ELSEVIER

Physica D 73 (1994) 277–304

PHYSICA D

Reversing k-symmetries in dynamical systems

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Received 14 June 1993; revised 24 January 1994; accepted 24 January 1994
Communicated by J.D. Meiss

Abstract

We generalize the concept of (reversing) symmetries of a dynamical system, i.e. we study dynamical systems that possess symmetry properties only if considered on a proper time scale. In particular (considering dynamical systems with discrete time), the kth iterate of a map may possess more (reversing) symmetries than the map itself. In this way the concepts of (reversing) symmetries and (reversing) symmetry groups are generalized to (reversing) k-symmetries and (reversing) k-symmetry groups.

Furthermore, a method is studied for finding orbits that are (k-) symmetric with respect to reversing (k-)-symmetries. Firstly an existing method for finding orbits that are symmetric with respect to one reversing symmetry is extended to the case of more than one reversing symmetry and secondly a generalization of this method to the case of reversing k-symmetries is introduced.

Some physically relevant examples of dynamical systems possessing reversing k-symmetries are discussed briefly.

1. Introduction

Interest in the effect of symmetries in dynamical systems is steadily growing. A lot of attention has been paid to Lie-symmetries of Hamilton systems that give rise to conserved quantities [1]. Also, more general methods of using Lie-symmetries have been developed in a wider context of differential and difference equations [2,3].

During the past two decades, the attention has been directed to discrete symmetries as well. In dynamical systems, discrete symmetries were considered in studies of bifurcation problems by e.g. Sattinger [4], Vanderbauwhede [5] and Golubitsky et al. [6].

In dynamical systems, *symmetries* are transformations in state space that leave the equations of motion invariant. On the other hand, *reversing symmetries* leave the equations of motion invariant if the direction of time is reversed [7].

Historically, dynamical systems possessing a reversing symmetry are called *weakly reversible* in general and *reversible* if the reversing symmetry is its own inverse, i.e. an involution [8]. For an overview of the work on reversible dynamical systems we recommend the review by Roberts and Quispel [9].

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SSDI 0167-2789(94)00015-I

Recently it has been recognized that there are dynamical systems that possess symmetry properties only if considered on a proper time scale: the time-τ map possesses symmetry properties only for certain values of τ, or in purely discrete systems iterates of a map possess symmetry properties that the map does not possess itself [10–13]. This phenomenon can be observed in various well known dynamical systems that occur in the context of problems in physics, e.g. the standard map [14], web maps [10,12,15], a class of trace maps [13,16], and the kicked rotator [11,17]. These examples will be discussed in section 4.

The outline of the rest of this paper is as follows. In the next section we briefly review the concepts of symmetries and reversing symmetries of dynamical systems, and also present some new results that are related to the case that a map possesses more than one reversing symmetry. Furthermore, we discuss a method to find periodic orbits that are symmetric, i.e. invariant, with respect to reversing symmetries.

Section 3 contains our main results. There we consider dynamical systems that possess symmetry properties only if considered on certain time scales. We discuss generalizations of the method for obtaining periodic orbits that are symmetric with respect to reversing symmetries, as established in section 2, to the case of reversing k-symmetries. A preliminary and abridged version of some of these results was given in [18].

Throughout this paper we confine ourselves to systems with discrete time, i.e. mappings. They occur as Poincaré sections or stroboscopic pictures of dynamical systems with continuous time, as well as in problems of a truly discrete nature.

2. Symmetries and reversing symmetries

In this section we first review the well known concepts of symmetries and reversing symmetries in dynamical systems. Thereafter we will discuss a method for finding periodic orbits that are symmetric with respect to reversing symmetries. This method is illustrated by an example.

We consider a discrete dynamical system, i.e. a map on a state space Ω,

$$L: \Omega \mapsto \Omega \,. \tag{1}$$

The map M is a *symmetry* of L if

$$M \circ L = L \circ M \,. \tag{2}$$

The map S is a *reversing symmetry* of L if

$$S \circ L = L^{-1} \circ S \,. \tag{3}$$

For the above definition to make sense we require L to be invertible. Throughout this paper we will assume that L is invertible, unless stated otherwise. Moreover, we will confine ourselves to invertible (reversing) symmetries. In that case, the set of symmetries of L is called the *symmetry group* of L, denoted by $\mathcal{G}$, and the set of symmetries and reversing symmetries of L is called the *reversing symmetry group* of L, denoted by $\mathcal{E}$[2]. In a previous publication [7] the group structure was already discussed. The following remarks were easily proved.

- The set of symmetries and the set of symmetries and reversing symmetries of a map form a group under composition. The composition of two (reversing) symmetries is a symmetry and the composition of a symmetry and a reversing symmetry is a reversing symmetry.
- A conjugacy class of a reversing symmetry group consists entirely of symmetries or entirely of reversing symmetries.
- L and L^{-1} are symmetries of L.

The symmetry group $\mathcal{G}$ is a normal subgroup of the reversing symmetry group $\mathcal{E}$. In case L possesses a reversing symmetry S, then

$$\mathcal{E} = \mathcal{G} \cup S\mathcal{G} \,, \tag{4}$$

[2] Sometimes it is convenient to consider instead of the full (reversing) symmetry group of a dynamical system, only a subgroup. Such a subgroup will be referred to as <u>a</u> (reversing) symmetry group in contrast to <u>the</u> (reversing) symmetry group, the latter term being reserved for the full group.

and $\mathscr{E}/\mathscr{G} \simeq \mathbb{Z}_2$, i.e. $\mathscr{G}$ is a subgroup of index 2.

If the equations of motion possess a group of symmetries and reversing symmetries it is of interest to analyze how these symmetries manifest themselves in the dynamics, for instance by mapping an invariant set $\Delta \subset \Omega$ of the dynamical system into itself[3]. An important example of an invariant set is the *orbit* of a point $\boldsymbol{x}_0$ in state space, which we denote by

$$\Gamma(x_0) := \{x \in \Omega \mid x = L^n x_0, n \in \mathbb{Z}\} . \tag{5}$$

Definition 2.1. Let L be a map, and $\mathscr{E}$ its reversing symmetry group. Then the *isotropy subgroup* $\Sigma_{\Gamma(\boldsymbol{x}_0)}$ of the orbit $\Gamma(\boldsymbol{x}_0)$ is the subgroup of $\mathscr{E}$ that leaves $\Gamma(\boldsymbol{x}_0)$ invariant, i.e.

$$\Sigma_{\Gamma(\boldsymbol{x}_0)} \leqslant \mathscr{E} \quad \text{and} \quad \Sigma_{\Gamma(\boldsymbol{x}_0)}\, \Gamma(\boldsymbol{x}_0) = \Gamma(\boldsymbol{x}_0) . \tag{6}$$

If an orbit $\Gamma(\boldsymbol{x}_0)$ has an isotropy subgroup $\Sigma_{\Gamma(\boldsymbol{x}_0)}$, then $\Gamma(\boldsymbol{x}_0)$ is called *symmetric* with respect to $\Sigma_{\Gamma(\boldsymbol{x}_0)}$.

Similarly to Definition 2.1 we may define the isotropy subgroup of a single point $\boldsymbol{x}_0$:

Definition 2.2. Let L be a map, and $\mathscr{E}$ its reversing symmetry group. Then the *isotropy subgroup* $\Sigma_{\boldsymbol{x}_0}$ of the point $\boldsymbol{x}_0$ is the subgroup of $\mathscr{E}$ that leaves $\boldsymbol{x}_0$ invariant, i.e.

$$\Sigma_{\boldsymbol{x}_0} \leqslant \mathscr{E} \quad \text{and} \quad \Sigma_{\boldsymbol{x}_0} \boldsymbol{x}_0 = \boldsymbol{x}_0 . \tag{7}$$

Symmetry properties of an orbit are completely specified by giving a point on the orbit and (the generators of) the isotropy subgroup of that point. It is easy to show that

$$\Sigma_{L^m \boldsymbol{x}_0} = L^m \Sigma_{\boldsymbol{x}_0} L^{-m} , \quad \text{for all } m \in \mathbb{Z} . \tag{8}$$

In fact, each finitely generated isotropy subgroup $\Sigma_{\boldsymbol{x}_0}$ can be generated either by symmetries alone, or (in case the isotropy subgroup contains reversing symmetries as well) by l symmetries $M_1, \ldots, M_l$ plus a single reversing symmetry S, see also (4). In the former case the isotropy subgroups are identical for all points on the orbit $\Gamma(\boldsymbol{x}_0)$ (cf. [6]):

$$\Sigma_{\boldsymbol{x}_0} = \Sigma_{L^m \boldsymbol{x}_0} . \tag{9}$$

In the latter case[4]

$$\begin{aligned} \Sigma_{\boldsymbol{x}_0} &= \langle M_1, \ldots, M_l, S \rangle \Leftrightarrow \Sigma_{L^m \boldsymbol{x}_0} \\ &= \langle M_1, \ldots, M_l, L^{2m} \circ S \rangle . \end{aligned} \tag{10}$$

(This argument of course also holds for isotropy subgroups that are not finitely generated.)

Proposition 2.1. Let $\Sigma_{\boldsymbol{x}_0} \leqslant \mathscr{E}$ be the isotropy subgroup of a point $\boldsymbol{x}_0$, and $\Sigma_{\Gamma(\boldsymbol{x}_0)} \leqslant \mathscr{E}$ be the isotropy subgroup of its orbit $\Gamma(\boldsymbol{x}_0)$. Then

$$\Sigma_{\Gamma(\boldsymbol{x}_0)} = \langle L \rangle \cdot \Sigma_{\boldsymbol{x}_0} , \tag{11}$$

where $\langle L \rangle$ is a normal subgroup of $\Sigma_{\Gamma(\boldsymbol{x}_0)}$. The product "$\cdot$" is semi-direct if and only if the orbit $\Gamma(\boldsymbol{x}_0)$ is nonperiodic.

Proof. Let $U \in \Sigma_{\Gamma(\boldsymbol{x}_0)}$, then there exists an $m \in \mathbb{Z}$ such that

$$U \boldsymbol{x}_0 = L^{-m} \boldsymbol{x}_0 , \tag{12}$$

implying that

$$L^m \circ U \in \Sigma_{\boldsymbol{x}_0} . \tag{13}$$

This verifies (11). Moreover $\langle L \rangle$ is a normal subgroup of $\Sigma_{\Gamma(\boldsymbol{x}_0)}$, for every $U \in \Sigma_{\Gamma(\boldsymbol{x}_0)}$ is a symmetry or reversing symmetry of L and hence

$$U \langle L \rangle U^{-1} = \langle L \rangle . \tag{14}$$

From this it follows that the product "$\cdot$" is a semi-direct product if and only if $\Sigma_{\boldsymbol{x}_0} \cap \langle L \rangle = \mathrm{Id}$. This last condition holds if and only if the orbit of $\boldsymbol{x}_0$ is nonperiodic. □

If a periodic orbit has an isotropy subgroup that contains a reversing symmetry then it can be tracked down easily by means of a simple method. Originally this method was introduced by DeVogelaere [19] for reversible maps and it has

[3] $\Delta \subset \Omega$ is called an invariant set of L if $L\Delta = \Delta$.

[4] $\langle a_1, \ldots, a_n \rangle$ denotes the group generated by $a_1, \ldots, a_n$.

been highly useful in several applications [9,17,20–23]. In [7] it was shown that the method works equally well in case a reversing symmetry is not an involution. Here we will discuss an extension of this method to the case that there is more than one independent[5] reversing symmetry in the reversing symmetry group.

Let us first define the set of fixed points of a map $A: \Omega \mapsto \Omega$, also called the fixed set of A,

$$\mathrm{Fix}(A) := \{\boldsymbol{x} \in \Omega \mid A\boldsymbol{x} = \boldsymbol{x}\} . \tag{15}$$

Proposition 2.2. $\Gamma(\boldsymbol{x}_0)$ has an isotropy subgroup containing the (reversing) symmetry U, i.e. $U \in \Sigma_{\Gamma(\boldsymbol{x}_0)}$, if and only if $\boldsymbol{x}_0 \in \mathrm{Fix}(L^m \circ U)$, i.e. $L^m \circ U \in \Sigma_{\boldsymbol{x}_0}$, for some value of $m \in \mathbb{Z}$.

Proof. If $U \in \Sigma_{\Gamma(\boldsymbol{x}_0)}$ then, by definition, there is an $m \in \mathbb{Z}$ such that $U\boldsymbol{x}_0 = L^{-m}\boldsymbol{x}_0$, cf. Definition 2.1. Conversely, if $L^m \circ U\boldsymbol{x}_0 = \boldsymbol{x}_0$ for some $m \in \mathbb{Z}$ then since U is an element of the reversing symmetry group of L, it is easily verified that $U\Gamma(\boldsymbol{x}_0) = \Gamma(\boldsymbol{x}_0)$. □

Proposition 2.3. Let S be a reversing symmetry of L and $m \in \mathbb{Z}$, then

$$\boldsymbol{x}_0 \in \mathrm{Fix}(L^m \circ S) \Rightarrow \boldsymbol{x}_0 \in \mathrm{Fix}(S^2) , \tag{16}$$

i.e.

$$L^m \circ S \in \Sigma_{\boldsymbol{x}_0} \Rightarrow S^2 \in \Sigma_{\boldsymbol{x}_0} . \tag{17}$$

Proof. $L^m \circ S\boldsymbol{x}_0 = \boldsymbol{x}_0 \Rightarrow$
$L^m \circ S \circ L^m \circ S\boldsymbol{x}_0 = \boldsymbol{x}_0 \Leftrightarrow S^2\boldsymbol{x}_0 = \boldsymbol{x}_0$. □

This implies that if S is a reversing symmetry only orbits of points that are fixed under S^2 can be symmetric with respect to S. As such, Proposition 2.3 imposes restrictions on the occurrence of orbits that are symmetric with respect to noninvolutory reversing symmetries.

[5] If S is a reversing symmetry, then $L^n \circ S$ is also a reversing symmetry. Two reversing symmetries that are both in the group generated by L and S, i.e. $\langle L, S\rangle$, are considered to be dependent.

Proposition 2.4. $\Gamma(\boldsymbol{x}_0)$ is a periodic orbit with period p and isotropy subgroup $\Sigma_{\Gamma(\boldsymbol{x}_0)}$, containing a reversing symmetry S, if and only if there exist $n, m \in \mathbb{Z}$ $(n \neq m)$ such that

$$\boldsymbol{x}_0 \in \mathrm{Fix}(L^n \circ S) \cap \mathrm{Fix}(L^m \circ S) \tag{18}$$

and $n - m = 0 \bmod p$.

