Nonlinearity **8** (1995) 1005–1026. Printed in the UK

Cyclic reversing k-symmetry groups

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Received 23 June 1994, in final form 20 July 1995
Recommended by R S MacKay

Abstract. We consider discrete invertible dynamical systems L with the property that the kth iterate L^k possesses (reversing) symmetries that are not possessed by L. A map U is called a (reversing) k-symmetry of L if k is the smallest positive integer for which U is a (reversing) symmetry of L^k. In this paper we discuss the particular case that L possesses a cyclic reversing k-symmetry group. We derive a decomposition property of maps that possess a cyclic reversing k-symmetry group and we classify the occurrence of such groups in invertible dynamical systems. We discuss the occurrence of nonsimultaneously linearizable nonisomorphic reversing k-symmetry groups in maps possessing cyclic reversing k-symmetry groups, illustrated by an example of a diffeomorphism on the plane $\mathbb{R}^2$. We also construct examples of diffeomorphisms with cyclic reversing k-symmetry groups on the circle S^1, on the two-torus T^2, and on the cylinder $S^1 \times \mathbb{R}$.

AMS classification scheme numbers: 58F99, 20F05, 58D07

1. Introduction and statement of the results

Symmetry properties of dynamical systems have become an important field of research, as symmetry properties may cause anomalous generic dynamical behaviour and are often encountered in dynamical systems of practical interest, cf e.g. [6, 9, 22, 24, 25].

In [19] we initiated investigations of the dynamics of maps L that possess less (reversing) symmetries than their kth iterate L^k, for some value of k. In this context we introduced the notion of so-called (reversing) k-symmetry groups, extending the conventional notion of symmetry groups and reversing symmetry groups [13].

The motivation for studying reversing k-symmetries stems from their occurrence in a range of dynamical systems of physical interest, including web maps [14, 12], trace maps occurring in the study of electronic spectra of one-dimensional quasicrystals [23, 26], and kicked rotors [10, 11]. More importantly, in [18] and [17] it has been shown that (reversing) k-symmetries naturally occur in driven systems, prevalently at resonances. The latter findings indicate a wide range of dynamical systems in which (reversing) k-symmetries may arise.

At present, it is not well understood whether—and in what way—specific groups can occur as reversing k-symmetry groups of nontrivial dynamical systems. Therefore, in

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this paper we have chosen to address this problem in (what we believe to be) the least complicated case of cyclic reversing k-symmetry groups. In this respect it is also interesting to note that in fact the *triangle map*, studied by Alexander *et al* [1] in connection with its riddled basin, possesses a cyclic 2-symmetry group of order 3.

Before we state the main results of this paper in more detail we first introduce some preliminaries from [19].

We consider invertible dynamical systems L with discrete time (homeomorphisms) defined on a state space Ω. We let $\mathrm{Inv}(\Omega)$ denote the group of invertible maps $\Omega \mapsto \Omega$, and $\circ$ denotes composition.

Definition 1.1. $M \in \mathrm{Inv}(\Omega)$ *is called a k-symmetry of L if k is the smallest positive integer such that*

$$M \circ L^k = L^k \circ M. \tag{1.1}$$

$S \in \mathrm{Inv}(\Omega)$ *is called a reversing k-symmetry of L if k is the smallest positive integer such that*

$$S \circ L^k = L^{-k} \circ S. \tag{1.2}$$

In this definition, the usual concept of a *symmetry*, resp. *reversing symmetry*, of L [13] is recovered in the case $k = 1$.

The set of all symmetries and reversing symmetries of a map L forms a group under composition. We call this group the *reversing symmetry group* of L [13]. The set of all symmetries of L is a normal subgroup of the reversing symmetry group and is called the *symmetry group* of L. We will adopt the notation $\mathcal{E}_k$ for the reversing symmetry group, $\mathcal{G}_k$ for the symmetry group, and $\mathcal{R}_k$ for the set of reversing symmetries of L^k:

$$\mathcal{G}_k := \{M \in \mathrm{Inv}(\Omega) \mid M \circ L^k = L^k \circ M\} \tag{1.3}$$

$$\mathcal{R}_k := \{S \in \mathrm{Inv}(\Omega) \mid S \circ L^k = L^{-k} \circ S\} \tag{1.4}$$

$$\mathcal{E}_k := \mathcal{G}_k \cup \mathcal{R}_k. \tag{1.5}$$

Note that $\mathcal{R}_k$ is not a group, and $\mathcal{R}_k = S\mathcal{G}_k$, with S being any reversing symmetry of L^k. Furthermore, $\mathcal{G}_k$ is a normal subgroup of $\mathcal{E}_k$, i.e. $\mathcal{G}_k \trianglelefteq \mathcal{E}_k$, and if $\mathcal{E}_k \neq \mathcal{G}_k$ then $\mathcal{E}_k/\mathcal{G}_k \simeq \mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the cyclic group of order 2. Thus $\mathcal{G}_k$ is a subgroup of index 2 in $\mathcal{E}_k$†.

The notion of (reversing) k-symmetries induces a map ϕ_L on $\mathcal{E}_k$,

$$\phi_L : \mathcal{E}_k \mapsto \mathcal{E}_k. \tag{1.6}$$

We define the action of ϕ_L on $\mathcal{G}_k$ as follows: let $M \in \mathcal{G}_k$, then‡

$$\phi_L(M) := L \circ M \circ L^{-1}. \tag{1.7}$$

For any $S \in \mathcal{R}_k$, we define the action of ϕ_L as

$$\phi_L(S) := L \circ S \circ L. \tag{1.8}$$

In this way, ϕ_L is defined both on the reversing and non-reversing part of $\mathcal{E}_k$,

$$\phi_L : \mathcal{G}_k \mapsto \mathcal{G}_k \quad \phi_L : \mathcal{R}_k \mapsto \mathcal{R}_k. \tag{1.9}$$

† This structure is also present in magnetic groups [21] and dichromatic colour groups [20]. For an illustration of the use of this analogy, see e.g. [14].

‡ It should be noted that there is a natural definition of ϕ_L also in the case of noninvertible maps possessing k-symmetries. Namely, with $\phi_L : \mathcal{G}_k \mapsto \mathcal{G}_k$ and $L \circ M = \phi_L(M) \circ L$ it follows that ϕ_L is a group automorphism of $\mathcal{G}_k$. Our restriction here to invertible maps is motivated by our interest in maps possessing next to k-symmetries also reversing k-symmetries.

With this definition of ϕ_L, (reversing) k-symmetries correspond to periodic orbits of ϕ_L of period k. We denote this $\#_L(M) = k$, resp. $\#_L(S) = k$. We will refer to orbits of ϕ_L as *ϕ_L-orbits*.

It is furthermore worthwhile to note that ϕ_L is a group automorphism of $\mathcal{G}_k$, but in general not of $\mathcal{E}_k$.

We now introduce the notion of a *(reversing) k-symmetry group.*

Definition 1.2. *Let $\mathcal{D} \leqslant \mathcal{E}_k$ and be closed under ϕ_L, i.e.*

$$\phi_L \;:\; \mathcal{D} \mapsto \mathcal{D} \tag{1.10}$$

and let k be the smallest positive integer for which all $U \in \mathcal{D}$ are (reversing) symmetries of L^k, then $\mathcal{D}$ is called a (reversing) k-symmetry group of L. We denote this by $\#_L(\mathcal{D}) = k$.

In definition 1.2 k is precisely the least common multiple of $\#_L(U)$ for all $U \in \mathcal{D}$. The way in which ϕ_L divides a (reversing) k-symmetry group into ϕ_L-orbits will be referred to as the *ϕ_L-orbit structure.*

It should be noted that not every subgroup of $\mathcal{E}_k$ is a (reversing) $\tilde{k}$-symmetry group, for some value of $\tilde{k}$, since not every subgroup of $\mathcal{E}_k$ will be closed under ϕ_L. Also $\#_L(\mathcal{E}_k)$ need not be equal to k, but may only be a divisor of k in case L does not possess a (reversing) k-symmetry.

Every (reversing) k-symmetry U of a map L has a natural embedding in a (reversing) k-symmetry group. We call this group the *ϕ_L-orbit group* of U†,

$$\mathcal{O}_L(U) := \langle U, \phi_L(U), \ldots, \phi_L^{k-1}(U)\rangle. \tag{1.11}$$

We will now formulate the main results of this paper. Our main concern in this paper is with maps possessing cyclic reversing k-symmetry groups. As can be read from the above definitions, a reversing k-symmetry S of a map L generates a cyclic reversing k-symmetry group if and only if $\langle S\rangle = \mathcal{O}_L(S)$.

Our first result consists of a decomposition property of maps possessing a cyclic reversing k-symmetry group.

Theorem 1.1. *A map L possesses a cyclic reversing k-symmetry group generated by S of order 2m if and only if L can be decomposed in the following way‡:*

$$L = U \circ S^{-1} \tag{1.12}$$

where

$$S^{2m} = Id \tag{1.13}$$

$$U^2 = S^{2q} \tag{1.14}$$

and

$$U \circ S^2 = S^{2p} \circ U \tag{1.15}$$

for some values of p and q ($0 < p, q \leqslant m$) satisfying

$$p^2 = 1 \bmod m \tag{1.16}$$

$$(p-1)q = 0 \bmod m. \tag{1.17}$$

The value of k is given by

$$\begin{array}{ll} k = 1 & \textit{if and only if } q = 1 \\ k = A & \textit{if } p = 1 \text{ and } q \neq 1 \\ k = \mathrm{lcm}(2, A) & \textit{if } p \neq 1. \end{array}$$

† $\langle \alpha_1, \ldots, \alpha_n\rangle$ denotes the group generated by the elements $\alpha_1 \ldots, \alpha_n$.

