Nonlinearity **8** (1995) 443–459. Printed in the UK

Factorizable Lie symmetries and the linearization of difference equations

G B Byrnes, R Sahadevan† and G R W Quispel‡

Department of Mathematics, La Trobe University, Bundoora 3083 Victoria, Australia

Received 8 November 1994

Recommended by P Constantin

Abstract. We show that an autonomous difference equation, of arbitrary order and with one or more independent variables, can be linearized by a point transformation if and only if it admits a symmetry vector field whose coefficient function is the product of two functions, one of the dependent variable u and one of the independent variables x: $X(x, u) = A(x)G(u)\frac{\partial}{\partial u}$. The factor depending on the independent variables, A, is required to satisfy some non-degeneracy conditions. This result is derived using a discrete jet space formalism for partial and ordinary difference equations, analogous to that used for the study of differential equations.

AMS classification scheme numbers: 39A12, 58F37

1. Introduction and examples

With the ever increasing ubiquity of computer-based simulations in mathematics, engineering and physics, the study of difference equations and other discrete models has assumed greater importance. Primarily this is due to the fact that computers can typically only solve discrete equations, but there are also fields of study where discrete structures are inherent, for example lattices in physics and cell structures in biology. On the other hand, since the time of Lie we have built up a very deep understanding of differential systems and their properties. It is therefore natural to attempt to transfer some of that understanding to analogous areas in the discrete theory. For example, integrable partial differential equations have been generalized to integrable partial difference equations [1, 11], integrable ordinary difference equations have been generalised to integrable mappings [2, 4, 12, 13, 14] and attempts have been made to find discrete analogues of the Painlevé equations and the Painlevé property [10, 15]. In particular, Lie's theory of continuous symmetries of differential equations has been shown to be applicable to difference equations [16].

This article summarises a further investigation of the rôle of Lie symmetries of difference equations. The results presented are rather surprising when held alongside the familiar differential theory. It is well known that if a system of n first-order ordinary differential equations (first-order ODE's) admit a k-parameter, solvable Lie group of symmetries, then the original system can be reduced to $n - k$ ODE's and k quadratures. A partial analogue of this result, restricted to autonomous symmetries, was given by Maeda [8] in

† Present address: Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chepauk, Madras-600 005, Tamilnadu, India.

‡ Email address: MATGRQ@lure.latrobe.edu.au

0951-7715/95/030443+17$19.50 © 1995 IOP Publishing Ltd and LMS Publishing Ltd

1986. Essentially the existence of such symmetries implies the possibility of a coordinate transformation in which some of the equations become linear with constant coefficients. In the case of a single first order ordinary difference equation (OΔE) and a single autonomous symmetry, Maeda gives the linearizing transformation explicitly, but matters appear to be far more complicated for higher order OΔE's or systems of several first order OΔE's. In any case, to linearise a single OΔE of order N one would expect to require N independent symmetries. Yet consider the second-order OΔE

$$u(x+2) = \frac{2u(x+1) + u(x)[u^2(x+1) - 1]}{1 - u^2(x+1) + 2u(x)u(x+1)} \quad \forall x \in \mathbf{R}. \tag{1}$$

The vector field

$$X(x,u) := x(1+u^2)\frac{\partial}{\partial u}$$

can be shown to be a symmetry vector field of (1). Note that X has the special form

$$X(x,u) = A(x)G(u)\frac{\partial}{\partial u}$$

in this case with $A(x) = x$, $G(u) = 1 + u^2$.

Suppose then that we introduce the new dependent variable w defined by

$$w(u) := \int^u \frac{d\zeta}{G(\zeta)}$$

so for equation (1)

$$w(u) := \arctan u.$$

Substituting this into (1) yields the (locally) transformed OΔE

$$w(x+2) - 2w(x+1) + w(x) = k\pi. \tag{2}$$

In fact for this example the linearizing transformation can be made global if w is taken to be a coordinate on the cylinder constructed as a quotient of the $\mathbf{C}$ by the relation $z \sim z + k\pi$, $k \in \mathbf{Z}$. The effect of this is that w is only defined $\bmod \pi$. The general solution of (2) is just

$$w(x) = a(x)x + b(x) + \frac{k}{2}\pi x^2 \bmod \pi \quad x \in \mathbf{R}$$

where a, b are unit periodic functions, that is $a(x+1) = a(x)$ and $b(x+1) = b(x)$ $\forall x \in \mathbf{R}$. It should be noted that the function $A(x) = x$ which is a factor of the symmetry vector field is a solution of the homogeneous part of the linearised equation (2). Moreover, observe that A is not the solution of any first-order, homogeneous, linear, constant coefficient OΔE: the significance of this will become clear with the statement of the general result.

Thus the original equation (1) has general solution

$$u(x) = \tan\left(\frac{k}{2}\pi x^2 + a(x)x + b(x)\right) \qquad x \in \mathbf{R}.$$

Given a single symmetry for a second-order equation we expect only a reduction of order by one, to a first order OΔE [3], whereas here we have completely solved the problem by reducing it to a linear, constant coefficient equation that can be solved by elementary means [7].

To see how this works before attempting to prove the most general result, consider an arbitrary autonomous second-order OΔE

$$u_{(2)} = F(u_{(1)}, u) \tag{3}$$

where we have introduced the notation

$$u_{(j)}(x) := u(x+j) \qquad x \in \mathbf{R}.$$

Let X be a vector field with factorizable coefficient function as before,

$$X(x,u) := A(x)G(u)\frac{\partial}{\partial u}.$$

The appropriate prolongation of X (see section 2) is

$$X^{(Q)}(x,u) = A(x)G(u)\frac{\partial}{\partial u} + A(x+1)G(u_{(1)})\frac{\partial}{\partial u_{(1)}} + A(x+2)G(u_{(2)})\frac{\partial}{\partial u_{(2)}}$$

so X is a symmetry vector field of (3) iff

$$X^{(Q)}(u_{(2)} - F([u]))|_{u_{(2)}=F} = 0$$

or in full with $F([u])$ substituted for $u_{(2)}$,

$$A(x+2)G \circ F([u]) = A(x+1)G(u_{(1)})\frac{\partial F}{\partial u_{(1)}}([u]) + A(x)G(u)\frac{\partial F}{\partial u}([u]). \quad (4)$$

Provided that $A \neq 0$ we can separate variables:

$$G(u)\frac{\partial F}{\partial u}([u]) = -\frac{A(x+1)}{A(x)}G(u_{(1)})\frac{\partial F}{\partial u_{(1)}}([u]) + \frac{A(x+2)}{A(x)}G \circ F([u]) \quad (5)$$

and differentiating with respect to x yields

$$\left(\frac{A_{(1)}}{A}\right)' G(u_{(1)})\frac{\partial F}{\partial u_{(1)}}([u]) = \left(\frac{A_{(2)}}{A}\right)' G \circ F([u])$$

or (assuming that A is not unit periodic)

$$\frac{(A_{(2)}/A)'}{(A_{(1)}/A)'} = \frac{G(u_{(1)})}{G \circ F([u])}\frac{\partial F}{\partial u_{(1)}}([u]) = \Gamma_1$$

where Γ_1 is a separation constant.

