

Linearisation of Difference Equations Using Factorisable Lie Symmetries

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ABSTRACT. We report that a difference equation can be transformed to an equivalent difference equation which is linear with constant coefficients if and only if it admits a symmetry vector field X which is factorisable,

$$X(x, u) = A(x)G(u)\frac{\partial}{\partial u},$$

with the factor A which depends on the independent variable satisfying a non-degeneracy condition. The ordinary difference equation case is presented in detail through an example. The case of difference equations with incommensurate step-sizes is also treated.

1. Symmetries of ordinary difference equations

With the ever increasing ubiquity of computer-based simulations in mathematics, engineering and physics, the study of difference equations and other discrete models has assumed greater importance. Primarily this is due to the fact that computers can typically only solve discrete equations, but there are also fields of study where discrete structures are inherent, for example lattices in physics and cell structures in biology. On the other hand, since the time of Lie we have built up a very deep understanding of differential systems and their properties. It is therefore natural to attempt to transfer some of that understanding to analogous areas in the discrete theory. For example, integrable partial differential equations have been generalised to integrable partial difference equations [1, 10], integrable ordinary difference equations have been generalised to integrable mappings [2, 5, 11, 12] and attempts have been made to find discrete analogues of the Painlevé equations and the Painlevé property [9, 14]. In particular, Lie's theory of continuous symmetries of differential equations has been shown to be applicable to difference equations [13].

This article summarises a further investigation of the rôle of Lie symmetries of difference equations. An article containing detailed proofs of the results of this paper will appear elsewhere [4]. The results presented are rather surprising when held alongside the familiar differential theory. It is well known that if a system of n

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first-order ordinary differential equations (first-order ODE's) admit a k -parameter, solvable Lie group of symmetries, then the original system can be reduced to $n - k$ ODE's and k quadratures. A partial analogue of this result, restricted to autonomous symmetries, was given by Maeda [7] in 1986. Essentially the existence of such symmetries implies the possibility of a coordinate transformation in which some of the equations become linear with constant coefficients. In the case of a single first order ordinary difference equation (ODE) and a single symmetry, Maeda gives the linearising transformation explicitly, but matters appear to be far more complicated for higher order ODE's or systems of several first order ODE. In any case, to linearise a single ODE of order N one would expect to require N independent symmetries. Yet consider the second-order ODE

$$(1.1) \quad u(x+2) = \frac{2u(x+1) + u(x)[u^2(x+1) - 1]}{1 - u^2(x+1) + 2u(x)u(x+1)}, \quad \forall x \in \mathbb{R}.$$

The vector field

$$X(x, u) := x(1 + u^2) \frac{\partial}{\partial u}$$

can be shown to be a symmetry vector field of (1.1). Note that X has the special form

$$X(x, u) = A(x)G(u) \frac{\partial}{\partial u},$$

in this case with $A(x) := x$, $G(u) := 1 + u^2$.

Suppose then that we introduce the new dependent variable w defined by

$$w(u) := \int^u \frac{d\zeta}{G(\zeta)},$$

so for equation (1.1)

$$w(u) := \arctan u + m\pi,$$

for some fixed $m \in \mathbb{Z}$. Substituting $\tan w$ for u in (1.1) yields the transformed ODE

$$(1.2) \quad w(x+2) - 2w(x+1) + w(x) = k\pi, \quad k \in \mathbb{Z}.$$

The general solution of (1.2) is just

$$w(x) = a(x)x + b(x) + \frac{k}{2}\pi x^2, \quad x \in \mathbb{R},$$

where a, b are unit periodic functions, that is $a(x+1) = a(x)$ and $b(x+1) = b(x)$ $\forall x \in \mathbb{R}$. It should be noted that the function $A(x) = x$ which is a factor of the symmetry vector field is a solution of the homogeneous part of the linearised equation (1.2). Moreover, observe that A is not the solution of any first-order, homogeneous, linear, constant coefficient ODE: the significance of this will become clear with the statement of the general result.

Thus the original equation (1.1) has general solution

$$u(x) = \tan\left(\frac{k}{2}\pi x^2 + a(x)x + b(x)\right), \quad x \in \mathbb{R}.$$

With the existence of a single symmetry for a second-order equation we expect only a reduction of order by one [3], to a first order ODE, whereas here we have completely solved the problem.