Proof. It is easily shown that (18) implies that $L^{n-m}\boldsymbol{x}_0 = \boldsymbol{x}_0$. Hence $n - m$ is a multiple of p. Conversely, Proposition 2.2 shows that there is an $n \in \mathbb{Z}$ such that $L^{-n}\boldsymbol{x}_0 = S\boldsymbol{x}_0$. Hence $L^{p-n}\boldsymbol{x}_0 = S\boldsymbol{x}_0$, since $\Gamma(\boldsymbol{x}_0)$ is a periodic orbit with period p. Setting $m = n - p$ completes the proof. □

Note that from Proposition 2.3 it follows that $\boldsymbol{x}_0 \in \mathrm{Fix}(L^n \circ S) \cap \mathrm{Fix}(L^m \circ S)$ implies $\boldsymbol{x}_0 \in \mathrm{Fix}(L^n \circ S^{o_1}) \cap \mathrm{Fix}(L^m \circ S^{o_2})$ for all odd o_1 and o_2.

Proposition 2.4 is quite powerful because of the fact that if S is a reversing symmetry of L, then [7]

$$\mathrm{Fix}(L^{2n} \circ S) = L^n(\mathrm{Fix}(S)) , \tag{19}$$

$$\mathrm{Fix}(L^{2n+1} \circ S) = L^n(\mathrm{Fix}(L \circ S)) . \tag{20}$$

Proposition 2.4 in combination with (19) and (20) thus provides us with a method for finding periodic orbits that are symmetric with respect to a reversing symmetry S [19,7]:

Theorem 2.5 (*fixed set iteration* (*FSI*) *method*). Periodic orbits of a map L are symmetric with respect to a reversing symmetry S if and only if all points of this orbit lie on intersections of iterates of $\mathrm{Fix}(S)$ and/or $\mathrm{Fix}(L \circ S)$. Moreover, a (periodic or non-periodic) orbit is symmetric with respect to a reversing symmetry S if and only if it has at least one point on either $\mathrm{Fix}(S)$ or $\mathrm{Fix}(L \circ S)$.

It is easily shown that a period p orbit has two points on either $\mathrm{Fix}(S)$ or $\mathrm{Fix}(L \circ S)$ if p is even, and one point on $\mathrm{Fix}(S)$ as well as one point on $\mathrm{Fix}(L \circ S)$ if p is odd. The occurrence of S and

$L \circ S$ in Theorem 2.5 can be understood in the context of the group structure of $\langle L, S \rangle$: S and $L \circ S$ are representatives of the two conjugacy classes containing reversing symmetries of the form $L^n \circ S$, for some $n \in \mathbb{Z}$.

The fixed set iteration method originally was developed within the context of systems possessing involutory reversing symmetries [19]. For elementary applications we refer the reader to [9, 17,20–23]. The validity of this method for noninvolutory reversing symmetries was given in [7].

Let us now extend our discussion to the case of more than one (reversing) symmetry. In this case the structure of the reversing symmetry group starts to play a role of some importance.

Proposition 2.6. Consider all orbits of a map L that are symmetric with respect to the (reversing) symmetry U. Then for all $\tilde{U}$ in the conjugacy class of U, all orbits of L that are symmetric with respect to $\tilde{U}$ are images of the orbits that are symmetric with respect to U under an element of the (reversing) symmetry group of L, i.e. a (reversing) symmetry V satisfying $\tilde{U} = V \circ U \circ V^{-1}$.

Proof. $\Gamma(\boldsymbol{x}_0)$ is symmetric with respect to U, if and only if $\Gamma(V\boldsymbol{x}_0) = V\Gamma(\boldsymbol{x}_0)$ is symmetric with respect to $V \circ U \circ V^{-1}$. □

The latter proposition provides us with guidelines on how to make economic use of the FSI method for finding symmetric periodic orbits. For example, if we have used the FSI method to find all orbits of some period that are symmetric with respect to the reversing symmetry S, it is easy to obtain all orbits of this period that are symmetric with respect to any other reversing symmetry group to the orbits that are symmetric with respect to S. However, if we want to know the orbits of this period that are symmetric with respect to a reversing symmetry that is not in the conjugacy class of S, we have to use the FSI method again, and there is no short-cut.

It is also interesting to find orbits that are symmetric with respect to more than one reversing symmetry. Essentially, we may use the FSI method again for finding these orbits. However, the determination of the period of an orbit of a point on the basis of an intersection of $L^n \circ S$ and $L^m \circ \tilde{S}$ is a little more complicated:

Theorem 2.7. Let S and $\tilde{S}$ be two independent reversing symmetries. Then $S, \tilde{S} \in \Sigma_{\Gamma(\boldsymbol{x}_0)}$ if and only if $\exists n, m \in \mathbb{Z}$ such that $\boldsymbol{x}_0 \in \mathrm{Fix}(L^n \circ S) \cap \mathrm{Fix}(L^m \circ \tilde{S})$. Let furthermore $\tau(S, S)$ be the greatest common divisor (gcd) of all integers τ, for which

$$\prod_{j=1}^{\tau} [\tilde{M}_j \circ \tilde{S} \circ M_j \circ S] = \mathrm{Id}\,, \tag{21}$$

for any

$$M_j, \tilde{M}_j \in \mathscr{H} := \langle S^2, \tilde{S}^2 \rangle\,. \tag{22}$$

Then, if $\tau(S, \tilde{S}) < \infty$ and $m \neq n$, the period of $\Gamma(\boldsymbol{x}_0)$ divides $(n-m)\tau(S, \tilde{S})$.

Proof. If $\boldsymbol{x}_0$ is an element of an orbit with an isotropy subgroup $\Sigma_{\Gamma(\boldsymbol{x}_0)}$, such that $S, \tilde{S} \in \Sigma_{\Gamma(\boldsymbol{x}_0)}$, then for some $n, m, \in \mathbb{Z}$ we have

$$S\boldsymbol{x}_0 = L^{-n}\boldsymbol{x}_0 \quad \text{and} \quad \tilde{S}\boldsymbol{x}_0 = L^{-m}\boldsymbol{x}_0\,. \tag{23}$$

For all $M \in \mathscr{H}$ we obtain

$$M \circ S\boldsymbol{x}_0 = M \circ L^{-n}\boldsymbol{x}_0 = L^{-n} \circ M\boldsymbol{x}_0 = L^{-n}\boldsymbol{x}_0\,. \tag{24}$$

In (24) the second equality follows from the commutativity of M with L, and the last equality follows from Proposition 2.3. Similarly one obtains for all $\tilde{M} \in \mathscr{H}$

$$\tilde{M} \circ \tilde{S}\boldsymbol{x}_0 = L^{-m}\boldsymbol{x}_0\,. \tag{25}$$

From this it follows by substitution that

$$\tilde{M} \circ \tilde{S} \circ M \circ S\boldsymbol{x}_0 = L^{n-m}\boldsymbol{x}_0\,, \tag{26}$$

and

$$\prod_{j=1}^{\tau} [\tilde{M}_j \circ \tilde{S} \circ M_j \circ S]\boldsymbol{x}_0 = L^{\tau(n-m)}\boldsymbol{x}_0\,, \tag{27}$$

concluding the proof. □

Note that in case the order of $\langle S, \tilde{S} \rangle$ is finite, τ is finite too. This implies that, in case $n \neq m$, the intersection points of the fixed sets are points of symmetric periodic orbits.

$\tau(S, \tilde{S})$ has the following fundamental properties:

$$\tau(S, \tilde{S}) = \tau(\tilde{S}, S) , \tag{28}$$

$$\tau(S, S) = 1 . \tag{29}$$

The second property (29) is related to the result of Proposition 2.4.

2.1. Example

To illustrate some of the results obtained above, let us consider the map on the plane[6]

$$L_\mu : \begin{cases} x' = \dfrac{x - \mu y}{1 + \varepsilon \cos(2\pi y)} , \\ y' = y[1 + \varepsilon \cos(2\pi x')] + \mu x' . \end{cases} \tag{30}$$

For all μ this map possesses a reversing symmetry group generated by the reversing symmetry M_v, denoting the mirror in the line $x = y$, and the symmetry $-\mathrm{Id}$. This group is called $2m'$, where '2' refers to the generator $-\mathrm{Id}$ and m' to the reversing generator M_v. In case $\mu = 0$ there is an additional reversing symmetry $R_{\pi/2}$, the rotation around the origin over an angle $\pi/2$. The resulting reversing symmetry group is generated by the reversing symmetry $R_{\pi/2}$ and the symmetry M_x, the mirror in the x-axis, and is called $4'm$[7]. In this way we constructed a one-parameter family of maps with $2m'$ symmetry, intersecting a point ($\mu = 0$) with $4'm$ symmetry. Note that $2m' < 4'm$.

Let us first consider the case $\mu \neq 0$. In that case we have two reversing mirrors: M_v and M_w, denoting the mirror in the line $x = -y$. These two mirrors are in different conjugacy classes of $2m'$. From Proposition 2.6 it then follows that the orbits that are symmetric with respect to M_v are not related via an element of the reversing symmetry group to the orbits that are symmetric with respect to M_w.

In case $\mu = 0$ this situation is drastically changed. Because of the new reversing symmetry $R_{\pi/2}$, the reversing mirrors M_v and M_w are related through conjugation with $R_{\pi/2}$ and therefore members of the same conjugacy class.

Periodic orbits that are symmetric with respect to a reversing symmetry can be tracked down with use of the FSI method of iterated fixed sets, as has been discussed before. In using this method, in the spirit of Theorem 2.7, the values of $\tau(S, \tilde{S})$ should be analyzed first. For this particular example, they are presented in Table 1 (observe the properties (28) and (29) in this table). However, we will not demonstrate the FSI method here in detail.

To illustrate the change of symmetry properties, in Fig. 1 we show two phase portraits of (30). Fig. 1a has $4'm$ symmetry and Fig. 1b has $2m'$ symmetry. These figures clearly illustrate how the equivalence of orbits that are symmetric with respect to M_v and M_w in the case of $4'm$ is broken in the case of $2m'$. Note in Fig. 1b the occurrence of attractor–repellor pairs that are typical in reversible dynamical systems that are not measure preserving (see also [9]).

Table 1
$\tau(S, \tilde{S})$ for the reversing symmetries of the map L_μ (30) in case $\mu = 0$

τ	$R_{\pi/2}$	M_v	M_w
$R_{\pi/2}$	1	2	2
M_v	2	1	2
M_w	2	2	1

[6] In Appendix A it is explained in detail how this example has been constructed.

[7] In classifying the structure of the reversing symmetry groups we make a distinction between reversing and non-reversing generators, borrowing the Shubnikov–Belov notation from magnetic point groups [10]. In this notation the reversing generators are labeled with a prime: $2m'$ in case $\mu \neq 0$ and $4'm$ if $\mu = 0$. Ignoring the primes we obtain $2m \simeq D_2$ and $4m \simeq D_4$.

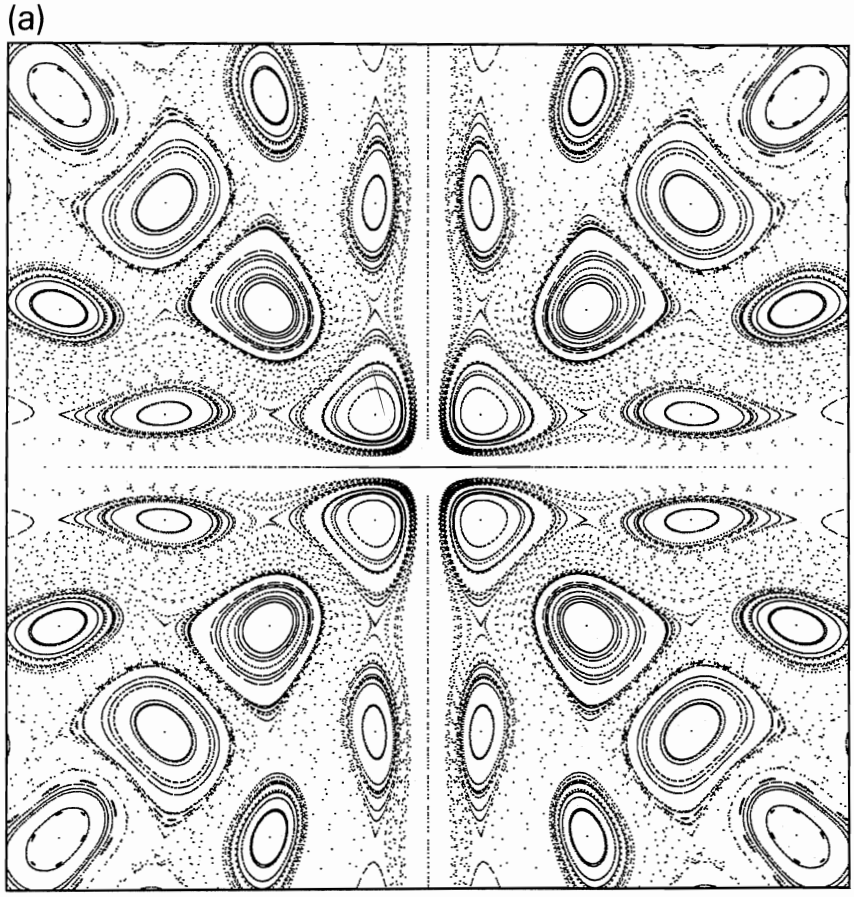

Fig. 1. Phase portraits of (30) on $[-2,2]\times[-2,2]$ with $\varepsilon=0.1$ and (a) $\mu=0$, (b) $\mu=0.02$.

3. *k*-symmetries and reversing *k*-symmetries

In this section we study the case that a map possesses some symmetry properties only if it is considered on the right time scale, e.g. L possesses fewer (reversing) symmetries than L^k. As we said in the introduction, this has been observed to occur in various dynamical systems, see e.g. section 4.