Therefore

$$\frac{\partial F/\partial u_{(1)}}{G(F)} = \frac{\Gamma_1}{G(u_{(1)}}$$

so integrating with respect to $u_{(1)}$ we obtain

$$\int^{F([u])} \frac{\mathrm{d}\zeta}{G(\zeta)} = \Gamma_1 \int^{u_{(1)}} \frac{\mathrm{d}\zeta}{G(\zeta)} + g(u).$$

Substituting $\Gamma_1 G \circ F([u])$ for $G(u_{(1)})\frac{\partial F}{\partial u_{(1)}}([u])$ in equation (5) and separating, we obtain by the same procedure

$$\int^{F([u])} \frac{\mathrm{d}\zeta}{G(\zeta)} = \Gamma_2 \int^{u} \frac{\mathrm{d}\zeta}{G(\zeta)} + h(u_{(1)})$$

so we have

$$\int^{F([u])} \frac{\mathrm{d}\zeta}{G(\zeta)} = \Gamma_1 \int^{u_{(1)}} \frac{\mathrm{d}\zeta}{G(\zeta)} + \Gamma_2 \int^{u} \frac{\mathrm{d}\zeta}{G(\zeta)} + \Gamma_3.$$

Making the coordinate transformation

$$w(u) := \int^{u} \frac{\mathrm{d}\zeta}{G(\zeta)}$$

as in the example, we have

$$w(u_{(2)}) = \Gamma_1 w(u_{(1)}) + \Gamma_2 w(u) + \Gamma_3 \tag{6}$$

or on restricting to solutions and substituting back for $u_{(2)}$,

$$w \circ F([u]) = \Gamma_1 w(u_{(1)}) + \Gamma_2 w(u) + \Gamma_3.$$

Note that this is a second-order constant coefficient linear O∆E, and can therefore be solved by elementary methods. It is important to note, however, that it is a linearization of the invariance condition (4) rather than of the original O∆E (3).

However if we assume *a priori* that X is a symmetry vector field, then (4) (and hence (6)) holds whenever (3) is satisfied. In section 4 it will be shown that the converse is also true, so (6) is in fact a linear, constant coefficient O∆E equivalent to (3).

From the separation procedure we also have

$$(A_{(2)}/A)' = \Gamma_1 (A_{(1)}/A)'$$

so integration and comparison with the dependent variable part shows that A will in fact be a solution of the homogeneous part of the linearised equation (6).

Several questions now arise. First, how general is this result? We will show in section 4 that it holds for autonomous O∆E's *and* P∆E's of arbitrary order. It also extends to difference equations with incommensurate spans, as will be shown in section 5.

Second, how important is the condition that the factor A not be unit-periodic and how does this condition generalize to higher-order O∆E's and P∆E's? It will be seen that the appropriate condition is that A must not be the solution of a linear difference equation with order strictly less (in any variable) than the equation we seek to linearize. This will be used in both of the main theorems of sections 3 and 4.

Finally, is there a converse: do all linearizable P∆E's admit a factorizable symmetry vector field satisfying appropriate non-degeneracy conditions? Again the answer is affirmative, as is shown in section 3.

Before attempting to prove these results, we need to introduce some machinery analogous to the jet spaces widely used in the study of differential equations. The analogous structure for discrete systems will be called a discrete jet space and is defined in section 2, along with some other notation and conventions.

A brief announcement of the results contained in this paper can be found in [17].

2. Discrete jet spaces

In this section we define the notion of a discrete jet space, in analogy to the jet spaces (locally equivalent to jet bundles) familiar in the study of differential equations [9, 18]. The discrete jet space is a differentiable manifold on which one can define vector fields, giving a more coherent structure to prolongation and coordinate transformation. The more general concept of a discrete jet *bundle* could be defined with no extra effort, by allowing the independent variables to be local coordinate functions on an n-dimensional manifold M, then defining E to be a line-bundle over M. Since the extra generality will easily be adduced by those familiar with such techniques, while the uninitiated would likely find the technology of fibre bundles to be wilfully obscure, we will forgo such indulgence.

We are interested in the case of a single partial difference equation (P∆E) with unit span, or as a special case an ordinary difference equation (O∆E). Also, it will be useful to work in the field of meromorphic functions to avoid problems of division by zero. Thus the dependent variable will be a coordinate on $\mathbf{C}$, the independent variables coordinates on $\mathbf{C}^n$,

$n \geqslant 1$. For convenience the product $\mathbf{C}^n \times \mathbf{C}$ will be denoted by E. The notation $\mathcal{M}(U)$, with $U \subset \mathbf{C}^n$, will stand for the field of functions meromorphic on U. The ring of analytic functions on U will be denoted $\mathcal{A}(U)$.

If $f \in \mathcal{M}(U)$, then the sub-set of U on which it vanishes is called the *zero-set* of f and is denoted by $Z(f)$:

$$Z(f) := \{x \in U \mid f(x) = 0\}.$$

Let $\mathbf{N}^* := \mathbf{N} \cup \{0\}$, that is the natural numbers and zero. The n-tuple $\alpha \in (\mathbf{N}^*)^n$, with components $(\alpha^1, \ldots, \alpha^n)$, will be used as a composite index. If $u \in \mathcal{M}(\mathbf{C}^n)$, then we define

$$u_{(\alpha)}(x) := u(x + \alpha) \qquad \forall x \in \mathbf{C}^n$$

with the addition defined by the obvious embedding $(\mathbf{N}^*)^n \hookrightarrow \mathbf{C}^n$. Written out in components this becomes

$$u_{(\alpha)}(x^1, \ldots, x^n) := u(x^1 + \alpha^1, \ldots, x^n + \alpha^n) \quad \forall (x^1, \ldots, x^n) \in \mathbf{C}^n.$$

A partial ordering $\preceq$ can be defined on $(\mathbf{N}^*)^n$ as follows. If $\alpha, \beta \in (\mathbf{N}^*)^n$, then we say

$$\alpha \preceq \beta \Leftrightarrow \exists \gamma \in (\mathbf{N}^*)^n \ \text{ s.t. } \alpha + \gamma = \beta.$$

It is clear that $\preceq$ is transitive and that $\forall \alpha \in (\mathbf{N}^*)^n$, $(0, \ldots, 0) \preceq \alpha$.