To see how this works before stating the general result, consider an arbitrary autonomous second-order OΔE

$$(1.3) \quad u_{(2)} = F(u_{(1)}, u),$$

where we have introduced the notation

$$u_{(j)}(x) := u(x + j), \quad x \in \mathbb{R}.$$

Let X be a vector field with factorizable coefficient function as before,

$$X(x, u) := A(x)G(u) \frac{\partial}{\partial u}.$$

Since the action of X on u induces an action on the shifts $u_{(j)}$ of u , we introduce the prolongation $X^{(2)}$ of X :

$$X^{(2)}(x, u) = A(x)G(u) \frac{\partial}{\partial u} + A(x+1)G(u_{(1)}) \frac{\partial}{\partial u_{(1)}} + A(x+2)G(u_{(2)}) \frac{\partial}{\partial u_{(2)}},$$

so X is a symmetry vector field of (1.3) iff

$$X^{(2)}(u_{(2)} - F([u]))|_{u_{(2)}=F} = 0,$$

where $[u]$ is short-hand for $(u_{(1)}, u)$. Written in full with $F([u])$ substituted for $u_{(2)}$,

$$(1.4) \quad A(x+2)G \circ F([u]) = A(x+1)G(u_{(1)}) \frac{\partial F}{\partial u_{(1)}}([u]) + A(x)G(u) \frac{\partial F}{\partial u}([u]).$$

Provided that $A \neq 0$ we have:

$$(1.5) \quad G(u) \frac{\partial F}{\partial u}([u]) = -\frac{A(x+1)}{A(x)}G(u_{(1)}) \frac{\partial F}{\partial u_{(1)}}([u]) + \frac{A(x+2)}{A(x)}G \circ F([u])$$

and differentiating with respect to x yields

$$\left(\frac{A_{(1)}}{A}\right)' G(u_{(1)}) \frac{\partial F}{\partial u_{(1)}}([u]) = \left(\frac{A_{(2)}}{A}\right)' G \circ F([u]),$$

hence (assuming that A is not unit periodic) we can separate variables:

$$\frac{(A_{(2)}/A)'}{(A_{(1)}/A)'} = \frac{G(u_{(1)})}{G \circ F([u])} \frac{\partial F}{\partial u_{(1)}}([u]) = \Gamma_1,$$

where Γ_1 is a separation constant.

Therefore

$$\frac{\partial F/\partial u_{(1)}}{G(F)} = \frac{\Gamma_1}{G(u_{(1)})},$$

so integrating with respect to $u_{(1)}$ we obtain

$$\int^{F([u])} \frac{d\zeta}{G(\zeta)} = \Gamma_1 \int^{u_{(1)}} \frac{d\zeta}{G(\zeta)} + g(u).$$

Substituting $\Gamma_1 G \circ F([u])$ for $G(u_{(1)}) \frac{\partial F}{\partial u_{(1)}}([u])$ in equation (1.5) and separating, we obtain by the same procedure

$$\int^{F([u])} \frac{d\zeta}{G(\zeta)} = \Gamma_2 \int^u \frac{d\zeta}{G(\zeta)} + h(u_{(1)}),$$

so we have

$$\int^{F([u])} \frac{d\zeta}{G(\zeta)} = \Gamma_1 \int^{u_{(1)}} \frac{d\zeta}{G(\zeta)} + \Gamma_2 \int^u \frac{d\zeta}{G(\zeta)} + \Gamma_3,$$

where Γ_2 and Γ_3 are further constants of integration. Making the coordinate transformation

$$w(u) := \int^u \frac{d\zeta}{G(\zeta)}$$

as in the example, we have

$$w \circ F([u]) = \Gamma_1 w(u_{(1)}) + \Gamma_2 w(u) + \Gamma_3,$$

or without restricting to solutions of (1.3),

$$(1.6) \quad w(u_{(2)}) = \Gamma_1 w(u_{(1)}) + \Gamma_2 w(u) + \Gamma_3.$$

From the separation procedure we also have

$$(A_{(2)}/A)' = \Gamma_1 (A_{(1)}/A)'$$

so after integration and comparison with the dependent variable part it follows easily that A will in fact be a solution of the homogeneous part of the linearised equation (1.6). Note that this is a second-order constant coefficient linear O Δ E, so the general solution can be obtained by elementary methods. It is important to note, however, that it is a linearisation of the invariance condition (1.4) rather than of the original O Δ E (1.3).