Consider the symmetry group $\mathcal{G}_k$ of L^k. All $M_0\in\mathcal{G}_k$ commute with L^k:

$$M_0\circ L^k = L^k\circ M_0\,. \tag{31}$$

It follows that

$$M_0 = L^k\circ M_0\circ L^{-k}\,. \tag{32}$$

We now define M_i by

$$M_i := L^i\circ M_0\circ L^{-i}\,. \tag{33}$$

Because of (31) we have $M_0=M_k$. Relation (33) induces a group automorphism ϕ_L of $\mathcal{G}_k$:

$$\phi_L\colon \mathcal{G}_k\mapsto\mathcal{G}_k\,. \tag{34}$$

The action of ϕ_L on $M\in\mathcal{G}_k$ is defined as

$$\phi_L(M) := L\circ M\circ L^{-1}\,. \tag{35}$$

A similar approach can be followed with the reversing symmetry group $\mathcal{E}_k$ of L^k. Let S_0 be a reversing element of $\mathcal{E}_k$, i.e.

$$S_0\circ L^k = L^{-k}\circ S_0\,, \tag{36}$$

(b)

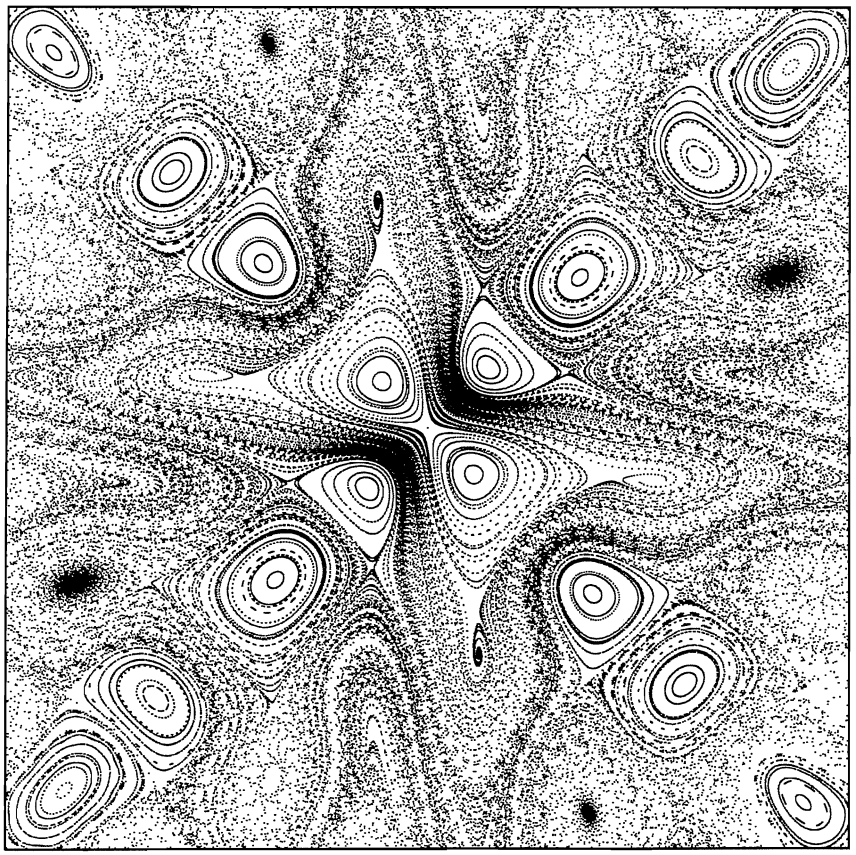

Fig. 1. (Continued).

from which it follows that

$$S_0 = L^k \circ S_0 \circ L^k \,. \tag{37}$$

We now define S_i by

$$S_i := L^i \circ S_0 \circ L^i \,. \tag{38}$$

By definition $S_0 = S_k$. Note that S_i is a reversing element of $\mathscr{E}_k$ for al i.

Relation (38) induces a permutation on the set of reversing elements of $\mathscr{E}_k$. If S is a reversing element of $\mathscr{E}_k$ then we define

$$\phi_L(S) := L \circ S \circ L \,. \tag{39}$$

For this permutation we used the same symbol as in (34). In this way we have defined ϕ_L on all of $\mathscr{E}_k$, via (35) and (39):

$$\phi_L : \mathscr{E}_k \mapsto \mathscr{E}_k \,. \tag{40}$$

Note that ϕ_L is a group automorphism of $\mathscr{G}_k$, but in general not of $\mathscr{E}_k$. In particular we find that ϕ_L has the following properties: let S and $\tilde{S}$ be reversing and M and $\tilde{M}$ non-reversing elements of $\mathscr{E}_k$, then for all $n \in \mathbb{Z}$

$$\begin{aligned}
&\text{(i)}\ \ \phi_L^n(M \circ \tilde{M}) = \phi_L^n(M) \circ \phi_L^n(\tilde{M}) \,, \\
&\text{(ii)}\ \ \phi_L^n(S \circ \tilde{S}) = \phi_L^n(S) \circ \phi_L^{-n}(\tilde{S}) \,, \\
&\text{(iii)}\ \ \phi_L^n(M \circ S) = \phi_L^n(M) \circ \phi_L^n(S) \,, \\
&\text{(iv)}\ \ \phi_L^n(S \circ M) = \phi_L^n(S) \circ \phi_L^{-n}(M) \,.
\end{aligned} \tag{41}$$

Having given some intuitive ideas, we will now present some more formal structures.

Definition 3.1. $U \in \mathscr{E}_{\tilde{k}}$ is called a (reversing) k-symmetry of L if k is the smallest positive integer such that

$$\phi_L^k(U) = U \,. \tag{42}$$

We denote this as

$$\#_L(U) = k \,. \tag{43}$$

Note that it follows that k divides $\tilde{k}$.

Proposition 3.1. Let U and $\tilde{U}$ be elements of the reversing symmetry group $\mathscr{E}_l$ of L^l, and let $\#_L(U) = k$ and $\#_L(\tilde{U}) = \tilde{k}$. Then $\#_L(U \circ \tilde{U})$ divides $\mathrm{lcm}(k, \tilde{k})$, i.e. the least common multiple of k and $\tilde{k}$.

Proof. This is a direct consequence of (41). □

Considering a (reversing) k-symmetry U of L, we find that the dynamics of the map L relates elements of $\mathscr{E}_k$ via the map ϕ_L. In this context we define the *orbit of U under* ϕ_L:

Definition 3.2. The *orbit* of a (reversing) k-symmetry U under ϕ_L is defined to be the set

$$\Phi_L(U) := \{\phi_L^i(U)\}_{i=0}^{k-1} \,. \tag{44}$$

The natural environment of an orbit of a (reversing) k-symmetry under ϕ_L is the *orbit group* in which it is embedded.

Definition 3.3. The orbit group of a (reversing) k-symmetry U of a map L, denoted $\mathscr{O}_L(U)$, is the smallest group that contains all (reversing) symmetries of the orbit of U under ϕ_L, i.e.

$$\mathscr{O}_L(U) := \langle U, \phi_L(U), \ldots, \phi_L^{k-1}(U) \rangle \,. \tag{45}$$

Similarly the orbit group of more (reversing) k_i-symmetries U_i $(i = 1, \ldots, n)$ is defined as

$$\mathscr{O}_L(U_1, \ldots, U_n) := \langle \mathscr{O}_L(U_1), \ldots, \mathscr{O}_L(U_n) \rangle \,. \tag{46}$$

Note that $\mathscr{O}_L(U_1, \ldots, U_n) \leqslant \mathscr{E}_k$ with $k = \mathrm{lcm}(k_1, \ldots, k_n)$.

The importance of the notion of orbit groups is illustrated by the following proposition that extends the findings of Proposition 3.1.

Proposition 3.2. Let U and $\tilde{U}$ be elements of the reversing symmetry group $\mathscr{E}_l$ of L^l, and let $\#_L(U) = k$ and $\#_L(\tilde{U}) = \tilde{k}$. Then the following implications hold:

$$\mathscr{O}_L(U) \cap \mathscr{O}_L(\tilde{U}) = \{\mathrm{Id}\} \Rightarrow \#_L(U \circ \tilde{U}) = \mathrm{lcm}(k, \tilde{k}) \,, \tag{47}$$

$$\tilde{k} = 1 \Rightarrow \#_L(U \circ \tilde{U}) = k \,, \tag{48}$$

$$k = 1 \Rightarrow \#_L(U \circ \tilde{U}) = \tilde{k} \,. \tag{49}$$

Proof. Let $\#_L(U \circ \tilde{U}) = q$, where q divides $\mathrm{lcm}(k, \tilde{k})$ (cf. Proposition 3.1), then we find using (41)

$$\phi_L^q(U \circ \tilde{U}) = \phi_L^q(U) \circ \phi_L^{\pm q}(\tilde{U}) = U \circ \tilde{U} \,. \tag{50}$$

This implies that there exists an $A \in \mathscr{E}_k$ such that

$$\phi_L^q(U) = U \circ A \,, \qquad \phi_L^{\pm q}(\tilde{U}) = A^{-1} \circ \tilde{U} \,, \tag{51}$$

and it follows that

$$A \in \mathscr{O}_L(U) \cap \mathscr{O}_L(\tilde{U}) \,. \tag{52}$$

Hence, in case $A = \mathrm{Id}$, we immediately find that $\#_L(U \circ \tilde{U}) = \mathrm{lcm}(k, \tilde{k})$. In case $\tilde{k} = 1$, or $k = 1$, we automatically have $A = \mathrm{Id}$. □

The most important property of an orbit group is the fact that it is mapped onto itself by ϕ_L. Moreover, each subgroup of $\mathscr{E}_k$ (for some k) that is closed under ϕ_L is in fact an orbit group of some (reversing) k-symmetries. To each such a group we can assign a number.

Definition 3.4. Let $\mathscr{D}$ be a subgroup of $\mathscr{E}_k$, i.e. the reversing symmetry group of L^k. Then $\#_L(\mathscr{D})$ is defined to be the smallest positive integer such that

$$\phi_L^{\#_L(\mathscr{D})}(U) = U \tag{53}$$

for all $U \in \mathscr{D}$. Note that $\#_L(\mathscr{D})$ is a divisor of k.

With this definition it follows that if U is a k-symmetry of a map L,

$$\#_L(\mathcal{O}_L(U)) = k\,. \tag{54}$$

Definition 3.5. If the symmetry group $\mathcal{G}_k$ of L^k satisfies $\#_L(\mathcal{G}_k) = k$, then $\mathcal{G}_k$ will also be called the *k-symmetry group of L*. Analogously, the reversing symmetry group $\mathcal{E}_k$ of L^k will also be called the *reversing k-symmetry group of L* if $\#_L(\mathcal{E}_k) = k$.

We can regard $\mathcal{E}_k$ as being composed of a reversing and a non-reversing part

$$\mathcal{E}_k = \mathcal{G}_k \cup S\mathcal{G}_k\,, \tag{55}$$

with S an arbitrary reversing element of $\mathcal{E}_k$. This implies that if a map possesses a reversing symmetry, it automatically possesses as many symmetries as reversing symmetries.

In practice it is hard and sometimes inconvenient to consider the entire (reversing) k-symmetry group of a map. In most cases we only deal with subgroups of the full (reversing) k-symmetry groups. In considering subgroups of the full (reversing) k-symmetry groups it is important to distinguish between subgroups that are closed under ϕ_L (like $\mathcal{E}_k$) and subgroups that are not.

Definition 3.6. In contrast to <u>*the*</u> k-symmetry group of a map L, we define <u>*a*</u> k-symmetry group $\mathcal{H}$ of L to be a subgroup of the symmetry group $\mathcal{G}_k$ of L^k, such that $\mathcal{H}$ is closed under ϕ_L and $\#_L(\mathcal{H}) = k$. Along the same lines we call $\mathcal{F}$ *a* reversing k-symmetry group if $\mathcal{F}$ is a subgroup of the reversing symmetry group $\mathcal{E}_k$ of L^k that is closed under ϕ_L and satisfies $\#_L(\mathcal{F}) = k$.

Note that every subgroup of the (reversing) k-symmetry group of L is a (reversing) symmetry group of L^k but not necessarily a (reversing) $\tilde{k}$-symmetry group of L, for some $\tilde{k}$ (dividing k); only the subgroups that are orbit groups (i.e. closed under ϕ_L) are called (reversing) $\tilde{k}$-symmetry groups.

Considering the composition properties of (reversing) k_i-symmetries we find that

$$\mathcal{G}_{\mathbb{N}} := \bigcup_{k\in\mathbb{N}} \mathcal{G}_k\,, \tag{56}$$

and

$$\mathcal{E}_{\mathbb{N}} := \bigcup_{k\in\mathbb{N}} \mathcal{E}_k\,, \tag{57}$$

are groups under composition. It is easily shown that the composition of a reversing element of $\mathcal{E}_{k_1}$ and a reversing element of $\mathcal{E}_{k_2}$ is an element of $\mathcal{G}_{\mathrm{lcm}(k_1,k_2)}$. The same conclusion follows for two non-reversing elements of reversing symmetry groups with different values of k. The composition of a reversing element of $\mathcal{E}_{k_1}$ and a non-reversing element of $\mathcal{E}_{k_2}$ is a reversing element of $\mathcal{E}_{\mathrm{lcm}(k_1,k_2)}$. If k_1 divides k_2 then $\mathcal{E}_{k_1} \leqslant \mathcal{E}_{k_2}$ and vice-versa. This fact underlies the subgroup structure of $\mathcal{E}_{\mathbb{N}}$.

After introducing the generalizations of (reversing) symmetries to (reversing) k-symmetries, we would like to see to what extent the celebrated FSI method for finding (periodic) orbits that are symmetric with respect to reversing symmetries can be extended to the case of reversing k-symmetries.

Before we consider extensions of the FSI method, we first have a close look at the way in which (reversing) k-symmetries manifest themselves in the dynamics.

Suppose Δ is an invariant set of L. Then if U is a (reversing) symmetry it follows that $U\Delta$ is an invariant set of L too, and we say that Δ is symmetric with respect to U if and only if $U\Delta = \Delta$. However, in case U is a (reversing) k-symmetry the ordinary notion of 'symmetric' is inappropriate, as illustrated by the following proposition.

Proposition 3.3. Let U be a (reversing) k-symmetry of a map L, then

$$U\Gamma(\boldsymbol{x}_0) = \Gamma(\boldsymbol{x}_0) \Leftrightarrow \quad \forall n \in \mathbb{Z},\ \phi_L^n(U)\,\Gamma(\boldsymbol{x}_0) = \Gamma(\boldsymbol{x}_0)\,. \tag{58}$$

Let S be a reversing k-symmetry of a map L, then

$$S\Gamma(\boldsymbol{x}_0) = \Gamma(\boldsymbol{x}_0)$$
$$\Rightarrow \forall n, \exists m \text{ such that } \boldsymbol{x}_0 \in \mathrm{Fix}(\phi_L^m(S) \circ \phi_L^n(S)) . \tag{59}$$

Proof. In proving (58), consider first the case that U is a reversing k-symmetry S. Then

$$S\Gamma(\boldsymbol{x}_0) = \Gamma(\boldsymbol{x}_0)$$
$$\Leftrightarrow \forall n \in \mathbb{Z}, \exists m \in \mathbb{Z} \text{ such that}$$
$$S \circ L^n \boldsymbol{x}_0 = L^m \circ L^n \boldsymbol{x}_0 , \tag{60}$$
$$\Leftrightarrow \forall n \in \mathbb{Z}, \exists m \in \mathbb{Z} \text{ such that}$$
$$\phi_L^n(S)\boldsymbol{x}_0 = L^m \boldsymbol{x}_0 . \tag{61}$$

Furthermore, for all integers n and p there exists an integer m such that

$$\phi_L^n(S) \circ L^p \boldsymbol{x}_0 = L^{-p} \circ \phi_L^{n+p}(S)\boldsymbol{x}_0$$
$$= L^{-p} \circ L^m \boldsymbol{x}_0 = L^{m-2p} \circ L^p \boldsymbol{x}_0 . \tag{62}$$

Hence,

$$\forall n, p \in \mathbb{Z}, \exists m \in \mathbb{Z} \text{ such that}$$
$$\phi_L^n(S) \circ L^p \boldsymbol{x}_0 = L^m \circ L^p \boldsymbol{x}_0 . \tag{63}$$

In case U is a k-symmetry, the proof of (58) is analogous.