Definition 1. *Let Q be a finite subset of $(\mathbf{N}^*)^n$ with the property that if $\beta \in Q$ and $\alpha \preceq \beta$, then $\alpha \in Q$. We call such a set Q a prolongation set.*

A prolongation set Q can be used to define an equivalence relation on $\mathcal{M}(\mathbf{C}^n)$. Let $u, v \in \mathcal{M}(\mathbf{C}^n)$ and $x \in \mathbf{C}^n$. Then we say that u is equivalent to v at x (or $u \sim_x v$) iff

$$u_{(\alpha)}(x) = v_{(\alpha)}(x) \quad \forall \alpha \in Q.$$

The *discrete Q-jet space* is then defined to be the union of quotients

$$J^Q_\Delta(E) := \cup_{x \in \mathbf{C}^n} \mathcal{M}(\mathbf{C}^n)/\sim_x$$

where the subscript Δ distinguishes the discrete jet space from the continuous version. The element of $J^Q_\Delta(E)$ determined by $u \in \mathcal{M}(\mathbf{C}^n)$ (the equivalence class of u) is called the Q-jet of u and is denoted by $[u]_Q$, or when no confusion can arise simply $[u]$.

The equivalence classes $[u]_Q$ can be shown to be invariant under smooth changes of coordinates which are *fibre preserving* and *mesh preserving*, which is to say that they must be of the form

$$\begin{aligned} y &= \phi(x) \qquad && x, y \in \mathbf{C}^n \\ w &= \theta(x, u) \qquad && x \in \mathbf{C}^n,\ u, w \in \mathbf{C} \end{aligned}$$

(fibre preserving), and satisfy

$$y + \alpha = \phi(x) + \alpha = \phi(x + \alpha) \qquad \forall \alpha \in Q$$

(mesh preserving).

With the above definitions and a suitable choice of prolongation set Q, any partial difference equation (PΔE) can be defined as a submanifold Φ of $J^Q_\Delta(E)$. Given coordinates for $J^Q_\Delta(E)$, Φ can be realised as the zero set of some function

$$F : J^Q_\Delta(E) \longrightarrow \mathbf{C}$$

so that the PΔE becomes an algebraic equation in the coordinates of $J^Q_\Delta(E)$:

$$F(x, [u]_Q) = 0.$$

It will always be assumed that Q is chosen for a given PΔE so that the PΔE cannot be written on any sub-prolongation set $P \subset Q$.

In order to deal with various types of degeneracies, the following definitions are made.

Definition 2. *We say that $F : E \to \mathbf{C}$ has maximal rank if there exists a vector field X on E such that*

$$X(F)|_{F=0} \neq 0.$$

This definition is equivalent to that given by Olver [9] in the context of differential equations, when restricted to our case of a single equation. We assume that all the equations considered have maximal rank.

Definition 3. *If Q is a prolongation set and $|Q| = M$, then it is always possible to define a complete (in general non-unique) ordering α on Q, $Q = \{\alpha(0), \ldots, \alpha(M-1)\}$, which respects the partial ordering $\preceq$ defined earlier: if $j < k$, then $\alpha(k) \npreceq \alpha(j)$. Such an ordering will be called good.*

Note that for any good ordering on any prolongation set Q, $\alpha(0) = (0, \ldots, 0)$.

Definition 4. *Let Q be a prolongation set with good ordering α and $|Q| = M$. We say that a function $A \in \mathcal{M}(\mathbf{C}^n)$ is Q-periodic if there exist constants $a_0, \ldots, a_{M-1}$, not all zero, such that*

$$0 = \sum_{j=0}^{M-1} a_j A_{\alpha(j)}.$$

We say A is minimally Q-periodic if A is not periodic on any prolongation set $P \subset Q$.

We also need to define a class of functions whose role in the study of difference equations is analogous to that of constant functions for differential equations.

Definition 5. *A function $u : \mathbf{C}^n \to \mathbf{C}$ is said to be totally unit periodic on Q if*

$$u_{(\alpha)} = u \qquad \forall \alpha \in Q.$$

3. Factorizable symmetries

A projectable vector field (one that generates a fibre preserving flow)

$$X(x, u) = \xi^j(x)\frac{\partial}{\partial x^j} + \eta(x, u)\frac{\partial}{\partial u}$$

on E has a Q-prolongation $X^{(Q)}$ defined by

$$X^{(Q)}(x, u) = \xi^j(x)\frac{\partial}{\partial x^j} + \sum_{\alpha \in Q} \eta(x + \alpha, u_{(\alpha)})\frac{\partial}{\partial u_{(\alpha)}}.$$

It follows that X is a point symmetry v.f. of a PΔE $F(x, [u]_Q) = 0$ iff

$$X^{(Q)} F|_{F=0} = 0.$$

To prove the following theorem, we need a lemma on the properties of linear, constant coefficient PΔE's.

Lemma 1. *Given a homogeneous, linear, constant coefficient PΔE written on $J_\Delta^Q(E)$ in the form*

$$0 = F([u]) := \sum_{j=0}^{M-1} a_j u_{(\alpha(j))} \tag{7}$$

with α a good ordering on Q, there is a solution $A \in \mathcal{M}(C^n)$ which is minimally Q-periodic.

Proof. A characteristic polynomial, analogous to that for a linear differential equation, can be obtained for (7) by substituting the test solution

$$u(x) = \mu^x := \mu_1^{x^1} \cdots \mu_n^{x^n} \quad \forall x \in \mathbf{C}^n.$$

First consider the case $n = 1$, that is an OΔE. In this case the lemma requires that there is at least one solution of $F = 0$ which is not the solution of any OΔE of lower order.

Solutions to linear, homogeneous, constant coefficient OΔE's are found by the same elementary method used for linear constant coefficient ordinary differential equations: that is we substitute a trial solution of the form $u(x) = \mu^x$, $x \in \mathbf{C}$ to obtain a characteristic equation [7] which can be factorized as

$$\prod_{j=1}^{s} (\mu - r_j)^{m_j} = 0. \tag{8}$$

A set of eigenfunctions $\{f_1, \ldots, f_s\}$ can then be constructed, each of the form

$$f_j(x) = x^{m_j} r_j^x \qquad \forall x \in \mathbf{C}$$

with m_j being an integer one less than the multiplicity of the root r_j. It can then be checked that the solution

$$u := \sum_{j=1}^{s} f_j$$

can only be the solution of an OΔE whose characteristic equation contains the left-hand side of (8) as a factor, but this cannot be the case for an OΔE of lower order and the lemma is proved for $n = 1$.

Assume now that $n \geqslant 2$. For each n-tuple μ solving the characteristic equation

$$S(\mu) := \sum_{j=0}^{M-1} a_j \mu^{\alpha(j)} = 0$$

there is an independent solution of (7) given by

$$f_\mu(x) := \mu^x \quad \forall x \in \mathbf{C}^n.$$

Now let g be a solution constructed as a sum of eigen-solutions derived from the characteristic equation,

$$g := \sum_{k=1}^{K} f_{\mu(k)}.$$

Here $\mu(k) \in Z(S)$, $k = 1, \ldots, K$ are distinct points.

