DYNAMICS AND K-SYMMETRIES

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1 INTRODUCTION

The presence of symmetries is common in many dynamical systems of interest, endowing the system with certain simplifying features. A lot of attention has been given to continuous symmetries, since they give rise to conservation laws allowing one to reduce the number of degrees of freedom. During the past two decades attention has been directed to discrete symmetries as well, often in the context of bifurcation problems (Sattinger, 1979; Vanderbauwhede, 1982; Golubitsky et al., 1988).

There are two types of discrete symmetries. *Symmetries* are transformations in phase space that leave the equations of motion invariant. *Reversing symmetries* leave the equations of motion invariant if the direction of time is also reversed (Lamb 1992). It is the latter case which is of primary interest here. For a review of systems with time reversal symmetry see Roberts and Quispel (1992).

When a discrete symmetry has a simple geometric interpretation in phase space its presence is quite manifest in the phase portrait. It has been realised recently (Lamb, 1993a; D'Ariano et al., 1992; Hoveijn, 1992) that it is possible to see manifest discrete symmetry in the phase portrait in the absence of a symmetry. An example of this is shown in Figure 1. The map which gives rise to this phase portrait, Eqn. (3.1), possesses symmetry properties only if considered on a proper time scale: i.e. it is only certain iterates of the map which possess the symmetry. This phenomenon can also be observed in various well known dynamical systems that occur in the context of problems in physics, e.g. the standard map (Chirikov, 1979), web maps (Lamb, 1993a; Zaslavsky et al., 1991; Hoveijn, 1992), a class of trace maps (Roberts and Baake, 1993), and the kicked rotator (D'Ariano et al., 1992). This last example will be discussed in section 4.

The outline of the rest of this paper is as follows. In the next section we briefly review the concepts of symmetries and reversing symmetries of dynamical systems and

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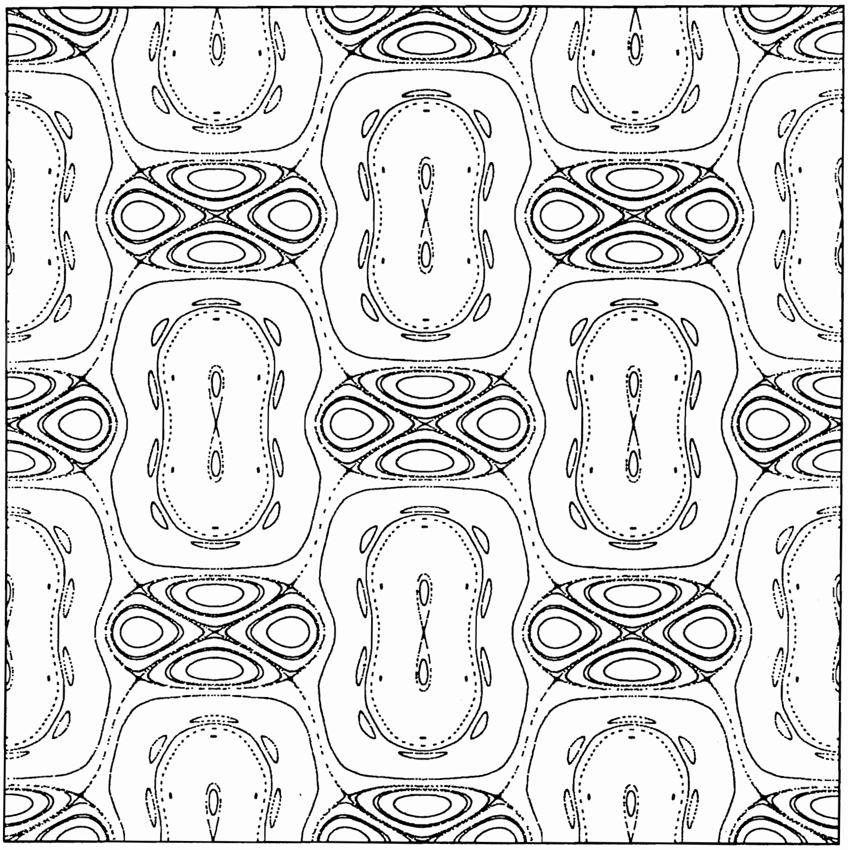

Figure 1. Phase portrait of (3.1) with $\lambda = -0.2$ and $\kappa = 0.34$ on $[-2, 2] \times [-2, 2]$

present some standard results. Section 3 contains our main results. There we consider dynamical systems that possess symmetry properties only if considered on certain time scales. We discuss a generalization of the method for obtaining periodic orbits to systems with these k-symmetries.