‡ The case that S generates a cyclic group of infinite order is discussed in appendix C.

where $A := 2m/\gcd(2m, 2q - p - 1)$†.

Notice in the above theorem that $p \neq 1$ implies $q \neq 1$ and that k always divides the order $2m$ of the cyclic group.

From theorem 1.1 with $k = 1$, we recapture the result of Lamb [13] that a map L possessing a reversing symmetry S can be written as

$$L = U \circ S^{-1} \tag{1.18}$$

with

$$U^2 = S^2. \tag{1.19}$$

Moreover, if S is an involution, we thus find that L is the composition of two involutions, as pointed out already by Birkhoff [3].

Decomposition properties of maps possessing reversing symmetries have been shown to be very useful in order to recognize and to construct such maps [24]. Decomposition properties have also proved useful for the study of symmetric periodic orbits, using a method due to DeVogelaere [7]. An extension of this method to the case of maps with reversing k-symmetries can be found in [5].

In order to study the ways reversing k-symmetry groups may occur in (invertible) dynamical systems, it is useful to study the group structure of reversing k-symmetry groups, including the map L as an element (following the approach of [13] in the case of reversing symmetry groups). The decomposition property presented in theorem 1.1 follows directly from this group structure.

Our next result concerns the ϕ_L-orbit structure of cyclic reversing k-symmetry groups. We address the question in what way ϕ_L can permute the elements of a cyclic reversing k-symmetry group. The ϕ_L-orbit structure of a (reversing) k-symmetry group contains important information on how symmetry properties interact with the dynamics. Examples of this interaction have been discussed in [16] in the context of local bifurcations and in [5] in relation to symmetry properties of periodic orbits.

Theorem 1.2. *The ϕ_L-orbit structures of cyclic reversing k-symmetry groups of a map L with the decomposition property of theorem 1.1 are uniquely labelled by the integers m, p and* $A := 2m/\gcd(2m, 2q - p - 1)$. *(See footnote, this page.)*

The above theorem states that if we have two maps with a cyclic reversing k-symmetry group and decomposition property characterized by integers m, p, and q as described in theorem 1.1, then the permutations that ϕ_L induces on the cyclic group $\langle S \rangle$ are equivalent if and only if m, p and A are identical in both cases. In this sense, theorem 1.2 provides a classification of possible ϕ_L-orbit structures of cyclic reversing k-symmetry groups.

Furthermore we find many regularities in the lengths and number of ϕ_L-orbits.

Theorem 1.3. *Let S be the generator of a cyclic reversing k-symmetry group of a map L. Then*
(i) $\#_L(S^n) \in \{k, k/2\}$ *if n is odd, and in particular* $\#_L(S) = k$,
(ii) $\#_L(S^n) \in \{1, 2\}$ *if n is even.*
Let r_k *and* $r_{k/2}$ *denote the number of reversing* ϕ_L*-orbits, i.e.* ϕ_L*-orbits in* $\mathcal{R}_k \cap \langle S \rangle$, *of length k and k/2. Then*
(iii) if $r_{k/2} \neq 0$ *then* $r_{k/2}$ *divides* r_k.

Notice that it follows from this theorem if k is odd, that $\langle S^2 \rangle$ is a symmetry group of L

† Whenever $2q - p - 1 = 0$ one should read $2q - p - 1 = 2m$, leading to $A = 1$.

and hence $\#_L(S^n) = k$ for all n odd. Necessary and sufficient conditions for the existence of reversing ϕ_L-orbits of length $k/2$ are given in proposition 2.7.

In appendix A we present a table containing for $m = 1, \ldots, 25$ all different ϕ_λ-orbit structures of cyclic reversing k-symmetry groups (with $k > 1$) represented by the integers m, p, and A. Also k and the ϕ_λ-orbit lengths are given, illustrating the content of theorem 1.3.

Theorem 1.1, theorem 1.2 and theorem 1.3 summarize our main results. Other results include the number of different values of A which are allowed for any given m and p (proposition 2.10). From this, the number of inequivalent ϕ_L-orbit structures can be computed as a function of the order $2m$ of the cyclic group.

An interesting property of maps with reversing k-symmetry groups is the coexistence of reversing k-symmetry groups. Namely, if S is a reversing k-symmetry of a map L then $U := L \circ S$ is also a reversing k-symmetry of L, and U is not contained in the ϕ_L-orbit group of S. It turns out that the ϕ_L-orbit group of U has a very nice structure. The ϕ_L-orbit group of a reversing k-symmetry S shares its non-reversing part with the ϕ_L-orbit group of $L \circ S$, and have the same number of elements. In particular, in the case of cyclic reversing k-symmetry groups we find that $\mathcal{O}_L(U) = \{U, U \circ S^2, \ldots, U \circ S^{2m-2}, S^2, S^4, \ldots S^{2m-2}, Id\}$. However, despite the similarities, $\mathcal{O}_L(S)$ and $\mathcal{O}_L(U)$ will in general not be isomorphic. In fact, in the case of cyclic reversing k-symmetry groups, if S is of order $2m$ then U is of order $2m/\gcd(2m, q)$. This implies that $\langle U \rangle \simeq \langle S \rangle$ (and $\mathcal{O}_L(U) \simeq \mathcal{O}_L(S)$) if and only if $\gcd(q, m) = 1$.

It is important to note that precisely *reversing* k-symmetries give rise to nice types of coexisting symmetry properties, non-reversing k-symmetries do not, cf [15].

The coexistence of reversing k-symmetry groups may lead to interesting dynamical consequences. As an illustration, let us consider the following example of a diffeomorphism $L : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with a cyclic reversing 2-symmetry group of order four:

$$L : \left\{ \begin{array}{lcl} x' & = & (\varepsilon y'^2 + 1)\left[x + f(x - \frac{y}{\varepsilon x^2+1})\right] \\ y' & = & \frac{-y}{\varepsilon x^2+1} - f(x - \frac{y}{\varepsilon x^2+1}), \end{array} \right. \tag{1.20}$$

where f is an odd function, i.e. $f(z) = -f(-z)$. It is readily verified that $R_{\pi/2} : (x, y) \mapsto (y, -x)$ is a reversing 2-symmetry of L, and that the ϕ_L-orbit structures are given by $\phi_L^2(R_{\pi/2}) = \phi_L(R_{\pi/2}^3) = R_{\pi/2}$ and $\phi_L(R_{\pi/2}^2) = R_{\pi/2}^2$, such that $\mathcal{O}_L(R_{\pi/2}) = \langle R_{\pi/2} \rangle$ †.

In figure 1, the phase portrait of the map L, with $f(z) = \kappa \sin(\pi z^3)$, $\kappa = 0.1$, and $\varepsilon = 0.2$ is depicted on the domain $[-1.5, 1.5] \times [-1.5, 1.5]$. It illustrates some dynamical consequences of the presence of the cyclic reversing 2-symmetry group $\langle R_{\pi/2} \rangle$. The folded black objects on the left and right side of the phase portrait are attractors. Because of the fact that $R_{\pi/2}$ is a reversing 2-symmetry, identically shaped strange repellors are present in the top and botom part. Their repelling nature makes them invisible in this picture (although they are there). Furthermore, by symmetry, the origin (0, 0) is forced to be a fixed point and due to the symmetry is generically a saddle point.

More interestingly, in figure 1 one observes four elliptic points (that can be recognized by the circles surrounding them). However, in a non-area-preserving map one would in general not expect to see elliptic periodic orbits, as they are not structurally stable (they tend to become attracting or repelling fixed points). However, they may arise in non-area-preserving context in a persistent way as (k-)symmetric orbits in systems possessing a mirror-type reversing (k-)symmetry.

The reason for their occurrence in the present example can be understood as a consequence of coexisting reversing 2-symmetry groups. Namely, $U := L \circ R_{\pi/2}$ is a

† The construction of this example is discussed in section 3.4.