However if we assume *a priori* that X is a symmetry vector field, then (1.4) (and hence (1.6)) holds whenever (1.3) is satisfied. It must be shown that the converse also holds, so that the linearised invariance equation is equivalent to the original O Δ E (in that it has the same solution set). To do this, we impose the restriction that in the O Δ E $u_{(2)} = F(u_{(1)}, u)$, F is a meromorphic function of the variables $u_{(1)}$ and u . We then introduce the function H ,

$$H(u_{(2)}, u_{(1)}, u) := u_{(2)} - F(u_{(1)}, u).$$

Solutions of the O Δ E (1.3) now correspond to a subset of the space with coordinates $(x, u, u_{(1)}, u_{(2)})$, analogous to the jet spaces used by Olver [8] in his treatment of symmetries of differential equations. The solutions lie in the subset where H vanishes, called the *zero-set* of H , $Z(H)$. From the fact that H is meromorphic, we know that $Z(H)$ is an analytic set (see e.g. [6]).

If the coefficient functions A and G of the symmetry vector field

$$X(x, u) = A(x)G(u) \frac{\partial}{\partial u}$$

are also meromorphic, then so is the derivative

$$K := X^{(2)}H.$$

Since we assume that X generates a symmetry of (1.3), we know that

$$K|_{H=0} = X^{(2)}H|_{H=0} = 0,$$

or in terms of zero sets:

$$Z(H) \subseteq Z(K).$$

To prove that the invariance equation, which we can linearise, is equivalent to the O Δ E, we now need to show that $Z(H) = Z(K)$.

From the theory of meromorphic functions of several variables, we know that $Z(H)$ is a union of complex submanifolds, with minimum co-dimension one. However the same argument applies to $Z(K)$, so if it can be shown that $Z(K)$ is irreducible then it will follow that $Z(H) = Z(K)$ [6]. Now every first-order polynomial is irreducible and we have shown that K can be linearised. Thus $Z(K)$ is irreducible

as an analytic set and so $Z(H) = Z(K)$: the invariance equation (1.6) has the same solution set as the OΔE (1.3).

1.1. The conditions on A and higher-order OΔE's. In the process of separation described above, there was a division by the derivative of the ratio $A_{(1)}/A$. Clearly the technique can only work if $A_{(1)}$ is not a constant multiple of A . Another way of stating this is that A must not satisfy a homogeneous, linear constant coefficient OΔE of first-order. In fact this is the only condition on A .

When it comes to higher-order OΔE's, the process of separation is simply an iteration of that described above for the second order case. However there is a string of divisions by complicated ratios of differentiated and shifted versions of A , so the conditions to be imposed appear more complicated at first. However the condition is in fact a direct analogue of the second-order condition: an OΔE of order N which has a symmetry vector field of the form

$$X(x, u) = A(x)G(u) \frac{\partial}{\partial u}$$

can be reduced to a linear OΔE with constant coefficients if and only if A does not satisfy any homogeneous, linear, constant coefficient OΔE of order less than N .

There is then the question of a converse: first, does every linearizable OΔE admit a factorizable symmetry vector field $A(x)G(u)\partial/\partial u$; second, if it does, will A satisfy the above non-degeneracy condition? It can be shown the first statement is true for arbitrary autonomous OΔE's. Moreover, if an autonomous OΔE can be reduced to one which is linear with constant coefficients, then the second is also true.

2. Generalisations

With rather more careful notation, it can be shown that the above results remain true for partial difference equations: we state the general theorem without any proof:

THEOREM 1. *A meromorphic autonomous partial difference equation of maximal rank can be reduced to a linear equation with constant coefficients by a fibre and mesh preserving coordinate transformation if and only if it admits a factorizable symmetry vector field*

$$X(x, u) = A(x)G(u) \frac{\partial}{\partial u}$$

with A a meromorphic function of the independent variables which does not satisfy any homogeneous, linear, constant coefficient partial difference equation lower in order, in any of the independent variables, than the original equation.