From (61) we find that for all integers n there exists an integer m such that

$$L^m \circ \phi_L^n(S)\boldsymbol{x}_0 = \boldsymbol{x}_0 , \tag{64}$$

implying (59)[8]. □

This proposition indicates that the ordinary notion of 'symmetric' is very restrictive when dealing with (reversing) k-symmetries. This originates from the fact that the image under a (reversing) k-symmetry U of an invariant set Δ, i.e. $U\Delta$, does not have to be an invariant set of L.

A more useful property becomes clear if one considers orbits of points in state space that are in the fixed set $\mathrm{Fix}(L^m \circ S)$ (for some integer value of m). These orbits have special symmetry properties.

Proposition 3.4. Let S be a reversing k-symmetry of a map L, then

$$\exists m \in \mathbb{Z} \text{ such that } \boldsymbol{x}_0 \in \mathrm{Fix}(L^m \circ S)$$
$$\Leftrightarrow \forall \boldsymbol{x}_i \in \Gamma(\boldsymbol{x}_0), \exists p \in \mathbb{Z} \text{ such that}$$
$$\phi_L^p(S)\boldsymbol{x}_i \in \Gamma(x_0) . \tag{65}$$

Proof.

$$\boldsymbol{x}_0 = L^m \circ S\boldsymbol{x}_0 \Leftrightarrow$$
$$\forall p \in \mathbb{Z}, \phi_L^{-p}(S) \circ L^p \boldsymbol{x}_0 = L^{-m-p}\boldsymbol{x}_0 \tag{66}$$

□

The symmetry property of orbits that is observed in the previous proposition is a natural generalization of the ordinary notion of 'symmetric' to *k-symmetric*.

Definition 3.7. An orbit $\Gamma(\boldsymbol{x}_0)$ is *k-symmetric* with respect to a (reversing) k-symmetry U if there is a point $\boldsymbol{x} \in \Gamma(\boldsymbol{x}_0)$ such that $U\boldsymbol{x} \in \Gamma(\boldsymbol{x}_0)$.

As in the case of ordinary (reversing) symmetries, we find again that all symmetry properties of an orbit are completely specified by giving a point on the orbit and (the generators of) the isotropy subgroup of that point. In the case of (reversing) k-symmetries however, in general, the isotropy subgroup of a point is not an orbit group, and therefore not a (reversing) k-symmetry group.

From Proposition 3.4 it follows that $\mathrm{Fix}(L^m \circ S)$ plays an important role in the analysis of k-symmetric orbits. As a generalization of (19) and (20) we find $\forall n, p \in \mathbb{Z}$

$$\mathrm{Fix}(L^{2n} \circ \phi_L^{p-n}(S)), = L^n(\mathrm{Fix}(\phi_L^p(S))) , \tag{67}$$

$$\mathrm{Fix}(L^{2n+1} \circ \phi_L^{p-n}(S)) = L^n(\mathrm{Fix}(L \circ \phi_L^p(S))) . \tag{68}$$

These properties allow us to generalize the FSI method to reversing k-symmetries.

[8] Note that (59) is a generalization of Proposition 2.3.

Theorem 3.5 (*fixed set iteration* (*FSI*) *method*). All points of periodic orbits of a map L that are k-symmetric with respect to a reversing k-symmetry S lie on intersections of iterates of $\mathrm{Fix}(\phi_L^p(S))$ and/or $\mathrm{Fix}(L\circ\phi_L^q(S))$ for some integer value of p and q. Moreover, an orbit $\Gamma(\boldsymbol{x}_0)$ is k-symmetric with respect to the reversing k-symmetry S if and only if $\Gamma(\boldsymbol{x}_0)$ has at least one point on either $\mathrm{Fix}(L\circ\phi_L^p(S))$ or $\mathrm{Fix}(\phi_L^p(S))$ for some integer value of p.

The above result generalizes the basis of the FSI method for finding orbits that are symmetric with respect to reversing symmetries, as discussed in the previous section, to a similar method for finding orbits that are k-symmetric with respect to reversing k-symmetries.

We will now focus on the dynamical implications of the intersections of the fixed sets. Firstly, we find as a direct generalization of Proposition 2.3 a condition on the domain in which the fixed sets can occur.

Proposition 3.6. Let S be a reversing k-symmetry of map L, then

$$\boldsymbol{x}_0\in\mathrm{Fix}(L^m\circ S)\Rightarrow\boldsymbol{x}_0\in\mathrm{Fix}(\phi_L^m(S)\circ S)\,. \tag{69}$$

Proof.

$$L^m\circ S\boldsymbol{x}_0=\boldsymbol{x}_0\Rightarrow L^m\circ S\circ L^m\circ S\boldsymbol{x}_0=\boldsymbol{x}_0\,. \tag{70}$$

□

Hence, points of orbits that are k-symmetric with respect to a reversing k-symmetry S are in $\mathrm{Fix}(\phi_L^m(S)\circ\phi_L^n(S))$ for some integer value of m and n.

Now let us consider what we can say about the dynamics of points that are in the intersections of the fixed sets. We consider right away the case of a reversing k-symmetry S and a reversing $\tilde{k}$-symmetry $\tilde{S}$ from which the case of one reversing k-symmetry can easily be deduced, setting $\tilde{S}$ equal to S. The following theorem generalizes Theorem 2.7.

Theorem 3.7. Let $\boldsymbol{x}_0$ be a point on an orbit that is k-symmetric with respect to the reversing k-symmetry S and $\tilde{k}$-symmetric with respect to the reversing $\tilde{k}$-symmetry $\tilde{S}$, such that

$$S\boldsymbol{x}_0=L^{-n}\boldsymbol{x}_0\quad\text{and}\quad\tilde{S}\boldsymbol{x}_0=L^{-m}\boldsymbol{x}_0\,. \tag{71}$$

Let furthermore $\tau_{n,m}(S,\tilde{S})$ be the greatest common divisor (gcd) of all integers τ, satisfying either

$$\prod_{j=1}^{\tau}[\phi_L^{(\tau-j)(n-m)+n}(\tilde{M}_j\circ\tilde{S})\circ\phi_L^{(\tau-j)(m-n)}(M_j\circ S)]=\mathrm{Id}\,, \tag{72}$$

or

$$\prod_{j=1}^{\tau}[\phi_L^{(\tau-j)(m-n)+m}(M_j\circ S)\circ\phi_L^{(\tau-j)(n-m)}(\tilde{M}_j\circ\tilde{S})]=\mathrm{Id}\,, \tag{73}$$

for any $M_j\in\mathscr{H}$ and $\tilde{M}_j\in\tilde{\mathscr{H}}$, where

$$\mathscr{H}:=\langle S\circ\phi_L^n(S),\phi_L^{m-n}(\tilde{S})\circ\phi_L^n(\tilde{S})\rangle\,, \tag{74}$$

$$\tilde{\mathscr{H}}:=\langle\tilde{S}\circ\phi_L^m(\tilde{S}),\phi_L^{n-m}(S)\circ\phi_L^m(S)\rangle\,. \tag{75}$$

Then, if $\tau_{n,m}(S,\tilde{S})<\infty$ and $m\neq n$, $\Gamma(\boldsymbol{x}_0)$ is a periodic orbit with a period that divides $(n-m)\tau_{n,m}(S,\tilde{S})$.

Proof. Given (71), we find

$$L^{n-m}\boldsymbol{x}_0=L^n\circ\tilde{S}\boldsymbol{x}_0=\phi_L^n(\tilde{S})\circ L^{-n}\boldsymbol{x}_0=\phi_L^n(\tilde{S})\circ S\boldsymbol{x}_0\,, \tag{76}$$

and in a similar way

$$L^{m-n}\boldsymbol{x}_0=\phi_L^m(S)\circ\tilde{S}\boldsymbol{x}_0\,. \tag{77}$$

Hence we obtain

$$L^{\tau(n-m)}\boldsymbol{x}_0=L^{(\tau-1)(n-m)}\circ\phi_L^n(\tilde{S})\circ S\boldsymbol{x}_0\,, \tag{78}$$

$$=\phi_L^{(\tau-1)(n-m)}(\phi_L^n(\tilde{S})\circ S)L^{(\tau-1)(n-m)}\boldsymbol{x}_0\,, \tag{79}$$

$$=\prod_{j=1}^{\tau}[\phi_L^{(\tau-j)(n-m)}(\phi_L^n(\tilde{S})\circ S)]\boldsymbol{x}_0\,. \tag{80}$$

With the help of the relations (41), this can be written as

$$L^{\tau(n-m)}\boldsymbol{x}_0 = \prod_{j=1}^{\tau} [\phi_L^{(\tau-j)(n-m)+n}(\tilde{S}) \circ \phi_L^{(\tau-j)(m-n)}(S)]\boldsymbol{x}_0 \,. \tag{81}$$

A more stringent condition is found, if one realizes that, from Proposition 3.6 and (71), it follows that

$$M \circ S\boldsymbol{x}_0 = L^{-n}\boldsymbol{x}_0 \quad \text{and} \quad \tilde{M} \circ \tilde{S}\boldsymbol{x}_0 = L^{-m}\boldsymbol{x}_0 \,, \tag{82}$$

for all $M \in \mathscr{H}$ and $\tilde{M} \in \tilde{\mathscr{H}}$. Incorporating this fact, leads us to (72). In an analogous way we derived condition (73). □

Note that it follows directly from the above proposition that

$$\tau_{n,m}(S, \tilde{S}) = \tau_{m,n}(\tilde{S}, S) \,, \tag{83}$$

$$\tau_{n,m}(S, S) = 1 \,. \tag{84}$$

If the orbit group $\mathcal{O}_L(S, \tilde{S})$ is finite, then $\tau_{n,m}(S, \tilde{S})$ is finite too. This implies that, in case $n \neq m$, the intersection points of the fixed sets are points of k- and $\tilde{k}$-symmetric period orbits[9].

In discussing the k-symmetry properties of a map L possessing a (reversing) k-symmetry, it is of interest to compare k-symmetric orbits of L with symmetric orbits of L^k. In this respect it is easily checked that if an orbit $\Gamma^{(k)}(\boldsymbol{x}_0)$ of L^k is symmetric with respect to a (reversing) k-symmetry U of L, then $\Gamma(\boldsymbol{x}_0)$ is k-symmetric with respect to U. However, if $\Gamma(\boldsymbol{x}_0)$ is k-symmetric with respect to a (reversing) k-symmetry U, it is not necessarily so that $\Gamma^{(k)}(\boldsymbol{x}_0)$ is symmetric with respect to U.

Let us consider an orbit $\Gamma(\boldsymbol{x}_0)$ that is k-symmetric with respect to a (reversing) k-symmetry U, such that (without loss of generality)

$$U\boldsymbol{x}_0 = L^m\boldsymbol{x}_0 \,, \tag{85}$$

for some $m \in \mathbb{Z}$. Furthermore, let $\Gamma^{(k)}(\boldsymbol{x}_0)$ be the orbit of $\boldsymbol{x}_0$ under the map L^k. Then it is obvious that $\Gamma^{(k)}(\boldsymbol{x}_0)$ is symmetric with respect to $L^{-m} \circ U$, because of the fact that L is a symmetry of L^k.

Proposition 3.8. Let $\Gamma(\boldsymbol{x}_0), \Gamma^{(k)}(\boldsymbol{x}_0)$ and U be defined as above, satisfying (85). Then $\Gamma^{(k)}(\boldsymbol{x}_0)$ is symmetric with respect to U if and only if

$$\exists j \in \mathbb{Z}, \text{ such that } (m + kj) = 0 \bmod p \,, \tag{86}$$

where p is the period of $\Gamma(\boldsymbol{x}_0)$ (read $p = \infty$ in case $\Gamma(\boldsymbol{x}_0)$ is nonperiodic).

Proof. If $\Gamma^{(k)}(\boldsymbol{x}_0)$ is symmetric with respect to U then

$$\forall j \in \mathbb{Z}, \exists q \in \mathbb{Z} \text{ such that } U \circ L^{kj}\boldsymbol{x}_0 = L^{kq}\boldsymbol{x}_0 \,. \tag{87}$$

From (85), however, it follows that

$$U \circ L^{kj}\boldsymbol{x}_0 = L^{m \pm kj}\boldsymbol{x}_0 \,, \tag{88}$$

where $\pm$ depends on U being reversing or not. This implies that $\Gamma^{(k)}(\boldsymbol{x}_0)$ is symmetric with respect to U if and only if

$$\forall j \in \mathbb{Z}, \exists q, t \in \mathbb{Z} \text{ such that } m \pm kj + pt = kq \,. \tag{89}$$

This is equivalent to

$$\exists j \in \mathbb{Z} \text{ such that } (m + kj) = 0 \bmod p \,. \tag{90}$$

□

An application of the above phenomenon is discussed in section 4.2.

The following proposition discusses some implications that follow in the case that $\Gamma(\boldsymbol{x}_0)$ is k-symmetric with respect to a (reversing) k-symmetry U and at the same time $\Gamma^{(k)}(\boldsymbol{x}_0)$ is symmetric with respect to U.

Proposition 3.9. Let $\Gamma(\boldsymbol{x}_0)$ be k-symmetric with respect to a (reversing) k-symmetry U and let $\Gamma^{(k)}(\boldsymbol{x}_0)$ be symmetric with respect to U. Then
– in case U is a k-symmetry:

[9] Note that this remark generalizes the remark given below Theorem 2.7.