Suppose then that P is a sub-prolongation set of Q and $H : J_\Delta^P(E) \to \mathbf{C}$ such that $H([g]) = 0$. Let the characteristic polynomial for H be S'. From the fact that each $\mu(k)$ is a simultaneous eigenfunction of all the shift operators $u \to u_{(\alpha(j))}$, $H([g]) = 0$ implies that $H([f_{\mu(k)}]) = 0$, $k = 1, \ldots, K$. Consequently, $\mu(k) \in Z(S')$, $k = 1, \ldots, K$.

Since H is necessarily a polynomial with monomials drawn from a pre-determined (by the fact that $P \subset Q$) finite set, while $Z(S)$ contains co-dimension one submanifolds of $\mathbf{C}^n$, we can choose K in the definition of g to be sufficiently large to force $Z(S) \subseteq Z(S')$. However since the degree of any Weierstrass polynomial from S' is strictly smaller than the order of the Weierstrass polynomial of S in the same coordinates, $Z(S) \not\subset Z(S')$, so $Z(S) = Z(S')$. From the *Nullstellansatz* for germs of analytic sets [6], it follows that S and S' can differ by at most the multiplicity of their prime factors.

Suppose therefore that π is a prime factor which occurs with multiplicity m in S and consider the zero-set $Z(\pi)$. Choose a point $p \in Z(\pi)$ which is regular (i.e. p is not contained in the discriminant locus of $Z(\pi)$, so there is an open neighbourhood U of p such that $Z(\pi) \cap U$ is an $n-1$ dimensional complex sub-manifold of $\mathbf{C}^n$). At least one coordinate vector field $\frac{\partial}{\partial \mu_j}$ must satisfy $\frac{\partial}{\partial \mu_j}\pi|_p \neq 0$: without loss of generality assume $j = n$, so that π is *normalized* with respect to μ_n (see [19]).

As p is a regular point of $Z(\pi)$ and π is prime, there is an open neighbourhood U of p in $\mathbf{C}^n$ such that π can be written in the form

$$\pi(\mu) = \sigma(\mu)(\mu_n - \tau(\mu_1, \ldots, \mu_{n-1}))$$

on U, where σ is a nowhere-zero analytic function on U and τ is an analytic function on $\mathbf{C}^{n-1} \cap U$ such that $\tau(p_1, \ldots, p_{n-1}) = p_n$. If we set $\mu_j = p_j$, $j = 1, \ldots, n-1$ and leave μ_n free, then S is equivalent to the characteristic polynomial $\tilde{S}$ of an OΔE obtained by fixing $x_1, \ldots, x_{n-1}$:

$$\tilde{S}(\mu_n) = (\mu_n - p_n)^m.$$

It follows that the function $\tilde{f}_1$ defined by

$$\tilde{f}_1(x) := x_n^{m-1} p_1^{x_1} \cdots p_n^{x_n} \qquad \forall x \in \mathbf{C}^n$$

is a solution of $F = 0$, but not of any PΔE whose characteristic polynomial does not contain π with multiplicity $\geqslant m$. Under the action of the shift operators appearing in F, $\tilde{f}_1$ maps to a sum, with positive coefficients, of solutions of the same form (with x_n having an equal or lower power, as for an ODE). So $g + \tilde{f}_1$, with g defined as above, can be the solution of a constant coefficient linear PΔE only if both g and $\tilde{f}_1$ are solutions.

The above process can be repeated for each prime factor of S, to obtain solutions $\tilde{f}_1, \ldots, \tilde{f}_t$. The sum

$$\tilde{g} := \sum_{k=0}^{t} \tilde{f}_k$$

is then a solution of $F = 0$ which can be a solution of $H = 0$ if and only if S' contains every prime factor of S with equal or greater multiplicity. However this is impossible since the degree of S' must be less than the degree of S, so $\tilde{g}$ is a minimally Q-periodic solution of $F = 0$. □

Theorem 2. *Suppose there exist coordinates*

$$(y, w) : E \to \mathbf{C}^n \times \mathbf{C}$$

such that a given PΔE has coordinate form $H(y, [w]_Q) = 0$ with $H : J_\Delta^Q(E) \longrightarrow \mathbf{C}$ linear in $[w]_Q$,

$$H(y, [w]_Q) = c + \sum_{\alpha \in Q} c_\alpha w_{(\alpha)}.$$

Then in any other coordinate system (x, u), where the PΔE has the coordinate representation $F(x, [u]_Q) = 0$ and satisfying

$$\begin{aligned} \frac{\partial F}{\partial x^j} &= \quad 0 \qquad j = 1, \ldots, n \\ x(y+\alpha) &= x(y) + \alpha \qquad \forall \alpha \in Q \\ \frac{\partial x}{\partial w} &= \quad 0 \end{aligned}$$

(that is in which F is autonomous and the transformation is mesh and fibre preserving), there is a factorizable symmetry vector field for $Z(F)$:

$$X(x, u) = A(x)G(u)\frac{\partial}{\partial u}$$

with $A \in \mathcal{M}(\mathbf{C}^n)$. Moreover if H is linear with constant coefficients and constant inhomogeneous term, then there will always be such a symmetry vector field with A minimally Q-periodic.

Proof. The mesh preservation condition eliminates the possibility of a similarity type transformation setting some coefficients to zero. The allowed transformations of the independent variables therefore have no effect on the linearity of the PΔE, and so there is no loss of generality in assuming that the transformation from (y, w) to (x, u) (and hence the inverse transform also) has the form

$$\begin{aligned} x^j &= y^j \\ u &= u(y, w) \end{aligned}$$

that is the transformation restricted to the independent variables is just the identity.Thus if we are given the PΔE $F(x, [u]_Q) = 0$ we can define its representation in the (y, w) coordinates by defining $H : J^Q_\Delta(E) \to \mathbf{C}$ to be

$$H(y, [w]_Q) := F(y, [u(y, w)]_Q)$$

then setting $H(y, [w]_Q) = 0$. This new expression is linear in $[w]_Q$ iff

$$c_\alpha := \frac{\partial H}{\partial w_{(\alpha)}} \qquad \frac{\partial c_\alpha}{\partial w_{(\beta)}} = 0 \quad \forall \alpha, \beta \in Q.$$

However from our assumption

$$\frac{\partial H}{\partial w_{(\alpha)}} = \sum_{\beta \in Q} \frac{\partial F}{\partial u_{(\beta)}} \frac{\partial u_{(\beta)}}{\partial w_{(\alpha)}} = \frac{\partial F}{\partial u_{(\alpha)}} (\frac{\partial u}{\partial w})_{(\alpha)}.$$