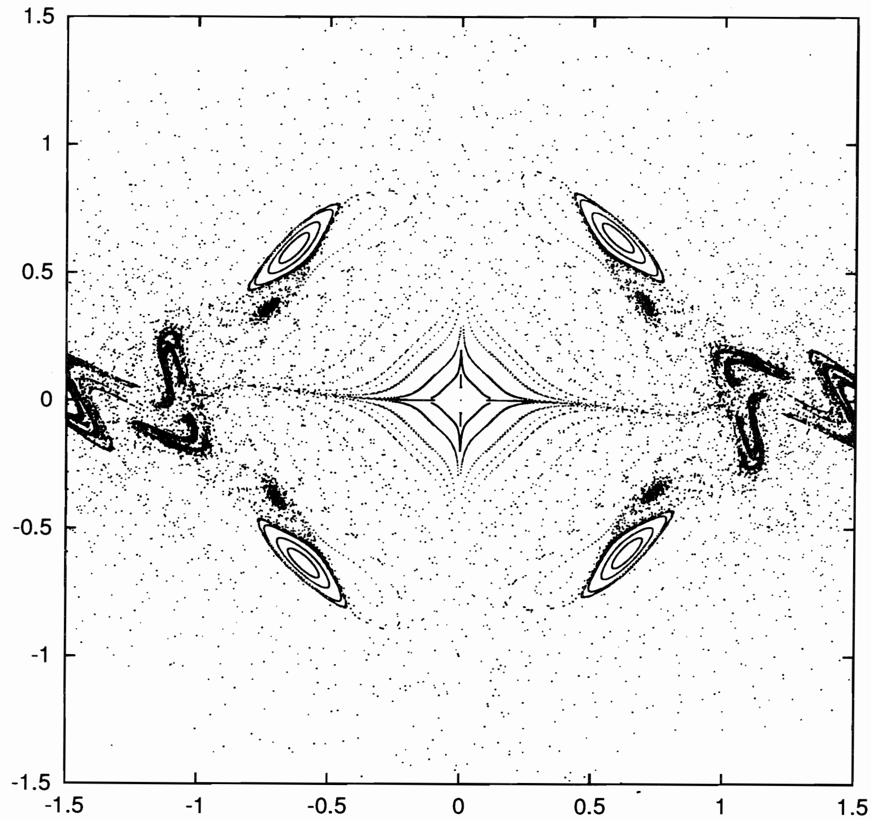

Figure 1. Phase portrait on the domain $[-1.5, 1.5] \times [-1.5, 1.5]$ of the map L (1.20), with $f(z) = \kappa \sin(\pi z^3)$, $\kappa = 0.1$, and $\varepsilon = 0.2$.

reversing 2-symmetry of the map L, and U is an involution acting as a nonlinear mirror in a curve. Its ϕ_L-orbit group furthermore consists of the reversing 2-symmetry $-Id \circ U$ (also a nonlinear mirror) and the symmetry $-Id$, forming together the dihedral group D_2. The elliptic points in the phase portrait precisely lie on the fixed sets of the reversing mirrors U and $-Id \circ U$.

The phase portrait of figure 1 clearly illustrates the rich dynamical behaviour that may occur in systems with reversing k-symmetries. It also shows the importance of recognizing nonlinear (reversing) symmetries. It is important to note that in this example it is impossible to find a coordinate frame in which the mirrors and the fourfold rotation simultaneously have a linear action. Namely, if one could, it would imply that the map L is essentially linear and thus has trivial dynamics, which is obviously not the case. In this context it is interesting to note that in [16] it has been observed that the *formal* normal forms governing the local bifurcations of the fixed point $(0, 0)$ have the nonsimultaneously linearizable reversing k-symmetry groups $\mathcal{O}_L(S)$ and $\mathcal{O}_L(U)$ acting effectively linearly, forming together the dihedral group D_4, and this can be understood from the ϕ_L-orbit structure of $\langle R_{\pi/2} \rangle$.

This paper focuses on one of the most elementary types of reversing k-symmetry groups, namely cyclic ones. It will be interesting to study also other types of reversing k-symmetry groups this way, especially in relation to applications. We think of dihedral and other finitely generated discrete groups, and in particular of crystallographic groups, because of their relevance to the formation of stochastic webs [14].

We moreover believe that the defining relations we obtained for the groups $\langle q; p, m\rangle_\lambda$ in section 2.2 are interesting in their own right, in relation to the work on generators and defining relations for discrete groups [4].

The remaining sections of this paper are devoted to the proofs of the theorems presented above (section 2) and to the construction of examples (section 3).

2. Group structure and ϕ_L-orbits

In this section we will work out the details of the ϕ_L-orbit structures of cyclic reversing k-symmetry groups and prove theorems 1.1–1.3. We first discuss briefly the case of cyclic k-symmetry groups in section 2.1. We deal with cyclic *reversing* k-symmetry groups in section 2.2.

2.1. Cyclic k-symmetry groups

We consider the case that a map L possesses a nontrivial cyclic k-symmetry group, generated by a map M that is of order m, i.e. $M^m = Id$†.

Proposition 2.1. *The group generated by a map L (of infinite order) and a map M of order 2m (generating a cyclic k-symmetry group of L) is isomorphic to the abstract group* $\langle p, m\rangle_\lambda$ *that is generated by two elements* λ *and* μ *satisfying the relations*

$$\mu^m = e \tag{2.1}$$

$$\mu\lambda = \lambda\mu^p \tag{2.2}$$

where e denotes the unit element of the group, and p and m are positive integers such that $p \leqslant m$ *and their greatest common divisor (gcd) is one, i.e.*

$$\gcd(p, m) = 1. \tag{2.3}$$

k is the smallest positive integer satisfying

$$p^k = 1 \bmod m. \tag{2.4}$$

Proof. The relations (2.1) and (2.2) are obvious. The relation (2.3) comes in as a necessary and sufficient condition for the order of μ to be m [4].

For all integers x and y we have $\mu^x\lambda^y = \lambda^y\mu^{x\cdot p^y}$ showing that $\#_\lambda(\mu^x)$ divides $\#_\lambda(\mu)$, proving (2.4). □

From (2.1) it follows that

$$\mu\lambda^{-1} = \lambda^{-1}\mu^{p'} \tag{2.5}$$

for some $p' \in \mathbb{N}$ with $1 \leqslant p' < m$ such that

$$\gcd(p', m) = 1 \tag{2.6}$$

and

$$p \cdot p' = 1 \bmod m. \tag{2.7}$$

The relation (2.7) uniquely defines p' because of (2.3) and (2.6).

† By nontrivial it is meant that there are no integers x, y such that $M^x L^y = Id$ and $y \neq 0$. The case that M generates a cyclic group of infinite order is discussed in appendix C.

In number theory k, as defined by (2.4), is known as the *exponent to which p belongs modulo m*. The problem of finding the exponent to which p belongs modulo m has been widely addressed, cf e.g. [8].

The ϕ_λ-orbit structure of a k-symmetry group is characterized by the way ϕ_λ permutes the elements of this group. In the definition of $\langle p, m\rangle_\lambda$ we see that m determines the order of the cyclic group, and that p determines the permutation of its elements. Note that if the permutations on the cyclic group $\langle\mu\rangle$ are determined, the permutations on $\langle\mu, \lambda\rangle \simeq \langle p, m\rangle_\lambda$ follow immediately.

However, in the defining relations of $\langle p, m\rangle_\lambda$ the value of p may depend on the choice we made in assigning an element of the cyclic group $\langle\mu\rangle$ as a generator. In order to investigate this ambiguity, we need to consider the automorphisms of the cyclic group $\langle\mu\rangle$ and see whether they induce connections between different values of p.

Definition 2.1. *The ϕ_λ-orbit structures of $\langle p, m\rangle_\lambda$ and $\langle \tilde{p}, m\rangle_\lambda$ are equivalent if these groups are related via an automorphism of the cyclic group $\langle\mu\rangle$.*

This notion of equivalence naturally extends to more complicated k-symmetry groups and reversing k-symmetry groups (see e.g. definition 2.2).

The above arguments imply that $\langle p, m\rangle_\lambda$ and $\langle \tilde{p}, m\rangle_\lambda$ may have equivalent ϕ_λ-orbit structures only if they are isomorphic. This is a necessary but in principle not a sufficient condition. Automorphisms of reversing k-symmetry groups which involve a (nontrivial) transformation of λ do not establish equivalence relations, as they in fact change the dynamical system under consideration.

It turns out that the automorphisms of the cyclic group ($\mu \mapsto \mu^x$, with $\gcd(x, m) = 1$) do not induce relations between different values of p. This leads us to the following characterization of ϕ_λ-orbit structures of cyclic k-symmetry groups:

Proposition 2.2. *The ϕ_λ-orbit structures of cyclic k-symmetry groups $\langle p, m\rangle_\lambda$ are uniquely labelled by the integers m and p.*

In the next section we will show that cyclic k-symmetry groups that occur as subgroups of cyclic reversing $\tilde{k}$-symmetry groups (with $k \leqslant \tilde{k}$) have $k \in \{1, 2\}$. The case $k = 1$ corresponds to an ordinary symmetry group, i.e. $p = 1$. The cyclic groups with $k = 2$ are more interesting. In this case p satisfies

$$\begin{cases} p^2 = 1 \bmod m \\ p \neq 1 \end{cases} \tag{2.8}$$

and all ϕ_λ-orbits have either period one or period two. In this case, we can express the lengths of the ϕ_λ-orbits in the cyclic group in terms of p and m.

Proposition 2.3. *If $\#_\lambda(\langle p, m\rangle_\lambda) = 2$, then $\langle p, m\rangle_\lambda$ has $\gcd(p-1, m)$ ϕ_λ-orbits of length 1 and $\frac{1}{2}(m - \gcd(p-1, m))$ ϕ_λ-orbits of length 2.*

Proof. If $\#_\lambda(\langle p, m\rangle_\lambda) = 2$ the ϕ_λ-orbits have length 1 or 2. Since

$$\mu^x\lambda = \lambda\mu^{xp} \tag{2.9}$$

we readily find that $\#_\lambda(\mu) = 1$ if and only if $x(p-1) = 0 \bmod m$. There are $\gcd(p-1, m)$ solutions to this equation: $x = 0 \bmod m/\gcd(p-1, m)$, proving the proposition. □

Note that groups with the same m but different p may have identical ϕ_λ-orbit lengths. This occurs for instance if $p \cdot \tilde{p} = 1 \bmod m$, i.e. when the induced permutations are each other's inverse, but it may also occur otherwise. For example, $\langle 3, 8\rangle_\lambda$ and $\langle 7, 8\rangle_\lambda$ both have two ϕ_λ-orbits of length 1 and three ϕ_λ-orbits of length 2 although their ϕ_λ-orbit structures are not equivalent.