$\forall i \in \mathbb{Z}, \Gamma^{(k)}(L^i \boldsymbol{x}_0)$ is symmetric with respect to

$$\phi_L^i(U) . \tag{91}$$

– in case U is a reversing k-symmetry: let $i \in \mathbb{Z}$, then

$$\Gamma^{(k)}(L^i \boldsymbol{x}_0) \text{ is symmetric with respect to } \phi_L^{-i}(U) \text{ if and only if } \frac{2i}{k} \in \mathbb{Z} . \tag{92}$$

Proof. Considering the proof of Proposition 3.8, it is easily shown that, in case U is a k-symmetry, (87) implies

$$\forall j \in \mathbb{Z}, \exists q \in \mathbb{Z} \text{ such that } \phi_L^i(U) \circ L^{kj+i} \boldsymbol{x}_0 = L^{kq+i} \boldsymbol{x}_0 . \tag{93}$$

In case U is a reversing k-symmetry, (87) leads to

$$\forall j \in \mathbb{Z}, \exists q \in \mathbb{Z} \text{ such that } \phi_L^{-i}(U) \circ L^{kj+i} \boldsymbol{x}_0 = L^{kq-i} \boldsymbol{x}_0 . \tag{94}$$

From the latter equation it follows that $\Gamma^{(k)}(L^i \boldsymbol{x}_0)$ is symmetric with respect to $\phi_L^{-i}(U)$ if and only if $2i/k$ is an integer. □

One could raise the question whether in case $\Gamma^{(k)}(\boldsymbol{x}_0)$ is not symmetric with respect to U, at least one point on the orbit of $\boldsymbol{x}_0$ will be symmetric with respect to an element of the orbit of U under ϕ_L. A natural candidate for the point $L^i \boldsymbol{x}_0$ is $\phi_L^{\pm i}(U)$, where the plus sign refers to the case that U is a k-symmetry, and the minus sign to the case that U is a reversing k-symmetry. The following proposition deals with this question.

Proposition 3.10. Let $\Gamma(\boldsymbol{x}_0), \Gamma^{(k)}(\boldsymbol{x}_0)$ and U be defined as above, satisfying (85). Let moreover $\Gamma^{(k)}(\boldsymbol{x}_0)$ not be symmetric with respect to U. Then

- in case U is a k-symmetry: for all $i \in \mathbb{Z}$, $\Gamma^{(k)}(L^i \boldsymbol{x}_0)$ is not symmetric with respect to $\phi_L^i(U)$.
- in case U is a reversing k-symmetry: there is always an $i \in \mathbb{Z}$ such that $\Gamma^{(k)}(L^i \boldsymbol{x}_0)$ is symmetric with respect to $\phi_L^{-i}(U)$ unless m is odd, k is even, and (in case $\Gamma(\boldsymbol{x}_0)$ is periodic) the period of $\Gamma(\boldsymbol{x}_0)$ is even.

Proof. The part of this proposition in which U is a k-symmetry follows as a corollary from Proposition 3.9.

In case U is a reversing k-symmetry, we obtain from (94) and Proposition 3.8 that $\Gamma^{(k)}(L^i \boldsymbol{x}_0)$ is symmetric with respect to $\phi_L^{-i}(U)$ if and only if

$$\exists j, i \in \mathbb{Z} \text{ such that } (m + kj + 2i) = 0 \bmod p . \tag{95}$$

The latter condition is always satisfied unless m is odd and k and p are even. □

Finally, we would like to emphasize that the importance of (reversing) k-symmetries for dynamical phenomena is not restricted to the FSI method for finding periodic orbits. For instance, they also determine the symmetries observed in the global phase portrait of a dynamical system. In fact, if $\mathscr{P} \in \Omega$ is the set of all (points on) periodic orbits of L, and $\mathscr{C} \in \Omega$ is the set of all (points on) chaotic orbits of L, then for all $U \in \mathscr{E}_{\mathbb{N}}$,

$$U\mathscr{P} = \mathscr{P} , \tag{96}$$

$$U\mathscr{C} = \mathscr{C} . \tag{97}$$

In fact, in [10], symmetry properties of stochastic webs were understood in the spirit of (97)[10].

3.1. Example

After having discussed several properties of dynamical systems possessing (reversing) k-symmetries, we will now concentrate on a simple

[10] For discussions on the way (reversing) symmetries manifest themselves in the global phase portrait, see e.g. also [24,25].

example. On this example we will demonstrate the use of the FSI method.

Let us consider the map L on the plane, given by[11]

$$L: \begin{cases} x' = \lambda \sin(\pi x) - y \\ \qquad + \kappa \sin\{\pi x - \pi\kappa \sin[\pi y - \pi\lambda \sin(\pi x)]\} , \\ y' = -x + \kappa \sin(\pi y - \pi\lambda \sin(\pi x)) \\ \qquad - \lambda \sin(\pi x') , \end{cases} \tag{98}$$

where $\kappa, \lambda \in \mathbb{R}$ are parameters. The map L possesses the mirrors in the x-axis, M_x, and y-axis, M_y, as reversing 2-symmetries. In fact, they form an orbit under ϕ_L:

$$\phi_L^2(M_x) = \phi_L(M_y) = M_x . \tag{99}$$

As a result of this, $-\mathrm{Id}$ is a symmetry of L. Moreover, (98) possesses the property that the translations with translation vector $(1, 1)$ and $(1, -1)$ are reversing 2-symmetries of L[12].

In Fig. 2 and Fig. 3 we present the global phase portraits of (98) in case $\lambda = -0.2$ and $\kappa = 0.2$, 0.34. Notice the way the (reversing) k-symmetries, that we discussed above, are apparent in these figures. The stochastic web, i.e. the collection of thin stochastic regions providing connections between saddle points all over the plane, possesses these (reversing) k-symmetries too (cf. [10]). Note that the symmetry of the web does not completely determine its topology. Comparing for instance the stochastic webs in Fig. 2 and Fig. 3 one observes a dramatic change of the form of the web which can be interpreted as a bifurcation of the stochastic web.

In discussing the FSI method, our main concern will be with the two orthogonal mirrors M_x and M_y. In Fig. 4 we zoomed in on the phase portrait of (98) in case $\kappa = 0.34$ and $\lambda = -0.2$. In this figure we also plotted some iterates of $\mathrm{Fix}(M_x)$ (lines) and $\mathrm{Fix}(M_y)$ (dashed lines). From the fact that $\langle M_x, M_y \rangle$ is a finite group it immediately follows that all intersection points of these curves are points on periodic orbits.

More information on the periods of these points can be achieved by calculating $\tau_{n,m}(S, \tilde{S})$ for all $S, \tilde{S} \in \{M_x, M_y\}$ in the spirit of Theorem 3.7. In Table 2 the results of this calculation are presented.

4. Some examples from physics: the standard map, web maps, trace maps, and the kicked rotator

In this section we present some examples of dynamical systems arising in the context of problems in physics, possessing (reversing) k-symmetries. However, a detailed discussion of these examples is beyond the scope of the present paper. Here, we mainly want to emphasize the occurrence and consequences of the (reversing) k-symmetries in these dynamical systems.

4.1. The standard map

In the study of the dynamics of a charged particle in a kicking inhomogeneous electric field the nontrivial part of the dynamics in phase space is modelled by the so-called standard map [14].

$$L: \begin{cases} x' = x + y , \\ y' = y + f(x') , \end{cases} \tag{100}$$

where $f(x) = f(x+1)$. Let $U_{a,b}$ be defined as a translation in the plane with translation vector (a, b), i.e.

[11] The construction of this example is explained in Appendix B.

[12] In fact, (98) is an example of map, having a so-called stochastic web possessing crystallographic symmetries, cf. [10]. In this case the map has a reversing 2-symmetry group that is isomorphic to the dichromatic crystallographic group $p_c' gg$, yielding a web with crystallographic symmetry group cmm (for a clear exposition on these groups see e.g. [26]).

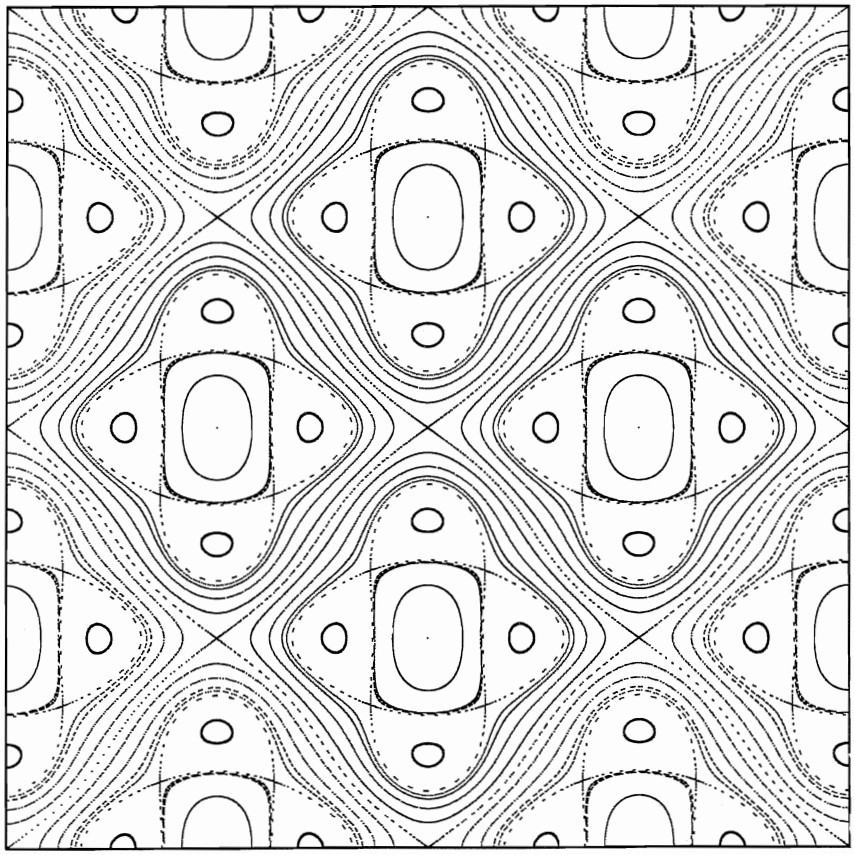

Fig. 2. Phase portrait of (98) with $\lambda = -0.2$ and $\kappa = 0.2$ on $[-2, 2] \times [-2, 2]$.

$$U_{a,b}: \begin{cases} x' = x + a \,, \\ y' = y + b \,. \end{cases} \tag{101}$$

Then for all $n, m \in \mathbb{Z}$ we have

$$U_{n,m} \circ L = L \circ U_{n-m,m} \,. \tag{102}$$

This implies that $U_{1,0}$ (and hence $U_{m,0}$ for all $m \in \mathbb{Z}$) is a symmetry of L.

Now we can perform a little trick by making use of the symmetry to define the map on a cylinder taking the x-part of the phase plane modulo k. Doing this, we find that $U_{0,1}$ is a k-symmetry and $U_{0,k}$ is a (1-)symmetry of the standard map defined on this cylindrical phase space. Hence, making use of the symmetry we can reduce the phase space further to the 2-torus $\mathbb{R}/k\mathbb{Z} \times \mathbb{R}/k\mathbb{Z}$. Choosing k equal to one, we obtain the usual reduction to the 2-torus $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, on which all translations $U_{n,m}$ $(n, m \in \mathbb{Z})$ are symmetries[13].

Besides these (k-)symmetries, L also possesses the reversing symmetry

$$S: \begin{cases} x' = x + y \,, \\ y' = -y \,. \end{cases} \tag{103}$$

Moreover, if f is odd, i.e. $f(x) = -f(-x)$, then $-\mathrm{Id}$ is an additional symmetry of the standard map.

Summarizing, the following groups are found to be reversing k-symmetry groups of the standard map L defined on $\mathbb{R}/k\mathbb{Z} \times \mathbb{R}/k\mathbb{Z}$:

$$f \text{ 1-periodic} \quad \langle S, U_{1,0}, U_{0,1} \rangle \,, \tag{104}$$

[13] In [27], this reduction is supported by a different argument. For a discussion of this other argument, see section 5.

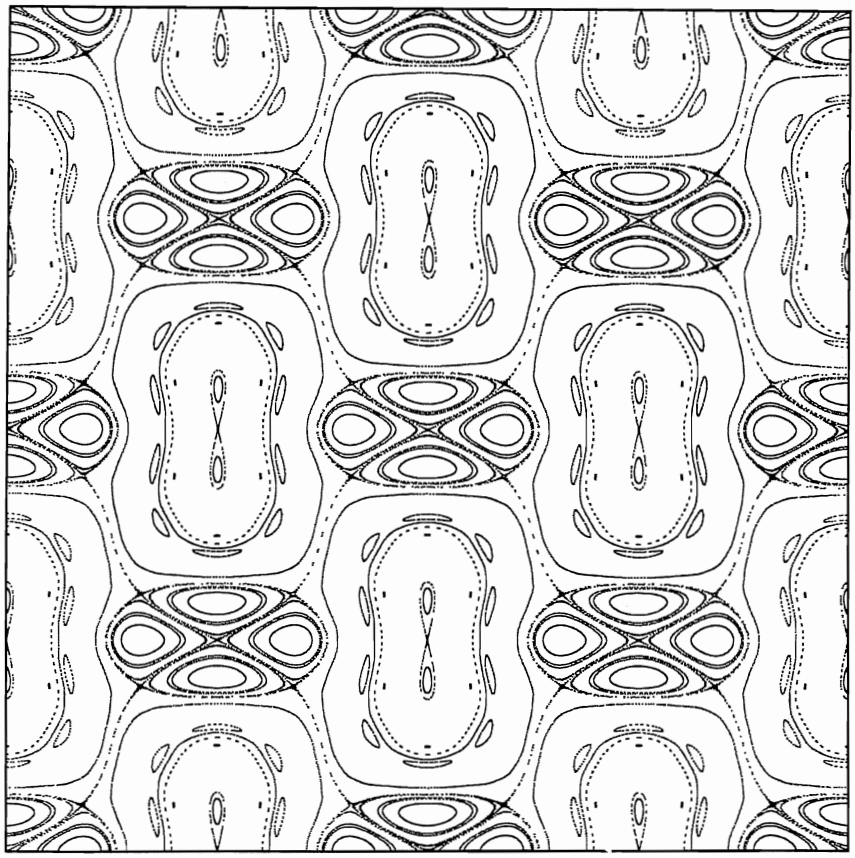

Fig. 3. Phase portrait of (98) with $\lambda = -0.2$ and $\kappa = 0.34$ on $[-2, 2] \times [-2, 2]$.

$$f \text{ 1-periodic and odd} \quad \langle S, -\mathrm{Id}, U_{1,0}, U_{0,1} \rangle \,. \tag{105}$$

On $\mathbb{R}/k \times \mathbb{R}/k$ we have the reversing $\tilde{k}$-symmetries ($\tilde{k}$ divides k) $S \circ U_{p,q}$ (if f is odd also $-S \circ U_{p,q}$). For each of these reversing $\tilde{k}$-symmetries there may be $\tilde{k}$-symmetric orbits. Note in this respect that the use of the reversing $\tilde{k}$-symmetries is limited by the fact that $\mathrm{Fix}(\phi_L^n(\pm S \circ U_{p,q}) \circ \pm S \circ U_{p,q})$ should be nonempty for some integer value of n (cf. Proposition 3.6). Here we find that

$$\mathrm{Fix}(\phi_L^n(S \circ U_{p,q}) \circ S \circ U_{p,q}) = \mathrm{Fix}(U_{2p+q(1-n),0}) \,, \tag{106}$$

$$\mathrm{Fix}(\phi_L^n(-S \circ U_{p,q}) \circ -S \circ U_{p,q}) = \mathrm{Fix}(U_{q(n-1),2q}) \tag{107}$$

yielding possibilities for the occurrence of $\tilde{k}$-symmetric orbits only if there exists an integer n such that $2p + (1-n)q = 0 \bmod k$ or (in case f is odd) $(q(n-1), 2q) = (0, 0) \bmod k$.