Since we are assuming H to be linear in $[w]_Q$, the above equation can be re-arranged to give

$$(\frac{\partial w}{\partial u})_{(\alpha)} = \frac{1}{c_\alpha} \frac{\partial F}{\partial u_{(\alpha)}}. \tag{9}$$

Also, with H linear we have

$$H(y, [w]_Q) = c(y) + \sum_{\alpha \in Q} c_\alpha(y) w_{(\alpha)}.$$

Suppose now that $B \in \mathcal{M}(\mathbf{C}^n)$ is a solution of the homogeneous part of $H = 0$, and in the case where c, c_α are constant, assume in addition that B is a minimally Q-periodic, which is known to be possible from lemma 1. That implies

$$\sum_{\alpha \in Q} c_\alpha(y) B(y + \alpha) = 0$$

from which it follows that

$$X(y,w) := B(y)\frac{\partial}{\partial w}$$

is a symmetry vector field of $Z(H)$. Consequently X is also a symmetry vector field of $F = 0$ (this being a different coordinate expression for the same subset of $J^Q_\Delta(E)$), with

$$X(x,u) = B(x)\frac{\partial}{\partial w} = B(x)\frac{\partial u}{\partial w}\frac{\partial}{\partial u} = B(x)\left(\frac{\partial w}{\partial u}\right)^{-1}\frac{\partial}{\partial u}.$$

However if we set $\alpha = 0$ in equation (9) above we obtain

$$X(x,u) = \frac{B(x)c_0(x)}{F_u[(u)]}\frac{\partial}{\partial u}$$

with $F_u := \frac{\partial F}{\partial u}$. Setting $A(x) := B(x)c_0(x)$ and $G(u) := 1/F_u$, we have proved the theorem. □

4. The converse

Of itself, theorem 2 is not particularly interesting. It allows us to confirm that a given PΔE is *not* linearizable if we find an inconsistency while searching for a factorizable symmetry. However, a converse to theorem 2 would make testing for linearizability equivalent to looking for a factorizable symmetry. We will eventually prove the necessary converse, but first require some additional tools.

Given $A \in \mathcal{M}(\mathbf{C}^n)$, let the functions $A^p_{\alpha(j);\mu}$ be defined inductively by

$$A^0_{\alpha(j)}(x) \quad := A(x+\alpha(j)) \qquad \forall x \in \mathbf{C}^n;$$

$$A^{p+1}_{\alpha(j);\mu(p+1)} := X\left(\frac{A^p_{\alpha(j);\mu(p)}}{A^p_{\alpha(p);\mu(p)}}\right) \qquad \mu(p+1) := \mu(p), X.$$

Here X is a meromorphic vector field on $\mathbf{C}^n$, $\mu(p)$ is a sequence of p vector fields ($\mu(0)$ is the empty list) and $\mu(p), X$ denotes the sequence constructed by appending X to $\mu(p)$.

Note that $A^p_{\alpha(q);\mu(p)} = 0$ whenever $q < p$.

Lemma 3. *Let $A \in \mathcal{M}(\mathbf{C}^n)$ be minimally Q-periodic, $|Q| = M$. Then there exists at least one sequence $\mu(p)$ such that for each $p = 0, \ldots, M-2$, $A^p_{\alpha(p);\mu(p)} \neq 0$.*

Proof. Suppose that for all sequences $\mu(p)$, $A^p_{\alpha(p);\mu(p)} = 0$. From the definition this implies that for all sequences $\mu(p-1)$ and all meromorphic vector fields X,

$$X\left(\frac{A^{p-1}_{\alpha(p);\mu(p-1)}}{A^{p-1}_{\alpha(p-1);\mu(p-1)}}\right) = 0.$$

Integrating and multiplying out we now have

$$A^{p-1}_{\alpha(p);\mu(p-1)} + \lambda_{p-1}A^{p-1}_{\alpha p-1;\mu(p-1)} = 0.$$

This process can be iterated until we obtain

$$A_{\alpha(p)} + \lambda_{p-1}A_{\alpha(p-1)} + \cdots + \lambda_0 A_{\alpha(0)} = 0. \tag{10}$$

However $p \leqslant M-2$ and it can be checked that if $\alpha(0), \ldots, \alpha(M-1)$ is a good ordering, then $P := \{\alpha(0), \ldots, \alpha(p)\}$ is a prolongation set such that $P \subset Q$ if $0 \leqslant p < M-1$. Hence (10) implies that A is P-periodic, contradicting the assumption that A is minimally Q-periodic, so there must be at least one sequence $\mu(p)$ such that $A^{p}_{\alpha(p);\mu(p)} \neq 0$. $\square$

Theorem 4. *If an autonomous, meromorphic PΔE of maximal rank, given by $F([u]_Q) = 0$, $F : J^{Q}_{\Delta}(E) \to \mathbf{C}$, has a factorizable symmetry vector field*

$$X(x,u) = A(x)G(u)\frac{\partial}{\partial u}$$

with $A \in \mathcal{M}(\mathbf{C}^n)$ minimally Q-periodic, then in the coordinates (y, w) defined by

$$\begin{aligned} y(x) \quad &:= x \\ w(x,u) &:= \int^{u} \frac{\mathrm{d}\zeta}{G(\zeta)} \end{aligned}$$

the PΔE is linear with constant coefficients.

Proof. The symmetry v.f. X has prolongation

$$X^{(Q)} = \sum_{j=0}^{M-1} A(x+\alpha(j))G(u_{(\alpha(j))})\frac{\partial}{\partial u_{(\alpha(j))}}$$

so the invariance condition satisfied by A and G is

$$\sum_{j=0}^{M-1} A(x+\alpha)G(u_{(\alpha)})\frac{\partial F}{\partial u_{(\alpha)}}([u])|_{F=0} = 0. \tag{11}$$

Now assume that the chosen good ordering on Q satisfies $\partial F/\partial u_{(\alpha(M-1))} \neq 0$. This is always possible by choice of Q and the maximal rank condition. Equation (11) then becomes

$$\sum_{k=0}^{M-1} A(x+\alpha(k))G(u_{(\alpha(k))})\frac{\partial F}{\partial u_{(\alpha(k))}}([u])|_{F=0} = 0. \tag{12}$$

Note that if we split off the first term and divide by $A(x+\alpha(1))$, which is not identically zero by the result of lemma 3, the left hand side of the equation can be made independent of x:

$$-G(u_{(\alpha(1))})\frac{\partial F}{\partial u_{(\alpha(1))}}|_{F=0} = \sum_{k=2}^{M} \frac{A(x+\alpha(k))}{A(x+\alpha(1))}G(u_{\alpha(k)})\frac{\partial F}{\partial u_{(\alpha(k))}}|_{F=0}.$$