2.2. *Cyclic reversing k-symmetry groups*

We now come to the main point of study in this paper. We consider a map L possessing a cyclic *reversing* k-symmetry group generated by a reversing k-symmetry S of even order $2m$. Note that S being of odd order would imply that L is of finite order too (which from the dynamical point of view is not very interesting)†.

Proposition 2.4. *The group generated by a map L (of infinite order) and a map S of order 2m, generating a cyclic reversing k-symmetry group of L, is isomorphic to the abstract group* $\langle q; p, m\rangle_\lambda$ *that is generated by two elements* λ *and* σ *satisfying the relations*

$$\sigma\lambda = \lambda^{-1}\sigma^{2q-1} \tag{2.10}$$

$$\sigma^2\lambda = \lambda\sigma^{2p} \tag{2.11}$$

$$\sigma^{2m} = e \tag{2.12}$$

where e denotes the unit element of the group, and p, q and m are positive integers such that $p \leqslant m$, $q \leqslant m$ *and*

$$p^2 = 1 \bmod m \tag{2.13}$$

$$(p-1)q = 0 \bmod m. \tag{2.14}$$

Proof. Consider a map L of infinite order and a reversing k-symmetry S of order $2m$. The ϕ_L-orbits of $\langle S\rangle$ are then completely determined by specifying the action on the reversing part

$$\phi_L(S) = S^{2q-1} \tag{2.15}$$

implying (2.10), and on the non-reversing part, implying (2.11). The last relation (2.12) follows from the fact that S is of order $2m$. The relations (2.13) and (2.14) are necessary and sufficient conditions for the order of σ to be $2m$, and for the order of λ to be not determined, i.e. it may be infinite. So (2.10)-(2.14) define a group that is genuinely isomorphic to $\langle L, S\rangle$. A detailed proof is presented in appendix B. □

If m is given, solutions of p and q from (2.13)–(2.14) are determined as follows: firstly, p is obtained from the quadratic congruence (2.13). For solving this congruence, efficient methods exist, cf [8]‡.

Considering a specific solution p, the possible values of q are determined by (2.14), i.e. $q = 0 \bmod m/\gcd(p-1, m)$ if $p \neq 1$ and $q = 0 \bmod m$ if $p = 1$.

In order to obtain a good understanding of the ϕ_λ-orbits we need to evaluate the consequences of the relations (2.10)–(2.14). We first note that using the relations (2.10)–(2.12) we can write any word of σ's and λ's as $\lambda^y\sigma^x$ where $x, y \in \mathbb{Z}$. Hence we may define functions f and g such that

$$\sigma^x\lambda^y = \lambda^{g(x,y)}\sigma^{f(x,y)}. \tag{2.16}$$

† The case that S generates a cyclic group of infinite order is discussed in appendix C.

‡ Let us write $m = 2^l\tilde{m}$ with $\tilde{m}$ odd. Then the number of different solutions for p is $\xi(l)2^{\tilde{m}_p}$, where $\tilde{m}_p$ denotes the number of different prime divisors of $\tilde{m}$ and $\xi(0) := 1$, $\xi(1) := 1$, $\xi(2) := 2$, and $\xi(l) := 4$ for $l \geqslant 3$. For a proof and further details on how to solve (2.13), see e.g. [8].

Let us furthermore define†

$$D := 2q - p - 1 \bmod 2m. \tag{2.17}$$

Then, with the relations (2.10)–(2.14) satisfied, it is easily checked that

$$f(x, y) = \varepsilon_{y+1}x + \varepsilon_y xp + \varepsilon_x yD \tag{2.18}$$

$$g(x, y) = (-)^x y \tag{2.19}$$

where $\varepsilon_z := 0$ if z is even and $\varepsilon_z := 1$ is z is odd. From (2.13), (2.14) we derive a very useful relation‡

$$pD = p \bmod m. \tag{2.20}$$

Let us study the ϕ_λ-orbits of $\langle q; p, m\rangle_\lambda$. In analogy to the definitions in section 2.1 we have

$$\phi_\lambda(\sigma) := \lambda\sigma\lambda, \tag{2.21}$$

since σ is a reversing k-symmetry of λ (for some value of k). The first questions arising concern the lengths of the ϕ_λ-orbits. Throughout this section we will take k to be defined as

$$k := \#_\lambda(\langle q; p, m\rangle_\lambda). \tag{2.22}$$

We use (2.16)–(2.19) to reveal some key properties of the ϕ_λ-orbit lengths:

Proposition 2.5.
(i) $k = \#_\lambda(\sigma)$.
(ii) k *divides* $2m$
(iii) $k = 2$ *if and only if* $D = m \bmod m$ *and* $q \neq 1$.
(iv) $\#_\lambda(\sigma^n) \in \{1, 2\}$ *if* n *is even.*
(v) $\#_\lambda(\sigma^n) \in \{k/2, k\}$ *if* n *is odd.*

Proof. The first assertion (i) follows from the fact that $\sigma\lambda^k = \lambda^{-k}\sigma$ implies $\sigma^x\lambda^k = \lambda^{(-)^x k}\sigma^x$ and hence $\#_\lambda(\sigma^x)$ divides $\#_\lambda(\sigma)$ for all $x \in \mathbb{Z}^+$. From equations (2.16)–(2.19) it can be directly shown that $\sigma\lambda^{2m} = \lambda^{-2m}\sigma$ (proving (ii)) and $\sigma\lambda^2 = \lambda^{-2}\sigma^{1+2D}$ (proving (iii)). Point (iv) reflects the fact that σ^2 commutes with λ^2. To prove point (v), assume that $\#_\lambda(\sigma^c) = y$, for some c odd, i.e. $\sigma^c\lambda^y = \lambda^{-y}\sigma^c$. This implies $\sigma^c\lambda^{2y} = \lambda^{-2y}\sigma^c$, leading to $\sigma\lambda^{2y} = \lambda^{-2y}\sigma$ (using (iv)). □

Thus the ϕ_λ-orbit of σ has length k. Note that all other ϕ_λ-orbit lengths divide k.

The following proposition shows how to obtain the value of k, for all reversing k-symmetry groups $\langle q; p, m\rangle_\lambda$:

Proposition 2.6. *Let* k *be defined by (2.22), and* $A := 2m/\gcd(2m, D)$. *Then* $k = A$ *if* $p = 1$, *and* $k = \operatorname{lcm}(2, A)$ *if* $p \neq 1$.

Proof. Let x be odd, then we have

$$\sigma\lambda^x = \lambda^{-x}\sigma^{p+xD}. \tag{2.23}$$

If x is even, we have

$$\sigma\lambda^x = \lambda^{-x}\sigma^{1+xD}. \tag{2.24}$$

† The domains of p and q yield $1 - m \leqslant D \leqslant 2m - 2$, but by convention we will let $1 \leqslant D \leqslant 2m$ as we are only concerned with D modulo $2m$.

‡ Note that the relations (2.10)–(2.14) are not equivalent to the relations (2.11)–(2.13) in combination with (2.20) and $\sigma\lambda = \lambda^{-1}\sigma^{D+p}$. Therefore, although D will turn out to be a useful quantity, we use q instead of D in the group definition.

Hence, if $p = 1$, k is the smallest positive integer satisfying

$$kD = 0 \bmod 2m. \tag{2.25}$$

We first note that k being odd implies $p = 1$. Namely, since $\#_\lambda(\sigma^2)$ divides k and $\#_\lambda(\sigma^2) \in \{1, 2\}$, $\#_\lambda(\sigma^2)$ must be 1 if k is odd. Hence, in case $p \neq 1$, k must be even yielding the result that (2.24) determines k, i.e. k satisfies the same equation (2.25). Note that we are interested in the smallest *even* integer k satisfying (2.25), i.e. if A is odd we have to multiply it by 2 to get k. □

Proof of theorem 1.1 The content of theorem 1.1 follows directly from proposition 2.4 and proposition 2.6. □

Proposition 2.6 gives us k. We know from proposition 2.5 (v) that all reversing ϕ_λ-orbits have either length k or length $k/2$. The following propositions deal with the occurrence of reversing ϕ_λ-orbits of length $k/2$.

Proposition 2.7. *The group $\langle q;\, p, m\rangle_\lambda$ has reversing ϕ_λ-orbits of length $k/2$ if and only if*
(i) k is even and $k/2$ is odd,
(ii) $p \neq 1$,
(iii) $m/\gcd(p-1, m)$ is odd,
(iv) $A + B$ is odd,
where A is defined as in proposition 2.6 and $B := (p-1)/\gcd(p-1, m)$.

Proof. If k is odd then there are obviously no ϕ_λ-orbits of length $k/2$.

If k and $k/2$ are both even (i.e. k is a multiple of 4), there are also no ϕ_λ-orbits of length $k/2$. This can be seen from the fact that for all c odd and x even we have

$$\sigma^c\lambda^x = \lambda^{-x}\sigma^{c+xD} \tag{2.26}$$

implying that in case all reversing ϕ_λ-orbit lengths are even, their length does not depend on the value of c, so they must all have equal length k. (Note that $k/2$ can be even, only if m is even.)

If $p = 1$, the fact that σ^2 commutes with λ causes all reversing ϕ_λ-orbits to be of equal length k.