We will not analyze the occurrence of the $\tilde{k}$-symmetric orbits here in detail. They can be tracked down using the FSI method as discussed in section 3[14].

4.2. Web maps

The physical system leading to the standard map, may be extended with a homogeneous magnetic field. In that case, the nontrivial part of

[14] For an FSI analysis with respect to the reversing symmetry (103), see [20,21].

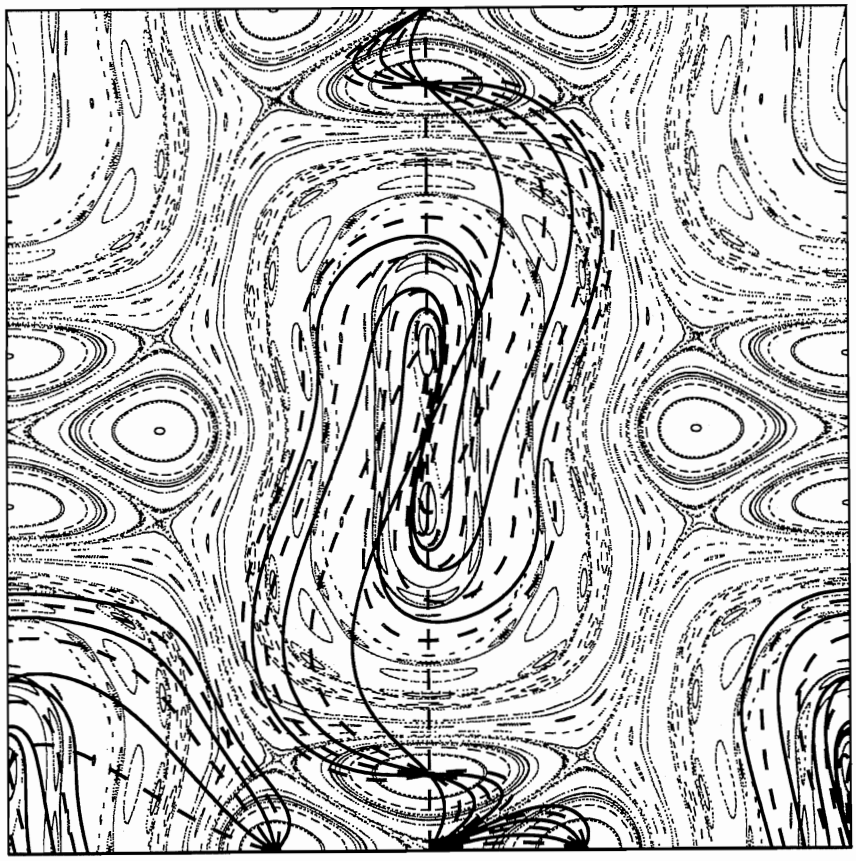

Fig. 4. Phase portrait of (98) with $\lambda = -0.2$ and $\kappa = 0.34$ on $[-1, 1] \times [0, 2]$. As an illustration of the FSI method for finding periodic orbits we plotted some iterates of $\mathrm{Fix}(M_x)$ (lines) and $\mathrm{Fix}(M_y)$ (dashed lines): the intersection points of these fixed sets are periodic points of (98).

the dynamics in phase space is modelled by the so-called web map [15][15]. This map is given by [15,12]

Table 2
The values of $\tau_{n,m}(S, \tilde{S})$ for all $S, \tilde{S} \in \{M_x, M_y\}$ that are reversing 2-symmetries of the map (98), cf. Theorem 3.2

		$\tau_{n,m}(M_x, M_x)$ $\tau_{n,m}(M_y, M_y)$	$\tau_{n,m}(M_x, M_y)$ $\tau_{n,m}(M_y, M_x)$
$n-m$ even	n, m even	1	2
	n, m odd	1	1
$n-m$ odd		1	1

[15] Web maps are named after their property of forming highly symmetric stochastic webs. However, there are more examples of maps that have this property, see e.g. section 3.1 and [29].

$$L_\alpha = R_\alpha \circ T \,, \tag{108}$$

where R_α and T are given by

$$R_\alpha\colon \begin{cases} x' = x\cos(\alpha) + y\sin(\alpha)\,, \\ y' = -x\sin(\alpha) + y\cos(\alpha)\,, \end{cases}$$

$$T\colon \begin{cases} x' = x + f(y)\,, \\ y' = y\,, \end{cases} \tag{109}$$

and $f(y) = f(y+1)$. In [10] it was shown that in case $\alpha = \alpha_{p,q} := 2\pi p/q$ with p and q coprime integers and $q \in \{1, 2, 3, 5, 6\}$, $L^q_{\alpha_{p,q}}$ possesses a reversing symmetry group that is a two-dimensional crystallographic space group. Moreover it was shown that these reversing symmetry groups are not reversing symmetry groups of $L_{\alpha_{p,q}}$ itself. Remains the question whether these crystallo-

graphic groups are reversing q-symmetry groups of $L_{\alpha_{p,q}}$.

Let us first recall the symmetries and reversing symmetries of L_α for arbitrary α. L_α is reversible, with reversing symmetry [10]

$$S = R_\alpha \circ M_y \,, \tag{110}$$

where M_y is the mirror in the y-axis, and has an additional symmetry $-\mathrm{Id}$ if f is odd, i.e. $f(-y) = -f(y)$.

We will analyze the case $q = 4$ and $\alpha_{p,q} = \pi/2$ in detail. In this case, the map is given by[16]

$$L_{\pi/2}: \begin{cases} x' = y \,, \\ y' = -x - f(y) \,. \end{cases} \tag{111}$$

For arbitrary (nonperiodic) f the only nontrivial (reversing) symmetry seems to be S, given by (110).

If f is unit-periodic, i.e. $f(y+1) = f(y)$, we find that for all $n, m, r \in \mathbb{Z}$

$$L^r_{\alpha_{p,q}} \circ U_{n,m} \circ L^{-r}_{\alpha_{p,q}} = R^r_{\alpha_{p,q}} \circ U_{n,m} \circ R^{-r}_{\alpha_{p,q}} \,, \tag{112}$$

where the translation $U_{n,m}$ is defined by (101). Hence it follows directly that the group of translations $\{U_{n,m}\}_{n,m\in\mathbb{Z}}$ is a 4-symmetry group of $L_{\alpha_{p,q}}$. There is more, however. We know that S, given by (110), is a reversing symmetry of $L_{\alpha_{p,q}}$. Moreover, from Proposition 3.2, it follows that $U_{n,m} \circ S$ are reversing 4-symmetries for all $n,m \in \mathbb{Z}$. These reversing 4-symmetries are of infinite order (unless $n = m = 0$). It follows that the group

$$\langle U_{1,0}, U_{0,1}, S \rangle \tag{113}$$

is a reversing 4-symmetry group of $L_{\alpha,p,q}$.

If f is odd, R_π is a symmetry of $L_{\alpha_{p,q}}$. Hence, if f is odd and unit-periodic the group

$$\langle U_{1,0}, U_{0,1}, S, R_\pi \rangle \tag{114}$$

is a reversing 4-symmetry group of $L_{\alpha_{p,q}}$. Note that $U_{n,m} \circ R_\pi$ is a rotation over π around the point $(n/2, m/2)$.

If f is odd and $f(y+1/2) = -f(y)$ (which implies that f is also unit-periodic) we find additional reversing 4-symmetries of order four. Let $\tilde{R}_{n,m}$ be defined as a rotation over $\pi/2$ around the point $(n/2, m/2)$ then it is easy to check that in case $n, m \in \mathbb{Z}$ and $(n+m)$ is odd

$$L_{\alpha_{p,q}} \circ \tilde{R}_{n,m} \circ L_{\alpha_{p,q}} = \tilde{R}^{-1}_{n,m} \,, \tag{115}$$

$$L_{\alpha_{p,q}} \circ \tilde{R}^{-1}_{n,m} \circ L_{\alpha_{p,q}} = \tilde{R}_{-n,-m} \,, \tag{116}$$

$$L_{\alpha_{p,q}} \circ \tilde{R}_{-n,-m} \circ L_{\alpha_{p,q}} = \tilde{R}^{-1}_{-n,-m} \,, \tag{117}$$

$$L_{\alpha_{p,q}} \circ \tilde{R}^{-1}_{-n,-m} \circ L_{\alpha_{p,q}} = \tilde{R}_{n,m} \,. \tag{118}$$

Similar relations are found in case $\alpha_{p,q} = -\pi/2$.

Furthermore, from Proposition 3.2 it follows that $\#_{\mathrm{L}}(\tilde{R}_{n,m} \circ U_{n',m'}) = 4$ for all $n, m, n', m' \in \mathbb{Z}$ with $n+m$ odd, because $\mathcal{O}_L(\tilde{R}_{n,m}) \cap \mathcal{O}_L(U_{n',m'}) = \{\mathrm{Id}\}$.

Hence, we find in case f is odd and $f(y+1/2) = -f(y)$ that

$$\langle U_{1,0}, U_{0,1}, S, R_\pi, \tilde{R}_{1,0} \rangle \tag{119}$$

is a reversing 4-symmetry group of $L_{\alpha_{p,q}}$ (in case $\alpha_{p,q} = \pm\pi/2$).

Let us consider the noninvolutory reversing 4-symmetries $\tilde{R}_{n,m}$. An orbit of $L_{\alpha_{p,q}}$ is 4-symmetric with respect to such a reversing 4-symmetry if and only if it has a point either on $\mathrm{Fix}(\tilde{R}_{n,m})$ or on $\mathrm{Fix}(L_{\alpha_{p,q}} \circ \tilde{R}_{n,m})$, for some $n, m \in \mathbb{Z}$ such that $n+m$ is odd.

We have

$$\mathrm{Fix}(\tilde{R}_{n,m}) = (n/2, m/2) \,. \tag{120}$$

The 4-symmetric orbits with a point on $(n/2, m/2)$, with $n+m$ odd, are points of period four. We even find for all r, n, m in $\mathbb{Z}$

$$L^r_{\pm\pi/2}(n/2, m/2) = R^r_{\pm\pi/2}(n/2, m/2) \,. \tag{121}$$

In fact, the point $(n/2, m/2)$ is precisely the only point in the phase space of $L^4_{\pm\pi/2}$ that is symmetric with respect to its reversing symmetry $\tilde{R}_{n,m}$. No other points can have this property, because

[16] The standard map is conjugate to a map of the form (111). A more complete study of the reversing k-symmetries of (111) will be given in a forthcoming paper [28].

they have to be in $\mathrm{Fix}(\tilde{R}^2_{n,m})$ (cf. Proposition 2.3).

Considering the other relevant fixed sets, we find in case $\alpha_{p,q} = \pi/2$ that

$$\mathrm{Fix}(L_{\alpha_{p,q}} \circ \tilde{R}_{n,m}) = ((m+n)/4, (m-n)/4 - f((m+n)/4)) . \tag{122}$$

Hence, the orbit of the point $((m+n)/4, (m-n)/4 - f((m+n)/4))$ under $L_{\alpha_{p,q}}$ is 4-symmetric with respect to $\tilde{R}_{n,m}$, with $n+m$ odd. However, since these points are not in $\mathrm{Fix}(\tilde{R}^2_{n,m})$, for any $n, m \in \mathbb{Z}$ with $n+m$ odd, we know that the orbits of these points under $L^4_{\alpha_{p,q}} r$ are not symmetric with respect to $\tilde{R}_{n,m}$, for any $n, m \in \mathbb{Z}$ with $n+m$ odd.

At this point let us illustrate the use of Proposition 3.8. From this proposition it follows that if the orbit of $((m+n)/4, (m-n)/4 - f((m+n)/4))$ under $L_{\pi/2}$ is periodic, with period p, then

$$\text{there exists no } j \in \mathbb{Z}, \text{ such that } (4j-1) = 0 \bmod p . \tag{123}$$

It is easy to check that the only restriction on p following from (123) is that p must be even. In case $\alpha_{p,q} = -\pi/2$ we find similar results.

Hence for every 4-symmetry $\tilde{R}_{n,m}$ (with $n+m$ odd), there are only two 4-symmetric orbits. One (generically hyperbolic [10]) orbit of period four, corresponding to a symmetric orbit of $L^4_{\pm\pi/2}$ and one orbit that does not correspond to a symmetric orbit of $L^4_{\pm\pi/2}$ that cannot have an odd period.

After the elaborate analysis of the case $q = 4$ we will be rather short on the remaining cases. Via a similar analysis we find in case $q = 3$ or $q = 6$, if f is unit-periodic that there is a translation group spanning a hexagonal lattice being a q-symmetry group of $L_{\alpha_{p,q}}$. If moreover f is odd, additional twofold reversing rotocenters are found [10]. The translations mentioned above together with these twofold rotocenters form again a q-symmetry group of $L_{\alpha_{p,q}}$.

For a detailed illustration of the conventional FSI method with respect to the reversing symmetry (110), see [22].

The consequences of the (reversing) k-symmetries on the chaotic dynamics of the web maps is reflected in the fact that the stochastic webs possess the symmetries of the reversing k-symmetry groups of the web maps[17]. For a discussion of this, in terms of crystallographic space groups, see [10].

Moreover, if a map possesses a crystallographic (reversing) k-symmetry group this may cause a disappearance of the threshold for stochastic diffusion [10,29].

4.3. Trace maps

Consider a one-dimensional system that admits invertible two-letter substitution rule renormalization group transformations. In physics, such systems occur e.g. as models for one-dimensional quasicrystals with two composites [30,31].