Now although we are restricted to the subset $Z(F)$, the autonomy of F implies that the x^j remain independent of the $u_{(\alpha)}$. Therefore differentiating the above equation with respect to any of the x^j gives zero on the left-hand side. Since we have assumed that A is not totally periodic on Q, there is at least one vector field X on $\mathbf{C}^n$ with which we can differentiate without annihilating the right-hand side as well. Thus equation (12) reduces to

$$\sum_{k=1}^{M-1} A^{1}_{\alpha(k);X}(x)G(u_{(\alpha(k))})\frac{\partial F}{\partial u_{(\alpha(k))}([u])}|_{F=0} = 0. \tag{13}$$

From lemma 3 we know that $\mu(p)$ can be chosen so that

$$A^{p}_{\alpha(p);\mu(p)} \neq 0 \qquad \forall p < M-1 \tag{14}$$

so the process of division and differentiation can be continued until we have completely separated variables and equation (13) has been reduced to

$$-\frac{A^{M-2}_{\alpha(M-1);\mu(M-2)}(x)}{A^{M-2}_{\alpha(M-2);\mu(M-2)}(x)}|_{F=0} = \frac{G(u_{(\alpha(M-2))})\frac{\partial F}{\partial u_{(\alpha(M-2))}}([u])}{G(u_{(\alpha(M-1))})\frac{\partial F}{\partial u_{(\alpha(M-1))}}([u])}|_{F=0}. \tag{15}$$

Consequently each side of equation (15) is equal to a constant we denote as Γ_{M-1}. Rearranging the right-hand side using the constraint $F = 0$ we have

$$\frac{1}{G(u_{(\alpha(M-1))})}\frac{\partial u_{(\alpha(M-1))}}{\partial u_{(\alpha(M-2))}} = \Gamma_{M-1}\frac{1}{G(u_{(\alpha(M-2))})}$$

which is then integrated with respect to $u_{(\alpha(M-2))}$ to give

$$\int^{u_{(\alpha(M-1))}} \frac{\mathrm{d}\zeta}{G(\zeta)} = \Gamma_{M-1}\int^{u_{(\alpha(M-2))}} \frac{\mathrm{d}\zeta}{G(\zeta)} + \Omega_{\alpha(M-2)}([u]) \tag{16}$$

the function $\Omega_{\alpha(M-2)}$ being independent of $u_{\alpha(M-2)}$.

The step preceding equation (15) in the reduction process is

$$\sum_{k=M-3}^{M-1} A^{M-3}_{\alpha(k);\mu(M-3)}(x)G(u_{(\alpha(k))})\frac{\partial F}{\partial u_{(\alpha(k))}}([u])|_{F=0} = 0. \tag{17}$$

Using the separation of equation (15) we can substitute $\Gamma_{M-1}G(u_{(\alpha(M-1))})\frac{\partial F}{\partial u_{(\alpha(M-1))}}([u])$ in place of $G(u_{(\alpha(M-2))})\frac{\partial F}{\partial u_{(\alpha(M-2))}}([u])$. The result is then separable:

$$\begin{aligned}\Gamma_{M-2} &= -\frac{\Gamma_{M-1}A^{M-3}_{\alpha(M-2);\mu(M-3)}(x) + A^{M-3}_{\alpha(M-1);\mu(M-3)}(x)}{A^{M-3}_{\alpha(M-3);\mu(M-3)}(x)}|_{F=0} \\ &= \frac{G(u_{\alpha(M-3)})\frac{\partial F}{\partial u_{(\alpha(M-3))}}([u])}{G(u_{(\alpha(M-1))})\frac{\partial F}{\partial u_{(\alpha(M-1))}}([u])}|_{F=0}.\end{aligned} \tag{18}$$

Integrating the right-hand side as in equation (16), we see that

$$\int^{u_{(\alpha(M-1))}} \frac{\mathrm{d}\zeta}{G(\zeta)} = \Gamma_{M-2}\int^{u_{(\alpha(M-3))}} \frac{\mathrm{d}\zeta}{G(\zeta)} + \Omega_{\alpha(M-3)}([u]). \tag{19}$$

Continuing in this way produces the sequence of equations

$$\begin{aligned}\int^{u_{(\alpha(M-1))}} \frac{\mathrm{d}\zeta}{G(\zeta)} &= \Gamma_{M-1}\int^{u_{(\alpha(M-2))}} \frac{\mathrm{d}\zeta}{G(\zeta)} + \Omega_{\alpha(M-2)}([u]) \\ \vdots \quad &= \vdots \\ \int^{u_{(\alpha(M-1))}} \frac{\mathrm{d}\zeta}{G(\zeta)} &= \Gamma_{1}\int^{u_{(\alpha(1))}} \frac{\mathrm{d}\zeta}{G(\zeta)} + \Omega_{\alpha(0)}([u]).\end{aligned}$$

In each case, $\Omega_{\alpha(k)}$ is independent of $u_{\alpha(k)}$.

Defining the new variable

$$w(u) := \int^{u} \frac{\mathrm{d}\zeta}{G(\zeta)} \tag{20}$$

the above sequence of equations becomes

$$w(u_{(\alpha(M-1))}) = \sum_{k=0}^{M-2} \Gamma_{k+1}w(u_{(\alpha(k))}) + \Omega([u]).$$

Here Ω can depend only on $u_{\alpha(M-1)}$, but if $\frac{\partial F}{\partial u_{(\alpha(M-1))}}|_{F=0} \neq 0$ then the constraint $F = 0$ implies that in $-\Gamma_0 := \Omega$ is constant. Thus we have shown that the coordinate change (20) reduces the original invariance equation (11) to the linear, constant coefficient PΔE

$$w(u_{(\alpha(M-1))}) - \Gamma_0 - \sum_{k=0}^{M-2} \Gamma_{k+1}w(u_{(\alpha(k))})|_{F=0} = 0. \tag{21}$$

However we assumed that X was in fact a symmetry, so the above equation is satisfied. Consequently if the linearized equation is written in terms of a function $H : J^Q_\Delta(E) \to \mathbf{C}$,

$$0 = H([w]_Q) := w_{(\alpha(M-1))} - \Gamma_0 - \sum_{k=0}^{M-2} \Gamma_{k+1} w_{(\alpha(k))}$$

then the set relation

$$Z(H) \supseteq Z(F)$$

holds. Since both H and F are meromorphic functions, their zero-sets are analytic sets (see for example [5]) whose component submanifolds have complex co-dimension down to one in $J^Q_\Delta(E)$. Now in the coordinates induced on $J^Q_\Delta(E)$ by (x, w) on E, H is multi-linear. Choosing some $w_{(\alpha(k))}$ for which

$$\frac{\partial H}{\partial w_{(\alpha(k))}} \neq 0$$

and dividing by a constant, we can take the linear expression for H to be a first-order Weierstrass polynomial in $w_{\alpha(k)}$, which is necessarily irreducible. It follows that $Z(H) = Z(F)$ (see [6]), so solutions of the original PΔE are precisely the solutions of the linear, constant coefficient PΔE $H = 0$. □

Combining the two theorems, we have proved the following result.