We will now consider the action of $\phi_\lambda^{k/2}$ on the reversing elements of $\langle\sigma\rangle$ to see whether ϕ_λ-orbits of length $k/2$ occur in the remaining cases.

Let Δ denote the shift on the exponent of σ induced by $\phi_\lambda^{k/2}$:

$$\phi_\lambda^{k/2}(\sigma) = \sigma^{1+\Delta} \tag{2.27}$$

then

$$\phi_\lambda^{k/2}(\sigma^{2x+1}) = \sigma^{1+2x+\Delta+2(p-1)x} \tag{2.28}$$

with $\Delta := p - 1 + kD/2$. Thus the ϕ_λ-orbit of σ^{2x+1} is of length $k/2$ if and only if x satisfies

$$\Delta + 2(p-1)x = 0 \bmod 2m. \tag{2.29}$$

It suffices to look for solutions for x of (2.29) within the domain

$$x \in \{1, \ldots, m/\gcd(p-1, m)\}. \tag{2.30}$$

In the domain $x \in \{1, \ldots, m\}$ one finds the same set of solutions $\gcd(p-1, m)$ times. Moreover, it is easily checked that no more than one solution can be found in the domain (2.30).

Let us now look for the actual solution. Using (2.25) we obtain from (2.29)

$$(2x+1)(p-1) = Dk/2 \bmod 2m. \tag{2.31}$$

The domain (2.30) leads to

$$2x+1 \in \{3, 5, \ldots, 2m/\gcd(p-1, m)+1\}. \tag{2.32}$$

Now, since (2.25) implies that

$$Dk/2 = 0 \bmod m \tag{2.33}$$

the only candidate for a solution within the domain (2.32) is

$$2x+1 = m/\gcd(p-1, m) \tag{2.34}$$

leading us to condition (iii). Substituting (2.34) into (2.31) we obtain

$$mB = Dk/2 \bmod 2m \tag{2.35}$$

with B defined as in proposition 2.7.

We have to consider the cases A odd and A even separately (cf proposition 2.6)†:

$$A \text{ is odd } \Rightarrow Dk/2 = 0 \bmod 2m, \tag{2.36}$$

$$A \text{ is even } \Rightarrow Dk/2 = m \bmod 2m. \tag{2.37}$$

Combining this with (2.35), we find that ϕ_λ-orbits with length $k/2$ exist if and only if $A+B$ is odd. □

Proposition 2.8. *Let r_k and $r_{k/2}$ denote the number of reversing ϕ_λ-orbits of period k, respectively $k/2$, in the cyclic group $\langle\sigma\rangle < \langle q; p, m\rangle_\lambda$, then in case the requirements (i)-(iv) of proposition 2.7 are met (i.e. $r_{k/2} \neq 0$), we find*

$$r_{k/2} = \frac{2}{k}\gcd(p-1, m) \tag{2.38}$$

and $r_{k/2}$ is a divisor of r_k.

Proof. In the proof of proposition 2.7 it is found that, in case conditions (i)–(iv) are met, the number of solutions for x of (2.29) (corresponding to points on reversing ϕ_λ-orbits of length $k/2$) is equal to $\gcd(p-1, m)$. Hence, if there exist reversing ϕ_λ-orbits with length $k/2$, $r_{k/2}$ is given by (2.38).

From the fact that all m reversing elements are divided in orbits of length k and $k/2$ we observe that $2r_k + r_{k/2} = 2m/k$. Using (2.38) we then find

$$\frac{r_k}{r_{k/2}} = \frac{1}{2}[m/\gcd(p-1, m) - 1]. \tag{2.39}$$

Condition (iii) of proposition 2.7 confirms that indeed $m/\gcd(p-1, m)$ is odd, and the quotient (2.39) is an integer. □

Proof of theorem 1.3. Point (i) and (ii) of theorem 1.3 follow directly from proposition 2.5, while point (iii) is proved in proposition 2.8 above. □

From proposition 2.3, proposition 2.6 and proposition 2.7 we now know for every group $\langle q; p, m\rangle_\lambda$ its value of k and its ϕ_λ-orbit lengths. Yet, we are left to classify the ϕ_λ-orbit structure of cyclic reversing k-symmetry groups, i.e. determine which groups $\langle q; p, m\rangle_\lambda$ have equivalent ϕ_λ-orbit structures. In analogy to definition 2.1, we have the following notion of equivalence:

† To derive (2.37), note that if A is even, $D/\gcd(2m, D)$ is odd.

Definition 2.2. *The ϕ_λ-orbit structures of $\langle q; p, m\rangle_\lambda$ and $\langle \tilde{q}; \tilde{p}, m\rangle_\lambda$ are equivalent if these groups are related via an automorphism of the cyclic group $\langle \sigma \rangle$.*

Therefore we are led to study the automorphisms of $\langle \sigma \rangle$, the cyclic group of order $2m$:

$$\sigma \mapsto \sigma^{2x+1} \quad \text{with} \ \gcd(2x+1, 2m) = 1. \tag{2.40}$$

Proposition 2.9. *The automorphisms (2.40) induce isomorphisms between the group $\langle q; p, m\rangle_\lambda$ and the groups $\langle \tilde{q}; \tilde{p}, \tilde{m}\rangle_\lambda$, with $m = \tilde{m}$, $p = \tilde{p}$, and*

$$\tilde{D} = \tilde{d} \cdot \gcd(2m, D) \tag{2.41}$$

with $\tilde{d}$ relatively prime to A.

Proof. Let $\tilde{\sigma} := \sigma^{2x+1}$, with $\gcd(2x+1, 2m) = 1$. Then $\tilde{\sigma}$ is of order $2m$, and

$$\tilde{\sigma}\lambda \ = \lambda^{-1}\sigma^{(2x+1)p+D} = \lambda^{-1}\tilde{\sigma}^{2\tilde{q}-1} \tag{2.42}$$

$$\tilde{\sigma}^2\lambda = \lambda\tilde{\sigma}^{2p}. \tag{2.43}$$

Hence, $\tilde{m} = m$, $\tilde{p} = p$, and

$$(2\tilde{q} - 1)(2x + 1) = (2x + 1)p + D \tag{2.44}$$

which implies that

$$\tilde{D}(2x + 1) = D \bmod 2m. \tag{2.45}$$

From the fact that $\gcd(2x + 1, 2m) = 1$ it now follows that $\tilde{D}$ is a multiple of $\gcd(2m, D)$. Hence with $d := D/\gcd(2m, D)$ and $\tilde{d} := \tilde{D}/\gcd(2m, D)$, and A as defined in proposition 2.6, we find

$$\tilde{d}(2x + 1) = d \bmod A. \tag{2.46}$$

From the fact that $\gcd(A, d) = \gcd(A, 2x + 1) = 1$ it follows that also $\gcd(A, \tilde{d}) = 1$. In fact, $\tilde{d} \in \{1, \ldots, A\}$ is a solution of (2.46) for some value of x (satisfying $\gcd(2x+1, 2m) = 1$) if and only if $\gcd(\tilde{d}, A) = 1$. □

The above proposition shows that there are precisely $\varphi_E(A)$ different groups $\langle q; p, m\rangle_\lambda$ that have the same ϕ_λ-orbit structure, where φ_E is the Euler totient function†. Note that these groups only differ in their value of q.

The transformation $\lambda \mapsto \lambda^{-1}$ relates the group $\langle q; p, m\rangle_\lambda$ with its corresponding value of D to the group $\langle \tilde{q}; p, m\rangle_\lambda$ with $\tilde{D} = 2m - D$. Note that this relation is also found in (2.41) with $\tilde{d} = A - D/\gcd(2m, D)$. Namely, since $D/\gcd(2m, D)$ is relatively prime to A, $\tilde{d}$ is relatively prime to A too.

Proposition 2.9 provides the basis for the proof of theorem 1.2.

Proof of theorem 1.2. From proposition 2.9 we find that the groups $\langle q; p, m\rangle_\lambda$ have equivalent ϕ_λ-orbit structures if and only if $m = \tilde{m}$, $p = \tilde{p}$, and D and $\tilde{D}$ satisfy (2.41). If (2.41) is satisfied we find

$$\gcd(2m, D) = \gcd(2m, \tilde{D}). \tag{2.47}$$

But with $m = \tilde{m}$ and $p = \tilde{p}$ fixed, the automorphism of the cyclic group cover all $(\varphi_E(A) - 1)$ possible $\tilde{D}$'s and hence all possible $\tilde{q}$'s satisfying (2.47). Hence, (2.41) is equivalent to (2.47) and hence equivalent to $A = \tilde{A}$. □

† Writing m as a product of primes $m = \prod_{i=1}^{l} p_i^{\alpha_i}$ $(\alpha_i \in \mathbb{N})$ the Euler totient function is given by $\varphi_E(m) = \prod_{i=1}^{l}(p_i^{\alpha_i} - p_i^{\alpha_i - 1})$, cf [8].

Given the order of the cyclic group, i.e. $2m$, we know how many different values of p will occur, cf second footnote on p 1013. The following proposition deals with the number of different values of A that arise for given m and p.

Proposition 2.10. *Let $N_A(m, p)$ denote the number of different values of A that arise for given values of m and p satisfying (2.13), then in case $p \neq 1$, $N_A(m, p)$ is equal to the number of odd divisors of $\gcd(p-1, m)$ in case $(p^2-1)/m$ is odd, and equal to the number of divisors of $\gcd(p-1, m)$ in case $(p^2-1)/m$ is even. In case $p = 1$, $N_A(m, p)$ is equal to the number of divisors of m*†.

