obtain the real form

$$\hat{f}(x, y, u, v) = \left(u, v, x + v + \frac{Bu}{A(u^2 + v^2)}, y - u - \frac{Bv}{A(u^2 + v^2)} \right).$$

This map is 2-symplectic (and hence 4-symplectic) with respect to either of

$$\Omega_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \Omega_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

In this case $\hat{f}_*^T \Omega_j \hat{f}_*^T = -\Omega_j$.

The original complex 4-integral I can be decomposed into real and imaginary parts, providing a pair of 4-integrals for \hat{f} :

$$I_1(x, y, u, v) = A(u(x^2 - y^2) + y(u^2 - v^2) + 2vx(u - y)) + B(x - v),$$

$$I_2(x, y, u, v) = A(v(x^2 - y^2) - x(u^2 - v^2) + 2uy(x + v)) + B(y + u).$$

Moreover, $\{I_1, I_2\} = 0$ is automatically satisfied as in Section 4.3.

The corresponding symmetries (using Ω_1) are

$$\begin{aligned} \xi_1(x, y, u, v) &= (2A(xy + vy - ux) + B) \frac{\partial}{\partial x} - A(x^2 - y^2 + 2vx + 2uy) \frac{\partial}{\partial y} \\ &\quad + A(u^2 - v^2 - 2uy - 2vx) \frac{\partial}{\partial u} + (2A(ux - vy + uv) + B) \frac{\partial}{\partial v}, \\ \xi_2(x, y, u, v) &= A(y^2 - x^2 - 2uy - 2vx) \frac{\partial}{\partial x} + (A(2ux - 2vy - 2xy) - B) \frac{\partial}{\partial y} \\ &\quad + (2A(uv + ux - vy) + B) \frac{\partial}{\partial u} + A(v^2 - u^2 + 2uy + 2xv) \frac{\partial}{\partial v}. \end{aligned}$$

Note also that because $I' = iI$, we have that

$$I'_1 = I_2, \quad I'_2 = -I_1.$$

Acknowledgements

This investigation is part of the research programme of the Australian Research Council. HWC gratefully acknowledges the support of a CRA-La Trobe Distinguished Visitor Fellowship.

References

- [1] M.J. Ablowitz and J.F. Ladik, A nonlinear difference scheme and inverse scattering, *Stud. Appl. Math.* 55 (1976) 213–229; On the solution of a class of nonlinear partial difference equations, *Stud. Appl. Math.* 57 (1977) 1–12.
- [2] V.I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer, New York, 1989).
- [3] M. Bruschi, O. Ragnisco, P.M. Santini and Tu Gui-Zhang, Integrable symplectic maps, *Physica D* 49 (1991) 273–294.
- [4] G.B. Byrnes, A geometric treatment of reduction of order of ordinary difference equations, *J. Phys. A: Math. Gen.* 28 (1995) 4925–4944.
- [5] G.B. Byrnes, R. Sahadevan and G.R.W. Quispel, Factorizable Lie symmetries and the linearization of difference equations, *Nonlinearity* 8 (1995) 443–459.
- [6] Y. Choquet-Bruhat, C. DeWitt-Morette with M. Dillard-Bleick, *Analysis, Manifolds and Physics*, 2nd Ed. (North-Holland, Amsterdam, 1982).
- [7] M. Crampin, A note on non-Noether constants of motion, *Phys. Lett. A* 95 (1983) 209–212.
- [8] B. Fuchssteiner and A.S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Physica D* 4 (1981) 47–66.
- [9] J.M. Greene, R.S. MacKay, F. Vivaldi and M.J. Feigenbaum, Universal behaviour in families of area preserving maps, *Physica D* 3 (1981) 468–486.
- [10] R. Hirota, Nonlinear partial difference equations I–III, *J. Phys. Soc. Japan* 43 (1977) 1424–1433, 2074–2078, 2079–2086; Discrete analogue of generalised Toda equation, *J. Phys. Soc. Japan* 50 (1981) 3785–3791.
- [11] J.S.W. Lamb and G.R.W. Quispel, Reversing k -symmetries in dynamical systems, *Physica D* 73 (1994) 277–304.
- [12] J.S.W. Lamb and G.R.W. Quispel, Cyclic reversing k -symmetry groups, *Nonlinearity* 8 (1995) 1005–1026.
- [13] D. Levi and P. Winternitz, Continuous symmetries of discrete equations, *Phys. Lett. A* 152 (1991) 335–338.
- [14] P. Libermann and C.-M. Marle, *Symplectic Geometry and Analytical Mechanics* (Reidel, Dordrecht, 1987).
- [15] J.D. Logan, First integrals in the discrete variational calculus, *Aequationes Math.* 9 (1973) 210–220.
- [16] S. Maeda, Canonical structure and symmetries for discrete systems, *Math. Japonica* 25 (1980) 405–420.
- [17] S. Maeda, The similarity method for difference equations, *IMA J. Appl. Math.* 38 (1987) 129–134.
- [18] F. Magri, A simple model of the integrable Hamiltonian equation, *J. Math. Phys.* 19 (1978) 1156–1162.
- [19] J.D. Meiss, Symplectic maps, variational principles and transport, *Rev. Mod. Phys.* 64 (1992) 795–848.
- [20] E. Noether, Invariante variationsprobleme, *Nachr. König. Gesell. Wissen. Göttingen, Math.-Phys. Kl.* (1918) 235–257 [English transl.: *Transport Theory and Stat. Phys.* 1 (1971) 186–207].
- [21] V.G. Papageorgiou, F.W. Nijhoff and H.W. Capel, Integrable mappings and nonlinear integrable lattice equations, *Phys. Lett. A* 147 (1990) 106–114.
- [22] V.G. Papageorgiou, F.W. Nijhoff, B. Grammaticos and A. Ramani, Isomonodromic deformation problems for discrete analogues of Painlevé equations, *Phys. Lett. A* 164 (1992) 57–64.
- [23] G.R.W. Quispel, H.W. Capel and R. Sahadevan, Continuous symmetries of differential-difference equations: the Kac-van Moerbeke equation and Painlevé reduction, *Phys. Lett. A* 170 (1992) 379–383.
- [24] G.R.W. Quispel, F.W. Nijhoff, H.W. Capel and J. van der Linden, Linear integral equations and nonlinear difference-difference equations, *Physica A* 125 (1984) 344–380.
- [25] G.R.W. Quispel, J.A.G. Roberts and C.J. Thompson, Integrable mappings and soliton equations I, II, *Phys. Lett. A* 126 (1988) 419–421; *Physica D* 34 (1989) 183–192.

- [26] J.A.G. Roberts and G.R.W. Quispel, Chaos and time-reversal symmetry. Order and chaos in reversible dynamical systems, *Phys. Rep.* 216 (1992) 63–177.
- [27] Yu.B. Suris, Discrete time generalised Toda lattices: complete integrability and relation with relativistic Toda lattices, *Phys. Lett. A* 145 (1990) 113–119.
- [28] Yu. B. Suris, A discrete-time Garnier system, *Phys. Lett. A* 189 (1994) 281–289.