Corollary 5. *A meromorphic autonomous partial difference equation of maximal rank written on the discrete jet space $J^Q_\Delta(E)$ can be reduced to a linear equation with constant coefficients by a fibre and mesh preserving coordinate transformation if and only if it admits a factorizable symmetry vector field*

$$X(x, u) = A(x)G(u)\frac{\partial}{\partial u}$$

with A a minimally Q-periodic meromorphic function of the independent variables.

It can be reduced to a linear equation (with possibly non-constant coefficients) only if it admits a factorizable symmetry vector field of the same form, but without the requirement that A be minimally Q-periodic.

As an example, consider the PΔE

$$\begin{aligned} 0 = &\left(u_{(20)}\sqrt{1+u_{(11)}^2} + u_{(11)}\sqrt{1+u_{(20)}^2}\right)\left(\sqrt{(1+u_{(01)}^2)(1+u^2)} + uu_{(01)}\right) \\ &- \left(u_{(01)}\sqrt{1+u^2} + u\sqrt{1+u_{(01)}^2}\right)\left(\sqrt{(1+u_{(20)}^2)(1+u_{(11)}^2)} + u_{(11)}u_{(20)}\right). \end{aligned} \tag{22}$$

Here Q is the set $\{(0,0), (0,1), (1,0), (1,1), (2,0)\}$ and there is a symmetry vector field

$$X(x, y, u) = (1 + 2^x 3^y \sin(\pi y) + (2)^x \sin(\pi x) + 2^y)\sqrt{1+u^2}\frac{\partial}{\partial u}.$$

The factor $A(x) = 1 + 2^x 3^y \sin(\pi y) + (2)^x \sin(\pi x) + 2^y$ is minimally Q-periodic, so the transformation

$$w(u) := \int^u \frac{\mathrm{d}\zeta}{G(\zeta)} = \operatorname{arcsinh} u$$

produces from (22) the linear, constant-coefficient PΔE

$$0 = w_{(20)} + w_{(11)} - w_{(01)} - w. \tag{23}$$

5. Incommensurate spans

In studying difference equations, either partial or ordinary, there arises the question of incommensurate spans. For example consider the OΔE

$$F(u(x), u(x+1), u(x+\gamma)) = 0 \qquad \forall x \in \mathbf{C} \tag{24}$$

where $\gamma \in \mathbf{R}\backslash\mathbf{Z}$. If γ is rational, it is straightforward to choose a new mesh whose span is equal to the greatest common divisor of 1 and γ. Rescaling this span to 1 then results in an OΔE or PΔE of higher order and unit span, which can be dealt with by the methods of the previous sections. For illustration, if in the example (24) $\gamma = 2/3$, it can be rewritten as

$$F(\tilde{u}(x), \tilde{u}(x+3), \tilde{u}(x+2)) = 0 \qquad \forall x \in \mathbf{C}$$

with $\tilde{u}(x) := u(x/3)$, $x \in \mathbf{R}$. The corresponding procedure for a PΔE should be apparent.

In the case where γ is irrational, the above procedure cannot be applied. Instead, the equation (24) can be replaced by the pair of equations

$$F(u(x, y), u(x+1, y), u(x+\gamma, y)) = 0 \tag{25}$$

$$\frac{\partial u}{\partial y} = 0. \tag{26}$$

The second of these conditions is preserved by any transformation of the form $u \to w(u)$, such as in the statement of theorem 4.

Now introduce an invertible linear transformation T of the independent variables, with new independent variables α, β defined by $(\alpha, \beta) := T(x, y)$. In these coordinates, with $\tilde{u} := u \circ T^{-1}$, equation (25) becomes

$$F(\tilde{u}(\alpha, \beta), \tilde{u}(\alpha + a, \beta + b), \tilde{u}(\alpha + c, \beta + d)) = 0$$

where

$$(a, b) = T(1, 0) \qquad (c, d) = T(\gamma, 0).$$

However since $u(x, y) = u(x, y+\delta)$ for all $x, y, \delta \in \mathbf{C}$, we can instead define

$$(a, b) = T(1, \delta) \qquad (c, d) = T(\gamma, \epsilon)$$

with δ, ϵ chosen to suit our needs. If we impose the requirement that $a = d = 1$, $b = c = 0$ then equation (25) transforms to

$$F(\tilde{u}(\alpha, \beta), \tilde{u}(\alpha + 1, \beta), \tilde{u}(\alpha, \beta + 1)) = 0$$

and we find that

$$\left[T(x, y)\right] = \frac{1}{\epsilon - \delta\gamma}\begin{bmatrix} \epsilon & -\gamma \\ -\delta & 1 \end{bmatrix}.$$

For convenience we choose $\delta = 0$, $\epsilon = 1$, so that

$$\alpha = x - \gamma y$$
$$\beta = y.$$

It follows that $Y := \frac{\partial}{\partial y} = -\gamma\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}$, so the system (25) and (26) can be replaced by

$$F(\tilde{u}(\alpha, \beta), \tilde{u}(\alpha + 1, \beta), \tilde{u}(\alpha, \beta + 1)) = 0 \tag{27}$$

$$Y\tilde{u} = 0. \tag{28}$$

Equation (27) is now in a form compatible with theorem 4. Although symmetry vector fields

$$A(\alpha, \beta)G(\tilde{u})\frac{\partial}{\partial \tilde{u}}$$

will not in general commute with Y, the resulting linearizing transformation

$$\tilde{w}(\tilde{u}) = \int^{\tilde{u}} \frac{\mathrm{d}\zeta}{G(\zeta)}$$

is generated by the vector field

$$G(\tilde{u})\frac{\partial}{\partial \tilde{u}}$$

which does satisfy

$$[Y, G(\tilde{u})\frac{\partial}{\partial \tilde{u}}] = 0.$$

Consequently

$$Y\tilde{u} = 0 \Longleftrightarrow Y\tilde{w} = 0.$$

Thus if we define $w := \tilde{w} \circ T$, $u \to w$ is a transformation linearizing (25) which satisfies $\frac{\partial w}{\partial y} = 0$, so suppressing the y variable provides a linearization of the original OΔE (24). Note, in particular, that although the linearization procedure is applied to a PΔE, the resulting linear equation is transformed back to an OΔE.

The above procedure generalizes immediately to OΔE's of higher order and to PΔE's. We summarise the general situation without proof in the following proposition.