Think of the substitution rule as giving an inflation rule for a chain, substituting 2×2 transfer matrices at each renormalization step according to a substitution rule. Then the trace of the product of the transfer matrices in subsequent renormalizable steps is given by one component of a three-dimensional trace map. Each substitution rule has a corresponding substitution matrix that is defined as follows. Let τ be a substitution on two letters a and b let $\#_a(\tau(b))$ denote the number of a's in the image of b under τ etcetera, then the substitution matrix R_τ is defined as[18]

$$R_\tau := \begin{pmatrix} \#_a(\tau(a)) & \#_b(\tau(a)) \\ \#_a(\tau(b)) & \#_b(\tau(b)) \end{pmatrix} . \tag{124}$$

The substitution matrices of invertible two-letter substitutions are 2×2 matrices with integer coefficients that have determinant ± 1.

[17] For examples see e.g. Fig. 2 and Fig. 3, or [15,10].

[18] We follow the conventions in notation of Baake et al. [31].

Note that different invertible substitution rules may have the same substitution matrix. The substitution matrices and the trace maps are two on one (homomorphism with kernel $\pm\mathrm{Id}$), and in case R_{τ_1} and R_{τ_2} are two substitution matrices of the substitution rules τ_1 and τ_2 and F_{τ_1} and F_{τ_2} their trace maps, then the trace map of $\tau_1 \circ \tau_2$ with substitution matrix $R_{\tau_1 \circ \tau_2} = R_{\tau_2} \circ R_{\tau_1}$ is given by

$$F_{\tau_1 \circ \tau_2} = F_{\tau_2} \circ F_{\tau_1} . \tag{125}$$

Hence, it is sufficient to consider the generators of the group of substitution matrices and their trace maps, because for any substitution matrix within the group, the trace map is simply given by a composition of trace maps of the generators of the group. In Table 3 the generators of the group of matrices with integer coefficients and determinant ± 1 and their trace maps are presented [13,31,32].

Now consider the group $\langle I_{xy}, I_{yz} \rangle$ generated by

$$I_{xy}: \begin{cases} x' = -x , \\ y' = -y , \\ z' = z , \end{cases} \quad I_{yz}: \begin{cases} x' = x , \\ y' = -y , \\ z' = -z . \end{cases} \tag{126}$$

Firstly we observe that $\langle I_{xy}, I_{yz} \rangle$ is isomorphic to the dihedral group D_2, i.e. the group has four elements and the generators satisfy

$$I_{xy}^2 = I_{yz}^2 = (I_{xy} \circ I_{yz})^2 = \mathrm{Id} . \tag{127}$$

Let us write $I_{xz} = I_{xy} \circ I_{yz}$, then we find that [32, 13, 16]

Table 3
Generators of the group of 2×2 matrices with integer coefficients and determinant ± 1, $R_U R_\sigma$ and R_P, and their trace maps

ρ	R_ρ	F_ρ
U	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} z \\ y \\ 2yz - x \end{pmatrix}$
σ	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 2xy - z \end{pmatrix}$
P	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \\ z \end{pmatrix}$

$$I_{xy} \circ F_U = F_U \circ I_{yz} , \tag{128}$$

$$I_{yz} \circ F_U = F_U \circ I_{xy} , \tag{129}$$

$$I_{xy} \circ F_\sigma = F_\sigma \circ I_{xy} , \tag{130}$$

$$I_{yz} \circ F_\sigma = F_\sigma \circ I_{yz} , \tag{131}$$

$$I_{xy} \circ F_P = F_P \circ I_{xy} , \tag{132}$$

$$I_{yz} \circ F_P = F_P \circ I_{xz} . \tag{133}$$

The above relations together with the fact that the order of $\langle I_{xy}, I_{yz} \rangle$ is four, ensures that $\langle I_{xy}, I_{yz} \rangle$ is a k-symmetry group of any trace map associated with a 2×2 substitution matrix with determinant ± 1. The value of k will depend on the actual substitution matrix, but in any way

$$k \in \{1, 2, 3\} . \tag{134}$$

For example consider the substitution rule $\tau_n = U^n \circ P$ (cf. [33], the case $n = 1$ corresponds to the Fibonacci chain [30]), with substitution matrix

$$R_{\tau_n} = \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix} , \tag{135}$$

then $k = 2$ in case n is even, and $k = 3$ in case n is odd.

For all n, F_{τ_n} is reversible, for we can write it as the product of two involutions [13]:

$$F_{\tau_n} = F_{\sigma \circ P} \circ F_{U^n \circ \sigma} . \tag{136}$$

As an illustration, let us consider the Fibonacci trace map, i.e.

$$L = F_{\tau_1}: \begin{cases} x' = y , \\ y' = z , \\ z' = 2yz - x . \end{cases} \tag{137}$$

L is reversible with respect to the reversing symmetry $F_{U \circ \sigma}$, that is precisely the mirror in the plane $x = z$,

$$P_{xz}: \begin{cases} x' = z , \\ y' = y , \\ z' = x . \end{cases} \tag{138}$$

Hence, let us consider the linear reversing 3-symmetry group

$$\langle P_{xz}, I_{xy}, I_{yz} \rangle . \tag{139}$$

Its orbits under ϕ_L are given by

$$\phi_L(P_{xz}) = P_{xz} , \tag{140}$$

$$\phi_L^3(I_{xy}) = \phi_L^2(I_{xz}) = \phi_L(I_{yz}) = I_{xy} , \tag{141}$$

$$\phi_L^3(I_{xy} \circ P_{xz}) = \phi_L^2(I_{xz} \circ P_{xz}) = \phi_L(I_{yz} \circ P_{xz}) = I_{xy} \circ P_{xz} . \tag{142}$$

The reversing conjugacy classes of (139) are

$$\{P_{xz}, I_{xz} \circ P_{xz}\} \quad \text{and} \quad \{I_{xy} \circ P_{xz}, I_{yz} \circ P_{xz}\} , \tag{143}$$

indicating that on the level of L^3 the conventional FSI method should be used with respect to two reversing symmetries, each one representing a conjugacy class, in the spirit of Proposition 2.6.

Of course, it would be more elegant to make full benefit of the FSI method of Theorem 3.7. This will involve the (nontrivial) calculation of $\tau_{n,m}(S, \tilde{S})$ for all $S, \tilde{S} \in \{P_{xz}, P_{xz} \circ I_{xz}, P_{xz} \circ I_{xy}, P_{xz} \circ I_{yz}\}$.

A discussion of scaling properties of energy spectra of one-dimensional quasiperiodic chains in relation to the occurrence of k-symmetries in trace maps is given in [16].

4.4. The kicked rotator

As a final example, the kicked rotator will be discussed in this section. The angular momentum of the rotator precesses freely around the y-axis while feeling a periodic strain of impulses inducing a rotation around the z-axis. In case the period of the kicks is one fourth of the period of the free precession the equations of motion (of the components of the angular momentum of the rotator) can be written as [11]

$$L: \begin{cases} x' = z\cos(kx) + y\sin(kx) , \\ y' = y\cos(kx) - z\sin(kx) , \\ z' = -x . \end{cases} \tag{144}$$

L possesses the reversing symmetry P_{xz} (cf. (138)), that we already discovered as a reversing symmetry of the Fibonacci trace map. L further possesses the symmetry I_{xz}. Moreover, L^2 possesses I_{xy} as a symmetry [11], implying that I_{xy} is a 2-symmetry of L. In fact,

$$\phi_L(I_{xy}) = I_{yz} . \tag{145}$$

Hence, we find that L possesses a reversing 2-symmetry group. This group is identical to the group (139) that we discussed in relation to the Fibonacci trace map, but has a different orbit structure under ϕ_L. In the case of the kicked rotator we find:

$$\phi_L(I_{xz}) = I_{xz} , \tag{146}$$

$$\phi_L^2(I_{xy}) = \phi_L(I_{yz}) = I_{xy} , \tag{147}$$

$$\phi_L(P_{xz}) = P_{xz} , \tag{148}$$

$$\phi_L(I_{xz} \circ P_{xz}) = I_{xz} \circ P_{xz} , \tag{149}$$

$$\phi_L^2(I_{xy} \circ P_{xz}) = \phi_L(I_{yz} \circ P_{xz}) = I_{xy} \circ P_{xz} . \tag{150}$$

Also here, it would be very interesting to study k-symmetric periodic orbits in the spirit of Theorem 3.7. Note at this point, that the use of the FSI method would imply iterating the similar sets as in the previous trace map example. However, the interpretation of the intersection points would be different because of the different ϕ_L orbit structures we find in both examples: it is to be expected that the values of $\tau_{n,m}(S, \tilde{S})$ are different from the ones in the trace map example.

A discussion of the occurrence of (reversing) k-symmetries, in case we do not have precisely four kicks every precession period, is discussed elsewhere [34].

5. Concluding remarks and summary

In this paper we generalized the concepts of (reversing) symmetries to (reversing) k-symmetries.

In setting up the frame-work for (reversing) k-symmetry groups, we carefully insisted on $k \in \mathbb{Z}$, i.e. $k \neq \infty$. We have chosen not to adopt the

notion of "commutation of a group $\mathcal{G}$ with a map L", as used by MacKay and others [27,35,13]. A group $\mathcal{G}$ commutes with a map L if [27]

$$\forall U \in \mathcal{G}, \exists \tilde{U} \in \mathcal{G}, \text{ such that } U \circ L = L \circ \tilde{U} . \tag{151}$$

Although from a group theoretical point of view (151) is a nice equivalence relation, this equivalence does not automatically include important dynamical features. In particular, $U \in \mathcal{G}$ satisfying (151) does not have to preserve the set of periodic orbits and the set of chaotic orbits, in contrast to (96) and (97). For instance, the group $\mathrm{Inv}(\Omega)$, consisting of all invertible maps of the phase space into itself, commutes with every $L \in \mathrm{Inv}(\Omega)$. Of course, this observation does not reveal much about the dynamics of L. The closure of the orbits under ϕ_L is essential for the preservation of dynamical features such as periodic and chaotic orbits.

In [27], the argument of "commutation with the group $\{U_{n,m}\}_{n,m\in\mathbb{Z}}$" was used in the case of the standard map to justify the reduction of the phase space to the 2-torus $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. However, one should be careful with this argument. For instance, consider the map

$$L: \begin{cases} x' = x + y , \\ y' = x + 2y . \end{cases} \tag{152}$$

This map is linear and not chaotic, has one fixed point $(0, 0)$ and no further other periodic orbits. It is easily seen that this map commutes with the group $\{U_{n,m}\}_{n,m\in\mathbb{Z}}$. However, a reduction of the phase space to the 2-torus $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ is not as harmless as in the case of the standard map. In fact, the map (152) restricted to the 2-torus $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ is Arnold's cat map [36], a classical example of a hard chaotic Anosov system possessing an infinite number of unstable periodic orbits.

In the case of the standard map in section 4.1, we used solely (1-)symmetries that preserve the important dynamical features to justify phase space reductions. However, this type of justification is not available in the case of the map (152) and the reduction to the 2-torus in this case leads to a severe change of dynamical features.

If one is interested in a property of a dynamical system that does not depend on the timescale (such as the extension of chaotic regions) we can safely study the system in a phase space that is reduced with the help of k-symmetries (see e.g. [10]). In case the properties to be studied are also independent of time-direction, also reversing k-symmetries may be used to reduce the phase space. One should be careful using elements of groups that "commute with the map" to reduce the phase space, for this may cause severe changes of the dynamics. In this respect it is useful to note that every finite group that commutes with the map is a k-symmetry group, for some value of k.

Let us now summarize the contents of this paper. After having generalized the FSI method for finding periodic orbits from the case of one reversing symmetry to the case of more than one reversing symmetry, we found a generalization of the FSI method also for the case of reversing k-symmetries. This provides us with powerful tools for calculating k-symmetric periodic orbits. Using these methods, reversing k-symmetries may also provide a nice tool for studying the relation between quantum mechanics and classical mechanics (cf. [11,17]).

The (reversing) k-symmetries also have a great impact on the global phase portrait, as indicated at the end of section 3 (see e.g. the example in section 3.1 and [10]).

By means of the examples in section 4, we have shown that (reversing) k-symmetries occur in well known dynamical systems (related to problems in physics). Recent work indicates that (reversing) k-symmetries occur in kicked systems (such as the standard map, web maps, and the kicked rotator) preferably at resonances [34]. However, more work needs, to be done to explore the full benefit of the (reversing) k-symmetries in these systems. This is far beyond the scope of the present paper, but we plan to

report on more specific case studies in the near future.

Of further interest are local dynamical phenomena, in connection to (reversing) k-symmetry properties. In Appendix C some consequences of (reversing) k-symmetries for the linearization around fixed points are presented. Starting from there, it would be of great interest to study (local) bifurcation phenomena in dynamical systems possessing (reversing) k-symmetries, in relation to the (local) bifurcation phenomena in dynamical systems with (reversing) symmetries [6,37].

Another point of interest, left nearly undiscussed in the present paper, is the group structure of reversing k-symmetry groups. In [7], reversing symmetry groups were already discussed from a more group theoretical point of view (see also [38] for an explicit discussion in the context of trace maps). In a forthcoming publication we will report on the group structure in the case that a map L possesses a cyclic reversing k-symmetry group [39].

Acknowledgements

It is a great pleasure to acknowledge various useful and stimulating discussions with M. Baake, H. Brands, H.W. Capel, C. Pisani, J.A.G. Roberts, H. de Vries and F. Wijnands. G.R.W.Q. is grateful to the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO) for supporting his visit to the Institute for Theoretical Physics of the University of Amsterdam, and to Professor Hans Capel for his hospitality. This investigation is part of the research program of the Australian Research Council.

Appendix A. Construction of a map on the plane with a reversing symmetry group isomorphic to $4'm$

In this appendix we show in detail how we constructed a map on the plane that has a reversing symmetry group isomorphic to $4'm$ where the generating elements are a reversing fourfold rotation around the origin and a mirror in the x-axis. We will embed this map in a one-parameter family of maps that possesses a reversing symmetry group isomorphic to $2m'$, generated by a twofold rotation around the origin and a mirror in the line $x=y$ denoted as v).