It can be reduced to a linear equation (with possibly non-constant coefficients) only if it admits a factorizable symmetry vector field of the same form, but without the requirement on A .

Note that a fibre preserving transformation is one where the action on the independent variables does not depend on u . The condition of maximal rank is analogous to that of Olver [8], restricted to our case of a single equation: the difference equation must depend essentially on all variables in a neighbourhood of the solutions.

As an example, consider the PΔE

$$(2.1) \quad 0 = (u_{(20)}\sqrt{1+u_{(11)}^2} + u_{(11)}\sqrt{1+u_{(20)}^2})(\sqrt{(1+u_{(01)}^2)(1+u^2)} + uu_{(01)}) \\ - (u_{(01)}\sqrt{1+u^2} + u\sqrt{1+u_{(01)}^2})(\sqrt{(1+u_{(20)}^2)(1+u_{(11)}^2)} + u_{(11)}u_{(20)}).$$

There is a symmetry vector field

$$X(x, y, u) = (1 + 2^x 3^y \sin(\pi y) + 2^x \sin(\pi x) + 2^y)\sqrt{1+u^2} \frac{\partial}{\partial u}.$$

The factor $A(x, y) := 1 + 2^x 3^y \sin(\pi y) + 2^x \sin(\pi x) + 2^y$ does not satisfy any lower order equation, so the transformation

$$w(u) := \int^u \frac{d\zeta}{G(\zeta)} = \operatorname{arcsinh} u$$

produces from (2.1) the linear, constant-coefficient PΔE

$$(2.2) \quad 0 = w_{(20)} + w_{(11)} - w_{(01)} - w.$$

2.1. Incommensurate step-sizes. The method does also work for difference equations with incommensurate step sizes. To illustrate this, consider the OΔE

$$(2.3) \quad u(x + \sqrt{2}) = F(u(x+1), u(x)).$$

Note that the ratio between step sizes is irrational: if it were rational it would suffice to re-scale to the largest common divisor as a single step size.

The technique used to reduce (2.3) to the unit step-size case is worth commenting upon. First, (2.3) is clearly equivalent to the pair of partial differential-difference equations

$$(2.4) \quad u(x + \sqrt{2}, y) = F(u(x+1, y), u(x, y)),$$

$$(2.5) \quad \frac{\partial u}{\partial y} = 0.$$

Note that the second of these equations is invariant under transformations $u \rightarrow w(u)$, as are used in the linearisation process described above.

Now introduce an invertible linear transformation T of the independent variables, defining the new variables $(\alpha, \beta) := T(x, y)$, so that

$$\alpha = x - \sqrt{2}y, \\ \beta = y$$

and set $\tilde{u} := u \circ T^{-1}$. Making use of equation (2.5) we find that (2.4) and (2.5) are equivalent to

$$(2.6) \quad \tilde{u}(\alpha, \beta + 1) = F(\tilde{u}(\alpha + 1, \beta), \tilde{u}(\alpha, \beta)),$$

$$(2.7) \quad Y\tilde{u} = 0,$$

with $Y := -\sqrt{2}\partial/\partial\alpha + \partial/\partial\beta = \partial/\partial y$. Equation (2.6) is now a conventional partial difference equation (PΔE), and as long as the only transformations made are of the form

$$\tilde{u} \rightarrow w(\tilde{u}) = \int^{\tilde{u}} \frac{d\zeta}{G(\zeta)},$$

then

$$Y\tilde{u} = 0 \iff Yw = 0.$$

Thus the linearised form of (2.6) can be used to generate solutions of the original incommensurate O Δ E (2.3).

The general result here is:

THEOREM 2. *Given a P Δ E with dependent variable u and n independent variables such that the spans of the k -th variable generate a degree N_k field extension of \mathbb{Q} , there is a unit-span P Δ E with $N := \sum_{k=1}^n N_k$ independent variables, which together with auxiliary conditions $Y_j u = 0$ for vector fields Y_1, \dots, Y_N is equivalent to the original P Δ E.*

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