Proof. We first consider the case $p \neq 1$. Both q and $(p+1)$ are multiples of $m/\gcd(p-1, m)$. Hence we may write

$$q = a \cdot m/\gcd(p-1, m) \quad (p+1) = b \cdot m/\gcd(p-1, m) \tag{2.48}$$

where $a \in \{1, \ldots, \gcd(p-1, m)\}$ and

$$b := \gcd\left(\frac{p^2-1}{m}, (p+1)\right) \tag{2.49}$$

as is easily verified from (2.13).

We now have to distinguish the two cases b is odd and b is even.

In the case where b is odd, we find from substituting (2.48) into the definition of A, that

$$A = 2\frac{\gcd(p-1, m)}{\gcd(\gcd(p-1, m), 2a-b)}. \tag{2.50}$$

Considering all $a \in \{1, \ldots, \gcd(p-1, m)\}$ we hence find as many different values for A, as there are odd divisors of $\gcd(p-1, m)$.

In the case where b is even, we find‡

$$A = \frac{\gcd(p-1, m)}{\gcd(\gcd(p-1, m), a-b/2)}. \tag{2.51}$$

Considering all $a \in \{1, \ldots, \gcd(p-1, m)\}$ we find that A may be any divisor of $\gcd(p-1, m)$.

The proof for the case $p \neq 1$ concludes by verification of the fact that b is odd if and only if $(p^2-1)/m$ is odd.

If $p = 1$, $A = 1$ if $q = 1$. Otherwise, A is given by

$$A = \frac{m}{\gcd(m, q-1)}. \tag{2.52}$$

Since in (2.52) $q \in \{2, \ldots, m\}$, we find that A may be any divisor of m. □

Note in the above proof that in case b is odd, A is even and it follows from proposition 2.6 that A and k are one-to-one. Only in case b is even it may happen that different values of A give rise to the same value of k, and sometimes even to completely identical ϕ_λ-orbit lengths.

† The number of divisors of the integer n, $d(n)$ can be found as follows: write n as a product of primes $n = \prod_{i=1}^{l} p_i^{\alpha_i}$ ($\alpha_i \in \mathbb{N}$), then $d(n) = \prod_{i=1}^{l}(\alpha_i + 1)$. Writing $n = 2^{\alpha_1}\tilde{n}$ with $\tilde{n}$ odd, one finds the number of odd divisors of n to be $d(\tilde{n})$. See also [8].

‡ If $a = b/2$, read $a - b/2 := m$.

3. Constructions of examples

In this section we will discuss the construction of faithful nonlinear representations of $\langle q;\, p, m\rangle_\lambda$, i.e. the construction of nontrivial dynamical systems (diffeomorphisms) possessing a cyclic reversing k-symmetry group. The main purpose of this section will be to illustrate the occurrence of cyclic reversing k-symmetry groups in explicit maps. In this respect it should be noted that nontrivial dynamical systems with cyclic (reversing) symmetry groups seem to occur more easily in lower dimensional phase spaces than nontrivial dynamical systems with cyclic (reversing) k-symmetry groups with $k > 1$. This is related to the minimal dimension needed for a faithful (nonlinear) representation of $\langle q;\, p, m\rangle_\lambda$. For instance, we manage to implement all cyclic reversing k-symmetry groups on the torus T^2, but not on the circle S^1, the cylinder $S^1 \times \mathbb{R}$, or the plane $\mathbb{R}^2$.

To obtain representations we will make use of the decomposition property obtained in theorem 1.1. This leaves us with the task to find two maps U and S that satisfy (1.13)–(1.17).

Before we present explicit representations we would like to point out a general method that we use in the constructions. We construct representations that are faithful and nonlinear (and dynamically interesting) from representations that are not. Suppose we have S and U satisfying (1.13)–(1.17). Then since only S^2 appears in (1.14), (1.15), a new representation is obtained from an old representation when we conjugate U by an invertible map C,

$$U \mapsto C \circ U \circ C^{-1}, \tag{3.1}$$

such that C commutes with S^2 (but preferably not with S). This method is analogous to the one used in [24] to construct (nontrivial) maps possessing reversing symmetries†.

3.1. Constructions on the circle S^1

Let S be of order $2m$ and a translation on the unit circle S^1, e.g.

$$S\ :\ \theta' = \theta + \frac{1}{2m}. \tag{3.2}$$

To construct a representation with $p = 1$, (1.14) tells us we should find a map U that commutes with S^2 and satisfies $U^2 = S^{2q}$. A not very interesting solution for U is given by $U = S^q$. We now construct faithful representations, in case q is odd, by conjugating U by an invertible circle map, e.g. [2]

$$C\ :\ \theta' = \theta + \omega + \kappa \sin(2\pi m\theta) \qquad \left(|\kappa| \leqslant \frac{1}{2\pi m}\right). \tag{3.3}$$

In this way we obtain a nontrivial circle map $L = C \circ S^q \circ C^{-1} \circ S$ that possesses a reversing k-symmetry group with $k = m/\gcd(q-1, m)$‡.

In a similar way, we can construct a representation with $p = m - 1$ and $q = m$, using the same S and C, but taking $U(\theta) = \frac{1}{m} - \theta$ such that $U \circ S^2 = S^{-2} \circ U$ (cf. (1.15)).

† Another method consists of taking tensor products of representations. Equations (1.14)–(1.15) are invariant under $S \mapsto S \otimes \tilde{S}$, and $U \mapsto U \otimes \tilde{U}$, where $\tilde{S}$ and $\tilde{U}$ also satisfy (1.13)–(1.15), e.g. $\tilde{S}$ and $\tilde{U}$ may be taken to be arbitrary involutions. This method can for instance be used to construct faithful matrix representations of $\langle q;\, p, m\rangle_\lambda$ in $\mathbb{R}^6$. One may of course also combine the two methods.

‡ Note that maps with $p = 1$ and q odd can always be obtained from a dynamical system L_0 possessing a reversing (1-)symmetry S of order $2m$. In that case, we have $L_0 = U \circ S^{-1}$, $U^2 = S^2$, and (1.15) is automatically satisfied. Now, the map $L_n := U \circ S^{2n-1}$ has the property that $L_n \circ S \circ L_n = S^{2q-1}$ with $q = 2n + 1$. Every map L possessing a cyclic reversing k-symmetry groups with a ϕ_L-orbit structure with $p = 1$ and q odd can be obtained in this way.

3.2. Constructions on the two-torus T^2

Representations of the groups $\langle q; p, m\rangle_\lambda$ are naturally constructed on the two-torus T^2. Choosing coordinates $(\theta, \psi) \in T^2$ (thus considering both coordinates θ and ψ modulo 1), we satisfy the relations (1.14)-(1.15) by choosing S and U to be translations on the torus: $S(\theta, \psi) = (\theta + \frac{1}{2m}, \psi + \frac{p}{2m})$, and $U(\theta, \psi) = (\psi + \frac{2q-1}{2m}, \theta + \frac{1}{2m})$. This representation can be made faithful and dynamically interesting by conjugating U by an invertible transformation C that commutes with S^2. For instance we may take $C(\theta, \psi) = (\theta + f(\psi), \psi)$, with $f(\psi + 1/m) = f(\psi)$ and $f(\psi + 1/2m) \neq f(\psi)$. If p is odd, C and S do not commute, and we obtain the nontrivial map

$$L := C \circ U \circ C^{-1} \circ S^{-1} \ : \ \begin{cases} \theta' = \psi + \frac{D}{2m} + f(\psi') \\ \psi' = \theta + f(\psi) \end{cases} \qquad (3.4)$$

with as usual $D := 2q - p - 1$.

Taking $C(\theta, \psi) = (\theta, \psi + f(\theta))$, with the same conditions on f as before, we obtain faithful and dynamically nontrivial representations for all groups $\langle q; p, m\rangle_\lambda$, yielding a family of maps

$$L := C \circ U \circ C^{-1} \circ S^{-1} \ : \ \begin{cases} \theta' = \psi + \frac{D}{2m} - f(\psi - 1/2m) \\ \psi' = \theta + f(\theta') \end{cases} \qquad (3.5)$$

in which p and q (or D) can take any of the values allowed by (2.13)–(2.14).

3.3. Constructions on the cylinder $S^1 \times \mathbb{R}$

In this section we present some faithful nonlinear representations of groups $\langle q; p, m\rangle_\lambda$ on the cylinder $S^1 \times \mathbb{R}$. We will confine ourselves to the cases $p = 1$, and $p = m - 1$ and $q = m$.

In [13, appendix A], a family of maps on the cylinder $S^1 \times \mathbb{R}$ possessing reversing symmetries was presented with which one can construct examples with $p = 1$ and q is odd, along the lines of the second footnote on p 1019. Here, however, we will describe slightly different constructions based on the strategy set out in [13, appendix A]. We use coordinates $(\theta, z) \in S^1 \times \mathbb{R}$ (thus we consider θ modulo 1). Our constructions use two maps: an involution $U(\theta, z) = (\theta, -z+g(\theta))$, and a map S of order $2m$, $S(\theta, z) = (\theta + \frac{1}{2m} + \omega(z), -z)$ with $\omega(z) = -\omega(-z)$. S^2 is a translation over $1/m$ in the θ-direction. We intend to construct examples with $p = 1$, having $\langle S\rangle$ as a cyclic reversing k-symmetry group. To this end, in order to satisfy (1.15) we take $g(\theta)$ to be $1/m$-periodic (i.e. $g(\theta) = g(\theta + 1/m)$), and we construct the family of maps

$$L_n := U \circ S^{2n-1} \ : \ \begin{cases} \theta' &= \theta + \frac{2n-1}{2m} + \omega(z), \\ z' &= z + g(\theta'). \end{cases} \qquad (3.6)$$

In the case where m is odd, S is a reversing 1-symmetry of L_n precisely if $2n - 1 = m$. In general, the map L_n has $q = 2n$. Note that these examples differ from the ones which may be constructed along the lines of the second footnote on p 1019. In the latter case q is restricted to be odd, whereas in the present examples there is no such restriction.