Proposition 6. *Given a PΔE with dependent variable u and n independent variables such that the spans of the kth variable generate a degree N_k field extension of* $\mathbf{Q}$*, there is a unit-span PΔE with $N := \sum_{k=1}^{n} N_k$ independent variables, which together with auxiliary conditions $Y_j u = 0$ for vector fields $Y_1, \ldots, Y_N$ is equivalent to the original PΔE.*

There remains a significant problem for incommensurate-span OΔE's, which is to solve the resulting linear equation. For example it is not clear how to obtain a general solution to

$$u(x + \sqrt{2}) + au(x + 1) + bu(x) = 0 \quad \forall x \in \mathbf{C}$$

when $a, b \in \mathbf{C}$ are arbitrary constants. However to finish this section we note that in special cases solutions can be found. Consider for example the OΔE

$$u(x + 1 + \gamma) - 3u(x + \gamma) - 2u(x + 1) + 6u(x) = 0 \quad \forall x \in \mathbf{C} \tag{29}$$

where $\gamma \in \mathbf{R}\backslash\mathbf{Z}$. The distinguishing feature is that (29) can be factorized by setting

$$v(x) := u(x + 1) - 3u(x) \quad \forall x \in \mathbf{C}.$$

Equation (29) can then be written as

$$v(x + \gamma) - 2v(x) = 0 \quad \forall x \in \mathbf{C}.$$

This is reflected in the characteristic equation of (29), which is

$$(\lambda^{\gamma} - 2)(\lambda - 3) = 0. \tag{30}$$

Thus there is a set of solutions of the form

$$u(x) = \alpha(x)2^{x/\gamma} + \beta(x)3^{x} \quad \forall x \in \mathbf{C}$$

where $2^{1/\gamma}$ is taken to be the real root and

$$\alpha(x+\gamma) = \alpha(x) \quad \beta(x+1) = \beta(x) \quad \forall x \in \mathbf{C}.$$

In fact it can be shown that all solutions are of this form.

Note that choosing one of the complex roots for $2^{1/\gamma}$ merely introduces a factor $\exp(2\pi i k x/a)$ in the first term, but this is a γ-periodic function of x and therefore corresponds to a different choice of α.

6. Concluding remarks

We have shown that for difference equations with an arbitrary number of independent variables and any order, reducibility to a linear difference equation with constant coefficients is equivalent to the existence of a single symmetry vector field of a particular type. Note that in the standard treatment of Lie symmetries of difference equations, as in the continuous theory on which it is based (see for example [9]), the presence of a single symmetry suffices only to reduce the order by one (OΔE case [8]), or to reduce the number of independent variables by one (PΔE case [15, 20]).

Reduction to a constant coefficient linear difference equation is important for at least two reasons. In the OΔE case, the linear equation can be completely solved by elementary methods completely analogous to those used for linear constant coefficient ODE's. In the PΔE case there does not appear to be any similar method for obtaining complete solutions, although some solutions can be produced, again using methods parallel to those for linear PDE's.

The second point is that if an OΔE can be globally linearized (if the symmetry vector field is globally defined without branch-points) then it cannot be chaotic.

Of course the usual caveat must be stated: as in the thery of differential systems, finding the appropriate symmetry may be no easier than solving the original problem. In particular, since we require a symmetry vector field

$$X(x,u) = A(x)G(u)\frac{\partial}{\partial u}$$

with A minimally q-periodic, we effectively need to find an appropriate number of symmetries

$$X_q(x,u) = A_q(x)G(u)\frac{\partial}{\partial u}$$

such that each A_q satisfies the same linear constant coefficient difference equation as A and $A = \sum_q A_q$. In fact this explains the rather mysterious process of completely linearizing an order M OΔE equation by means of a single symmetry: the condition of minimal Q-periodicity imposes that A may in fact be required to be the sum of M functionally independent symmetry vector fields corresponding to M independent solutions of the linear equation. In particular, this is the case when the characteristic equation of the OΔE has distinct roots. However at least in the case of repeated roots we may still get something for free, as in the example of section 1.

Acknowledgment

This investigation is part of the research programme of the Australian Research Council.

References

[1] Ablowitz M and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)

[2] Bruschi M, Ragnisco O, Santini P M and Tu Gui-Zhang 1991 Integrable symplectic maps *Physica* **49D** 273–94

[3] Byrnes G B and Quispel G R W 1994 Non-autonomous Lie symmetries of difference equations, in preparation

[4] Capel H W, Nijhoff F W, Papageorgiou V G and Quispel G R W 1991 Integrable mappings and soliton lattices *Solitons and Chaos* ed F Lambert (Berlin: Springer)

[5] Field M 1982 *Several Complex Variables and Complex Manifolds 1 (London Math. Soc. Lecture Note Series)* **65** (Cambridge: Cambridge University Press)

[6] Gunning R C and Rossi H 1965 *Analytic Functions of Several Complex Variables* (Englewood Cliffs, NJ: Prentice-Hall)

[7] Kelley W G and Peterson A C 1991 *Difference Equations* (San Diego, CA: Academic)

[8] Maeda S 1987 The similarity method for difference equations *IMA J. Appl. Math.* **38** 129–34

[9] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (New York: Springer)

[10] Papageorgiou V G, Nijhoff F W, Grammaticos B and Ramani A 1992 Isomonodromic deformation problems for discrete analogues of Painlevé equations *Phys. Lett.* **164A** 57–64

[11] Quispel G R W, Nijhoff F W, Capel H W and van der Linden J 1984 Linear integral equations and nonlinear partial difference equations *Physica* **125A** 344

[12] Quispel G R W, Roberts J A G and Thompson C J 1988 Integrable mappings and soliton equations I *Phys. Lett.* **126A** 419–21

[13] Quispel G R W, Roberts J A G and Thompson C J 1989 Integrable mappings and soliton equations II *Physica* **34D** 183–92

[14] Quispel G R W, Capel H W, Papageorgiou V G and Nijhoff F W 1991 Integrable mappings derived from soliton equations *Physica* **173A** 243–66

[15] Quispel G R W, Capel H W and Sahadevan R 1992 Continuous symmetries of differential-difference equations: the Kac–van Moerbeke equation and Painlevé reduction *Phys. Lett.* **170A** 379–83

[16] Quispel G R W and Sahadevan R 1993 Lie Symmetries and the integration of difference equations *Phys. Lett.* **184A** 64–70

[17] Sahadevan R, Byrnes G B and Quispel G R W 1994 Linearisation of difference equations using factorisable Lie smmetries *Proc. Workshop on Symmetries and Integrability of Difference Equation 1994, Estérel* ed D Levi, L Vinet and P Winternitz (CRM Proceedings and Lecture Notes Series) (Providence, RI: Am. Math. Soc.)

[18] Saunders D J 1989 *The Geometry of Jet Bundles (London Math. Soc. Lecture Note Series)* **142** (Cambridge: Cambridge University Press)

[19] Whitney H 1972 *Complex Analytic Varieties* (Reading, MA: Addison-Wesley)

[20] Winternitz P and Levi D 1991 Continuous symmetries of discrete equations *Phys. Lett.* **152A** 335–8