To construct a map L on the plane that possesses a reversing fourfold rotation around the origin, $R_{\pi/2}$, we can simply use the fact that any map L that has a reversing symmetry K_0 has the form $L=K_0\circ K_1$, where $K_0^2\circ K_1^2=\mathrm{Id}$ [7]. It follows that if $K_0=R_{\pi/2}$, L can be written:

$$L=R_{\pi/2}\circ A\ , \tag{153}$$

where $A^2=-\mathrm{Id}$ is also of order four. Adding a mirror in the x-axis, M_x, as a symmetry of L, implies that $R_{\pi/2}\circ M_x$ is an involutory reversing symmetry of L. Using again the decomposition property, we can write

$$L=R_{\pi/2}\circ M_x\circ B\ , \tag{154}$$

where B is an involution, i.e. $B^2=\mathrm{Id}$. Since $-\mathrm{Id}=R_{\pi/2}^2$ (i.e. the composition of two reversing symmetries), $-\mathrm{Id}$ is a symmetry and hence commutes with L. Since $-\mathrm{Id}$ also commutes with $R_{\pi/2}$ and M_x, it must also commute with B

$$B\circ -\mathrm{Id}=-B\ . \tag{155}$$

Since M_x commutes with L and anticommutes with $R_{\pi/2}$, M_x must anticommute with B,

$$B\circ M_x=-M_x\circ B\ . \tag{156}$$

There is a linear solution for B that is the mirror in the line $x = y$,

$$M_v: \begin{cases} x' = y \,, \\ y' = x \,. \end{cases} \tag{157}$$

However, in discussing dynamical systems, we are interested in a nonlinear solution of B. From the linear solution M_v we can obtain a nonlinear solution

$$B = C \circ M_v \circ C^{-1} \,, \tag{158}$$

provided that C is nonlinear, invertible, and commutes with $-\mathrm{Id}$ and with M_x. Moreover, we do not want C to commute with M_v, for that would imply that L is a linear map. It is easily verified that

$$C: \begin{cases} x' = xp(y) + q(y) \,, \\ y' = -y \,, \end{cases} \tag{159}$$

is invertible and commutes with $-\mathrm{Id}$ and M_x, but not with M_v, if and only if p is an even function and q is zero. However, if p is even and q is odd but nonzero, we find that the symmetry $-\mathrm{Id}$ and the reversing symmetry M_v are preserved, but that the symmetry M_x and reversing symmetry $R_{\pi/2}$ are broken.

Thus, we constructed a map with a reversing symmetry group isomorphic to $4'm$, embedded in a family of maps with a reversing symmetry group isomorphic to $2m'$.

Choosing

$$q(y) = -\mu y \,, \tag{160}$$

$$p(y) = 1 + \varepsilon \cos(2\pi y) \,, \tag{161}$$

one obtains the example of section 2.1.

Appendix B. Construction of a map on the plane with a reversing 2-symmetry group isomorphic to $2m'$.

In this appendix we will construct a family of maps on the plane that have a reversing 2-symmetry group isomorphic to $2m'$, i.e. a reversing 2-symmetry group generated by a reversing 2-symmetry that is a mirror, and a twofold rotocenter that is a (1-)symmetry. Let us denote the mirror in the x-axis M_x and the mirror in the y-axis M_y. Then we are looking for a map L, satisfying

$$L \circ M_x \circ L = M_y \,, \quad L \circ M_y \circ L = M_x \,. \tag{162}$$

Writing

$$L = M_x \circ B \,, \tag{163}$$

it follows that L satisfies (162) if and only if

$$B^2 = -\mathrm{Id} \,. \tag{164}$$

The family of maps

$$B: \begin{cases} x' = (r(x) - y)p(y' - r(x')) + q(y' - r(x')) \,, \\ y' = \dfrac{(x - q(y - r(x)))}{p(y - r(x)) + r(x')} \,, \end{cases} \tag{165}$$

satisfies this property, if p is an even function and q and r are odd functions [7]. Choosing

$$p(x) = 1\,, \quad r(x) = \lambda \sin(\pi x)\,, \quad q(x) = \kappa \sin(\pi x)\,, \tag{166}$$

we obtain the example map (98) of section 3.1.

Appendix C. Linearizations around fixed points

In this appendix we investigate the consequences, at linear order, of (reversing) symmetries and (reversing) k-symmetries for a (real) map L. For simplicity we consider only the consequences on linearizations of L around a fixed point $\boldsymbol{x}_0$.

C.1. (Reversing) symmetries

Let L be a map possessing a reversing symmetry group $\mathscr{E}$, and let $\boldsymbol{x}_0$ be a fixed point of L. Then, for any $U \in \mathscr{E}$, the point

$$\boldsymbol{y}_0 := U\boldsymbol{x}_0 \tag{167}$$

is also a fixed point of L.

We use the notation $[\kappa]_m$ to denote a Jordan m-block corresponding to an eigenvalue κ, i.e.

$$[\kappa]_m := \begin{pmatrix} \kappa & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & \kappa \end{pmatrix}. \tag{168}$$

The Jordan blocks in the linearizations of L around $\boldsymbol{y}_0$ come in block singlets and block duplets that are simply related to those in the linearization of L around $\boldsymbol{x}_0$. With the above notation, this relationship can be expressed as follows:

	$dL\|_{x_0}$	$dL\|_{y_0}$
Block-singlets	$\{[\kappa]_m\}$	$\{[\kappa^\varepsilon]_m\}$
Block-duplets	$\{[\mu]_m, [\mu^*]_m\}$	$\{[\mu^\varepsilon]_m, [\mu^{*\varepsilon}]_m\}$

(169)

Here $\kappa \in \mathbb{R}$, $\mu \in \mathbb{C}$ and ε is 1 if U is a symmetry, and -1 if U is a reversing symmetry.

If $\boldsymbol{x}_0$ is symmetric with respect to some reversing symmetry S, i.e. $S\boldsymbol{x}_0 = \boldsymbol{x}_0$ then the Jordan blocks of $dL|_{x_0}$ occur in

Block-singlets	$\{[1]_m\}, \quad \{[-1]_m\}$
Block-duplets	$\{[\lambda]_n, [\lambda^{-1}]_m\}$,
Block-quadruplets	$\{[\mu]_m, [\mu^*]_m, [\mu^{-1}]_m, [\mu^{*-1}]_m\}$.

(170)

Here $\lambda \in \mathbb{R} \cup \mathrm{S}^1$ (where S^1 denotes the unit circle in the complex plane), and $\mu \in \mathbb{C}$.

In case $\boldsymbol{x}_0$ is symmetric with respect to some symmetry, there are in general no extra restrictions on the linear part of L.

C.2. (Reversing) k-symmetries

Let L be a map possessing a (reversing) k-symmetry U, and let $\boldsymbol{x}_0$ be a fixed point of L. Then, the point

$$\boldsymbol{y}_0 := U\boldsymbol{x}_0 \tag{171}$$

will in general *not* be a fixed point, but will be periodic with a period p that divides k. The Jordan blocks in the linearization of L^p around $\boldsymbol{y}_0$ come in block-singlets and block-duplets that are related to those in the linearization of L around $\boldsymbol{x}_0$ via

$$(dL|_{\boldsymbol{x}_0})^k = (dL^p|_{\boldsymbol{y}_0})^n\,, \tag{172}$$

where $n = k/p$. This relationship can be expressed as follows:

	$dL\|_{\boldsymbol{x}_0}$	$dL^p\|_{\boldsymbol{y}_0}$
Block-singlets	$\{[\kappa]_m\}$	$\{[(\pm)^{1+n}[\kappa^{p\varepsilon}]_m\}$
Block-duplets	$\{[\mu]_m, [\mu^*]_m\}$	$\{[\mu^{p\varepsilon}\,\mathrm{e}^{2\pi\mathrm{i}l/n}]_m, [\mu^{*p\varepsilon}\,\mathrm{e}^{-2\pi\mathrm{i}l/n}]_m\}$

(173)

Here $\kappa \in \mathbb{R}$, $\mu \in \mathbb{C}$, $l \in \mathbb{Z}$, and ε is 1 in case U is a k-symmetry and -1 if U is a reversing k-symmetry. (Note that p and l may depend on which point $\boldsymbol{x}_0$ and which (reversing) k-symmetry U we are considering.) If $\boldsymbol{x}_0$ is symmetric with respect to a (reversing) k-symmetry S, i.e. $S\boldsymbol{x}_0 = \boldsymbol{x}_0$, then the Jordan blocks of $dL|_{\boldsymbol{x}_0}$ occur in

Block-singlets	$\{[1]_m\}\,,\quad \{[-1]_m\}$
Block-duplets	$\{[\kappa]_m, [(\pm)^{1+k}[\kappa^{-1}]_m\}\,,\quad \{[\nu]_m, [\nu^{-1}]_m\}$
Block-quadruplets	$\{[\mu]_m, [\mu^*]_m, [\mu^{-1}\,\mathrm{e}^{2\pi\mathrm{i}l/k}]_m, [\mu^{*-1}\,\mathrm{e}^{-2\pi\mathrm{i}l/k}]_m\}\,.$

(174)

Here $\kappa \in \mathbb{R}$, $\mu \in \mathbb{C}$, $l \in \mathbb{Z}$, and $\nu \in \mathrm{S}^1$.

Note added in proof

On the basis of our results in sections 2 and 3, recently an alternative approach towards the FSI method for obtaining k-symmetric periodic orbits has been developed [40]. In this approach the evaluation of $\tau_{n,m}$ (cf. Theorem 3.7) is circumvented and a systematic search for k-symmetric periodic orbits has been realized

References

[1] H. Goldstein, Classical Mechanics, 2nd Ed. (Addison-Wesley, Reading, 1980).

[2] P.J. Olver, Applications of Lie groups to differential equations, Graduate Texts in Mathematics, vol. 107 (Springer, New York, 1986).

[3] G.R.W. Quispel, H.W. Capel and R. Sahadevan, Phys. Lett. A 170 (1992) 379.

[4] D.H. Sattinger, Group theoretic methods in bifurcation theory, Lecture Notes in Mathematics, vol. 762 (Springer, Berlin, 1979).

[5] A. Vanderbauwhede, Local bifurcation and symmetry, Research Notes in Mathematics, vol. 75 (Pitman, London, 1982).

[6] M. Golubitsky, I. Stewart and D.G. Schaeffer, Singularities and Groups in Bifurcation Theory, vol. 2, Applied Mathematical Sciences, vol. 69 (Springer, New York, 1988).

[7] J.S.W. Lamb, J. Phys. A: Math. Gen. 25 (1992) 925.

[8] M.B. Sevryuk, Reversible systems, Lecture Notes in Mathematics, vol. 1211 (Springer, Berlin, 1986).

[9] J.A.G. Roberts and G.R.W. Quispel, Phys. Rep. 216 (1992) 63.

[10] J.S.W. Lamb, J. Phys. A: Math. Gen. 26 (1993) 2921.

[11] F. Haake, M. Kús and R. Scharf, Z. Phys. B 65 (1987) 381.
[12] I. Hoveijn, Chaos Solitons Fractals 2 (1992) 81.
[13] J.A.G. Roberts and M. Baake, J. Stat. Phys. 74 (1994) 829.
[14] B.V. Chirikov, Phys. Rep. 52 (1979) 265.
[15] G.M. Zaslavsky, R.Z. Sagdeev, D.A. Usikov and A.A. Chernikov, Weak chaos and quasi-regular patterns, Cambridge Nonlinear Science Series, vol. 1 (Cambridge Univ. Press, Cambridge, 1991).
[16] Z.Y. Wen, F. Wijnands and J.S.W. Lamb, A natural class of generalized Fibonacci chains, preprint University of Amsterdam ITFA 93-27, J. Phys. A: Math. Gen. 27 (1994), in press.
[17] G.M. D'Ariano, L.R. Evangelista and M. Saraceno, Phys. Rev. A 45 (1992) 3646.
[18] G.R.W. Quispel and J.S.W. Lamb, Dynamics and k-symmetries, preprint University of Amsterdam ITFA 93-20 (1993), in: Hamiltonian Systems: Integrability and Chaotic Behaviour, ed. J. Seimenis, NATO ASI series B: Physics (Plenum Press, New York), to appear.
[19] R. DeVogelaere, Contribution to the Theory of Nonlinear Oscillations, vol. 4 (Princeton Univ. Press, Princeton, 1958) p. 53.
[20] Y.H. Ichikawa, T. Kanimura, T. Hatori and S.Y. Kim, Prog. Theor. Phys. Suppl. 98 (1989) 1.
[21] E. Piña and L. Jiménez Lara, Physica D 26 (1987) 369.
[22] E. Piña and E. Cantoral, Phys. Lett. A 135 (1989) 190.
[23] P.H. Richter, H. Scholz and A. Wittek, Nonlinearity 3 (1990) 45.
[24] P. Chossat and M. Golubitsky, Physica D 32 (1988) 423.
[25] J.C. Kimball and H.S. Dumas, Phys. Lett. A 144 (1990) 201.
[26] A.L. Loeb, Color and Symmetry (Wiley, New York, 1971).
[27] R.S. MacKay, Phys. Lett. A 106 (1984) 99.
[28] G.R.W. Quispel and J.S.W. Lamb, in preparation.
[29] J.S.W. Lamb, Stochastic webs with fourfold rotation symmetry, preprint University of Amsterdam ITFA 93-21 (1993), in: Hamiltonian Systems: Integrability and Chaotic Behaviour, ed. J. Seimenis, NATO ASI series B: Physics (Plenum Press, New York), to appear.
[30] M. Kohmoto, L.P. Kadanoff and C. Tang, Phys. Rev. Lett. 50 (1983) 1870.
[31] M. Baake, U. Grimm and D. Joseph, Int. J. Mod. Phys. B 7 (1993) 1527.
[32] J. Peyrière, Z.Y. Wen and Z.X. Wen, l'Enseignement de Mathématiques 39 (1993) 153;
J. Peyrière, Z.X. Wen and Z.Y. Wen, Algebraic properties of trace mappings associated with substitutive sequences, Modern Math. (China, 1993), in press.
[33] P.A. Kalugin, A.Y. Kitaev and L.S. Levitov, Sov. Phys. JETP 64 (1986) 410.
[34] J.S.W. Lamb and H. Brands, Symmetries and reversing symmetries in kicked systems, preprint University of Amsterdam ITFA 93-28 (1993), in: Dynamics, Bifurcation and Symmetries, ed. P. Chossat (Kluwer, Dordrecht), to appear.
[35] J. Llibre and R.S. MacKay, Math. Proc. Camb. Phil. Soc. 112 (1992) 539.
[36] V.I. Arnold and A. Avez, Problèmes Ergodiques de la Mécanique Classique (Gauthier-Villars, Paris, 1967).
[37] J.S.W. Lamb and H.W. Capel, Local bifurcations on the plane with reversing point group symmetry, preprint University of Amsterdam ITFA 93-17, Chaos Solitons Fractals (1994), in press.
[38] M. Baake and J.A.G. Roberts, Symmetries and reversing symmetries of trace maps, Tübingen preprint TPT-QC-93-08-I, Int. J. Mod. Phys. B (1994), to appear.
[39] J.S.W. Lamb and G.R.W. Quispel, Cyclic reversing k-symmetry groups, preprint University of Amsterdam ITFA 94-10 (1994).
[40] H. Brands, J.S.W. Lamb and I. Hoveÿn, in preparation.