In a similar fashion as with the circle maps, we may use the above framework to construct a representation with $p = m - 1$ and $q = m$. We only have to replace U by $\tilde{U}(\theta, z) = (\frac{1}{m} - \theta, z + g(\theta))$, with $g(\theta) = -g(-\theta)$.

3.4. Constructions on the plane $\mathbb{R}^2$

In this section we construct a family of faithful nonlinear representations of groups $\langle q;1,m\rangle_\lambda$ on the plane $\mathbb{R}^2$. We also explain the construction of the map (1.20) presented in section 1.

In the spirit of the second footnote on p 1019 we construct a family of maps on the plane that have a cyclic reversing symmetry group of order $2m$. If m is odd, one such a nontrivial map suffices to construct nontrivial maps L with ϕ_L orbit structures that are characterized by $p=1$ and any value of q. In case m is even, in this way one can only construct maps with ϕ_L orbit structures having $p=1$ and q odd.

To construct a map possessing a reversing symmetry of order $2m$ we need to find two maps U and S that are both of order $2m$, and that satisfy $S^2=U^2$, cf (1.19).

We use polar coordinates (θ,ρ) on the plane. We take S to be a $2m$-fold rotation around the origin: $S(\theta,\rho)=(\theta+\frac{\pi}{m},\rho)$. We take U to be conjugate to S, i.e. $U:=C\circ S\circ C^{-1}$ with $C(\theta,\rho)=(g(\theta,\rho),\rho f(\theta))$, where $f(\theta)=f(\theta+2\pi/m)>0$, $g(\theta,\rho)+2\pi/m=g(\theta+2\pi/m,\rho)$, and g should be such that C is invertible. For instance we may take

$$f(\theta)=1+a\sin(m\theta),\quad (|\kappa|<1) \tag{3.7}$$
$$g(\theta,\rho)=\theta+b\cos(m\theta+\rho^2),\quad (|b|<1/m). \tag{3.8}$$

The maps $L_n:=C\circ S\circ C^{-1}\circ S^{2n-1}$ now have $\langle S\rangle$ as a reversing k-symmetry group of order $2m$, with $q=2n+1$, and $\langle S^2\rangle$ as a symmetry group (i.e. $p=1$).

As a last example, we discuss the construction of a representation of the specific group $\langle 2;1,2\rangle_\lambda$, with $k=2$, that was presented in section 1. From theorem 1.1 it follows that for a representation of $\langle 2;1,2\rangle_\lambda$ we need a map of order four, and an involution.

The map of order four will be taken to be a rotation over $\pi/2$ around the origin $R_{\pi/2}(x,y)=(y,-x)$ and for the involution we will take $U(x,y)=(y+f(x+y),x-f(x+y))$. Moreover (1.15) imposes an additional condition on f, namely that $f(z)=-f(-z)$.

In this way we have constructed a faithful representation of $\langle 2;1,2\rangle_\lambda$. The map $L:=U\circ R_{\pi/2}^{-1}$ is area-preserving.

An example with dissipative dynamical behaviour is obtained by conjugating U with a transformation C (such that C commutes with $R_{\pi/2}^2$). Taking $C(x,y)=(x(\varepsilon y^2+1),y)$ we obtain an involution $\tilde{U}:=C\circ U\circ C^{-1}$, and the map $L:=\tilde{U}\circ R_{\pi/2}^{-1}$, as presented in (1.20).

Appendix A. ϕ_L-orbit structures

In this appendix we present table 1, containing all possible ϕ_L-orbit structures of cyclic reversing k-symmetry groups of order $2m$ with $m=1,\ldots,25$ (labelled by the integers p and A, with reference to theorem 1.2) and the lengths of the reversing and non-reversing ϕ_L-orbits in the cyclic group. Note that according to theorem 1.3 the non-reversing ϕ_L-orbits are always of length 1 or 2, and the reversing ϕ_L-orbits are always of length $k/2$ or k.

Table 1. All inequivalent ϕ_L-orbit structures occurring in cyclic reversing k-symmetry groups of order $2m$, for $m = 1, \ldots, 25$, with $k > 1$. The ϕ_L orbit structures are uniquely represented by the integers m, p, and A (cf theorem 1.2). For each ϕ_L-orbit structure, the number of non-reversing ϕ_L-orbits with length 1 and 2 are indicated by s_1, resp. s_2. The number of reversing ϕ_L-orbits with length $k/2$ and k are indicated by $r_{k/2}$, resp. r_k.

m	p	A	k	s_1	s_2	$r_{k/2}$	r_k
2	1	2	2	2	0	0	1
3	1	2	3	3	0	0	1
3	2	2	2	1	1	1	1
4	1	2	2	4	0	0	2
4	1	4	4	4	0	0	1
4	3	1	2	2	1	0	2
4	3	2	2	2	1	0	2
5	1	5	5	5	0	0	1
5	4	2	2	1	2	1	2
6	1	2	2	6	0	0	3
6	1	3	3	6	0	0	2
6	1	6	6	6	0	0	1
6	5	1	2	2	2	2	2
6	5	2	2	2	2	0	3
7	1	7	7	7	0	0	1
7	6	2	2	1	3	1	3
8	1	2	2	8	0	0	4
8	1	4	4	8	0	0	2
8	1	8	8	8	0	0	1
8	3	4	4	2	3	0	2
8	5	8	8	4	2	0	1
8	7	1	2	2	3	0	4
8	7	2	2	2	3	0	4
9	1	3	3	9	0	0	3
9	1	9	9	9	0	0	1
9	8	2	2	1	4	1	4
10	1	2	2	10	0	0	5
10	1	5	5	10	0	0	2
10	1	10	10	10	0	0	1
10	9	1	2	2	4	2	4
10	9	2	2	2	4	0	5
11	1	11	11	11	0	0	1
11	10	2	2	1	5	0	5
12	1	2	2	12	0	0	6
12	1	3	3	12	0	0	4
12	1	4	4	12	0	0	3
12	1	6	6	12	0	0	2
12	1	12	12	12	0	0	1
12	5	1	2	4	4	0	6
12	5	2	2	4	4	4	4
12	5	4	4	4	4	0	3
12	7	1	2	6	3	0	6
12	7	2	2	6	3	0	6
12	7	3	6	6	3	0	2
12	7	6	6	6	3	0	2
12	11	1	2	2	5	0	6
12	11	2	2	2	5	0	6
13	1	13	13	13	0	0	1
13	12	2	2	1	6	1	6
14	1	2	2	14	0	0	7
14	1	7	7	14	0	0	2
14	1	14	14	14	0	0	1
14	13	1	2	2	6	2	6
14	13	2	2	2	6	0	7
15	1	3	3	15		0	5
15	1	5	5	15	0	0	3
15	1	15	15	15	0	0	1
15	4	2	2	3	6	3	6
15	4	6	6	3	6	1	2
15	11	1	2	5	5	5	5
15	11	5	10	5	5	1	1
15	14	2	2	1	7	1	7
16	1	2	2	16	0	0	8
16	1	4	4	16	0	0	4
16	1	16	16	16	0	0	1
16	7	4	4	2	7	0	4
16	9	16	16	8	4	0	1
16	15	1	2	2	7	0	8
16	15	2	2	2	7	0	8
17	1	17	17	17	0	0	1
17	16	2	2	1	8	1	8
18	1	2	2	18	0	0	9
18	1	3	3	18	0	0	6
18	1	6	6	18	0	0	3
18	1	9	9	18	0	0	2
18	1	18	18	18	0	0	1
18	17	1	2	2	8	2	8
18	17	2	2	2	8	0	9
19	1	19	19	19	0	0	1
19	18	2	2	1	9	1	9

m	p	A	k	s_1	s_2	$r_{k/2}$	r_k
20	1	2	2	20	0	0	10
20	1	4	4	20	0	0	5
20	1	5	5	20	0	0	4
20	1	10	10	20	0	0	2
20	1	20	20	20	0	0	1
20	9	1	2	4	8	4	8
20	9	2	2	4	8	0	10
20	9	4	4	4	8	0	5
20	11	1	2	10	5	0	10
20	11	2	2	10	5	0	10
20	11	5	10	10	5	0	2
20	11	10	10	10	5	0	2
20	19	1	2	2	9	0	10
20	19	2	2	2	9	0	10
21	1	3	3	21	0	0	7
21	1	7	7	21	0	0	3
21	1	21	21	21	0	0	1
22	21	1	2	2	10	2	10
21	8	2	2	7	7	7	7
21	8	14	14	7	7	1	1
21	13	1	2	3	9	3	9
21	13	3	6	3	9	1	3
21	20	2	2	1	10	1	10
22	1	2	2	22	0	0	11
22	1	11	11	22	0	0	2
22	1	22	22	22	0	0	1
22	21	2	2	2	10	0	11
23	1	23	23	23	0	0	1
23	22	2	2	1	11	1	11
24	1	2	2	24	0	0	12
24	1	3	3	24	0	0	8
24	1	4	4	24	0	0	6
24	1	6	6	24	0	0	4
24	1	8	8	24	0	0	3
24	1	12	12	24	0	0	2
24	1	24	24	24	0	0	1
24	5	8	8	4	10	0	3
24	7	1	2	6	9	0	12
24	7	2	2	6	9	0	12
24	7	3	6	6	9	0	4
24	7	6	6	6	9	0	4
24	11	4	4	2	11	0	6
24	13	8	8	12	6	0	3
24	13	24	24	12	6	0	1
24	17	1	2	8	8	8	8
24	17	2	2	8	8	0	12
24	17	4	4	8	8	0	6
24	17	8	8	8	8	0	3
24	19	4	4	6	9	0	6
24	19	12	12	6	9	0	2
24	23	1	2	2	11	0	12
24	23	2	2	2	11	0	12
25	1	5	5	25	0	0	5
25	1	25	25	25	0	0	1
25	24	2	2	1	12	1	12

Appendix B. Proof of group property

In this appendix we show that the relations (2.10)–(2.12) given in proposition 2.4 define a group $\langle q\ ;\ p, m\rangle_\lambda$ generated by σ of order $2m$ (leaving the order of λ free, i.e. infinite in general) if and only if (2.13) and (2.14) are satisfied. First we will show that (2.13) and (2.14) are necessary conditions. Subsequently we will show that (2.13) and (2.14) are also sufficient conditions.

From (2.10) and (2.11) it follows that

$$\sigma^{2q-1}\lambda^{-1} = \lambda\sigma \tag{B.1}$$

$$\sigma^{2(p-q)+1}\lambda = \lambda^{-1}\sigma \tag{B.2}$$

and hence

$$\sigma^{2p}\lambda = \lambda\sigma^2. \tag{B.3}$$

Combining (2.10) with (B.3) we obtain

$$\sigma^2\lambda^2 = \lambda^2\sigma^2. \tag{B.4}$$

But from (2.10) alone we find

$$\sigma^2\lambda^2 = \lambda\sigma^{2p}\lambda = \lambda^2\sigma^{2p^2}. \tag{B.5}$$

This is consistent with (B.4) if and only if

$$p^2 = 1 \bmod m \tag{B.6}$$

i.e. precisely (2.13).

Combining (2.10) with (B.3) we find

$$\sigma\lambda^{-1} = \lambda\sigma^{2(1-q)p+1} \tag{B.7}$$

$$\Rightarrow \quad \sigma^2\lambda^{-1} = \lambda^{-1}\sigma^{2(1-q)p+2q}. \tag{B.8}$$

To be consistent with (B.3) we must therefore require

$$2(1-q)p + 2q = 2p \bmod 2m \tag{B.9}$$

$$\Leftrightarrow \quad pq \quad = q \bmod m \tag{B.10}$$

i.e. precisely (2.14).

After showing that (2.13) and (2.14) are necessary conditions for the relations (2.10)–(2.12) to define a group, such that the order of σ is $2m$, we will now show that these conditions are also sufficient.

To show that a set with a binary operation is a group, one must prove:

(i) There is a unique unit element e.
(ii) Any element has an inverse.
(iii) The group operation is associative.

In order to prove these properties, we use the fact that

$$\sigma^x\lambda^y = \lambda^{g(x,y)}\sigma^{f(x,y)} \tag{B.11}$$

where $f(x, y) = \varepsilon_{y+1}x + \varepsilon_y xp + \varepsilon_x yD$, and $g(x, y) = (-)^x y$, with $D := 2q - p - 1 \bmod 2m$ and $\varepsilon_z := 0$ if z is even and $\varepsilon_z := 1$ is z is odd, cf. (2.16)–(2.19).

(i) The unit element of the group is $\lambda^0\sigma^0$. This unit element is unique since $f(x, y) = 0 \bmod n$ and $q(x, y) = 0$ if and only if $y = 0$ and $x = 0 \bmod m$.

(ii) The inverse of $\lambda^y\sigma^x$ is given by

$$(\lambda^y\sigma^x)^{-1} = \sigma^{-x}\lambda^{-y} = \lambda^{g(-x,-y)}\sigma^{f(-x,-y)}. \tag{B.12}$$

The uniqueness of the unit element provides uniqueness of the inverses.

(iii) In order to establish associativity we must prove

$$\left(\lambda^y\sigma^x\lambda^{y'}\sigma^{x'}\right)\lambda^{y''}\sigma^{x''} = \lambda^y\sigma^x\left(\lambda^{y'}\sigma^{x'}\lambda^{y''}\sigma^{x''}\right). \tag{B.13}$$

In terms of the functions $f(x, y)$ and $g(x, y)$ this condition becomes

$$f(f(x, y') + x', y'') = f(x', y'') + f(x, y' + g(x', y'')) \bmod m \tag{B.14}$$

$$g(x, y' + g(x', y'')) = g(x, y') + g(x' + f(x, y'), y''). \tag{B.15}$$

In order to prove that (B.14) and (B.15) are satisfied, we first derive the following relations from (2.18) and (2.19),

$$(-1)^{f(x,y)} = (-1)^x \tag{B.16}$$

$$(-1)^{g(x,y)} = (-1)^y. \tag{B.17}$$

With the relations (B.16), (B.17) and (2.20), and noting that $\varepsilon_z = (1 - (-1)^z)/2$, it is not difficult to check that the expressions (2.18) and (2.19) for $f(x, y)$ and $g(x, y)$ indeed satisfy (B.14) and (B.15), completing the proof of associativity.

Appendix C. Cyclic (reversing) k-symmetry groups of infinite order

In this appendix we discuss cyclic (reversing) k-symmetry groups of infinite order. Because these groups have an infinite number of elements, it is important to note that we speak of (reversing) k-symmetries only if $k < \infty$.

Let us first consider the case that a map L possesses a cyclic k-symmetry group of infinite order, i.e. $\langle M\rangle \simeq \mathbb{Z}_\infty$. In that case we have $M \circ L = L \circ M^p$, and the value of $k := \#_L(M)$ is the smallest positive integer satisfying $p^k = 1$. So, we find finite values of k if and only if $p = \pm 1$. The case $p = 1$ corresponds to the trivial case that $\langle M\rangle$ is a symmetry group of L. In case $p = -1$, every element of the cyclic group is mapped to its inverse by ϕ_L, and $k = 2$.

In case L possesses a cyclic *reversing* k-symmetry group of infinite order, $\langle S\rangle \simeq \mathbb{Z}_\infty$ we find in a similar way to the above discussion that $S^2 \circ L = L \circ S^{2p}$ implies $p = \pm 1$. Let furthermore $S \circ L = L^{-1} \circ S^{2q-1}$, then from (B.8) we find that $pq = q$, implying that $q = 0$ if $p \neq 1$. In case $p = 1$ we have $\phi_L(S^{2x+1}) = S^{2(x+q-1)+1}$. Hence, we find a finite value of k if and only if $q = 1$, leading to $k = 1$ (i.e. a reversing symmetry group).

There remains only one type of cyclic reversing k-symmetry group of infinite order with $k > 1$, having $p = -1$ and $q = 0$ (this is comparable to $p = m - 1$, $q = m$ in case S is of finite order $2m$). It has $k = 2$, and ϕ_L maps all elements of the cyclic group to their inverse.

A map L that possesses a cyclic reversing 2-symmetry group of infinite order can be decomposed as follows:

$$L = U \circ S^{-1} \tag{C.1}$$

where

$$U^2 = Id \tag{C.2}$$

and

$$U \circ S^2 = S^{-2} \circ U. \tag{C.3}$$

The derivation of this decomposition property is analogous to that of theorem 1.1. The relations (C.2) and (C.3) imply that S is 2-reversible with respect to U.

As an example in $\mathbb{R}^2$, consider the map $S(x, y) := (x + 1 + \omega(y), -y)$, with $\omega(y) = -\omega(-y)$. It is easily checked that $S^2(x, y) = (x + 2, y)$, and that $-Id$ is a reversing symmetry of S, i.e. $(-Id) \circ S \circ (-Id) = S^{-1}$. However, to construct an explicit map, we need an involutory U that is a reversing 2-symmetry of S. From the reversing symmetry $-Id$ we may obtain a reversing 2-symmetry U by composing $-Id$ with a map C that is a 2-symmetry of S, i.e. $U := C \circ (-Id)$. For instance, we may take $C(x, y) := (x, y + \kappa\cos(\pi x))$. With this choice of C we obtain a reversing 2-symmetry U of S that is an involution.

From the decomposition property discussed above we construct the map

$$L := U \circ S^{-1} \; : \; \begin{cases} x' = -x + 1 - \omega(y) \\ y' = y + \kappa\cos(\pi x') \end{cases} \tag{C.4}$$

having $\langle S\rangle$ as a cyclic reversing 2-symmetry group of infinite order.

Acknowledgements

It is a great pleasure to acknowledge various interesting discussions with H de Vries. We also thank M Baake for useful comments in the early stages of this work. JSWL is grateful to the Department of Mathematics of La Trobe University for support and hospitality. GRWQ is grateful to the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO) for support during the early stages of this work. This investigation is part of the research program of the Australian Research Council.

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