Throughout this paper we confine ourselves to systems with discrete time, i.e. mappings. These occur as Poincaré sections of dynamical systems with continuous time, as well as in problems of a truly discrete nature.

The reader who is interested in a more extended motivation and theoretical treatment of k-symmetries may wish to consult Lamb and Quispel (1993) of which the present paper is an abridged version.

2 SYMMETRIES AND REVERSING SYMMETRIES

Let us first review some familiar concepts and results both in order to make our presentation more self contained, but also so that we may make a comparison with results of the next section.

We consider a discrete dynamical system, i.e. a map on a state space Ω

$$L : \Omega \mapsto \Omega. \tag{2.1}$$

Let us first give some notation.

The map M is a symmetry of L if

$$M \circ L = L \circ M. \tag{2.2}$$

The map S is a reversing symmetry of L if

$$S \circ L = L^{-1} \circ S. \tag{2.3}$$

The set $\mathcal{G}$ of symmetries of L is called the *symmetry group* of L.[1] We denote the set comprising both the symmetries and the reversing symmetries the *reversing symmetry group* $\mathcal{E}$. Discussion of the group structure of $\mathcal{E}$ and $\mathcal{G}$ can be found in Lamb (1992).

We denote the orbit of a point $\mathbf{x}_0$ by

$$\Gamma(\mathbf{x}_0) = \{\mathbf{x} \in \Omega | \mathbf{x} = L^n \mathbf{x}_0, n \in \mathbb{Z}\} \tag{2.4}$$

and the set of fixed points of a map $A: \Omega \mapsto \Omega$ by

$$\text{Fix}(A) = \{\mathbf{x} \in \Omega \mid A\mathbf{x} = \mathbf{x}\}. \tag{2.5}$$

The presence of discrete symmetries naturally has consequences for the dynamics. For instance, given an invariant set $\Delta \subset \Omega$ it follows that $U\Delta$ is also an invariant set if U is a discrete symmetry of the equations of motion. If $\Delta = U\Delta$ also, then we say that Δ is a *symmetric* invariant set. A more fruitful consequence of reversing symmetries is that their presence permits one to readily locate symmetric periodic orbits via the Fixed Set Iteration (FSI) Method (DeVogelaere (1958), Piña and Jiménez Lara (1987), Piña and Cantoral (1989), Ichikawa et al. (1989), Richter et al. (1990), D'Ariano et al. (1992), Roberts and Quispel (1992), Lamb (1992)). Prior to stating the method let us first present some Lemmas which are of independent interest.

Lemma 1 $\Gamma(\mathbf{x}_0)$ *is symmetric with respect to the reversing symmetry* S *if and only if* $\mathbf{x}_0 \in \text{Fix}(L^m \circ S)$ *for some value of* $m \in \mathbb{Z}$.

Proof.
For all $n \in \mathbb{Z}$ we have

$$S\mathbf{x}_0 = L^{-m}\mathbf{x}_0 \Rightarrow L^n \circ S\mathbf{x}_0 = L^{n-m}\mathbf{x}_0 \Rightarrow S \circ L^{-n}\mathbf{x}_0 = L^{n-m}\mathbf{x}_0. \tag{2.6}$$

The converse follows by definition. □

Lemma 2 *Let* S *be a reversing symmetry of* L *and* $m \in \mathbb{Z}$, *then* $\mathbf{x}_0 \in \text{Fix}(L^m \circ S) \Rightarrow \mathbf{x}_0 \in \text{Fix}(S^2)$.

Proof.
$L^m \circ S\mathbf{x}_0 = \mathbf{x}_0 \Rightarrow L^m \circ S \circ L^m \circ S\mathbf{x}_0 = \mathbf{x}_0 \Leftrightarrow S^2\mathbf{x}_0 = \mathbf{x}_0$. □

Now we present the fixed set iteration (FSI) method.

Theorem 3 $\Gamma(\mathbf{x}_0)$ *is a symmetric periodic orbit with respect to a reversing symmetry* S *if and only if there are* $n, m \in \mathbb{Z}$ $(n \neq m)$ *such that* $\mathbf{x}_0 \in \text{Fix}(L^n \circ S) \cap \text{Fix}(L^m \circ S)$.

Proof.

$$\begin{aligned}
\mathbf{x}_0 &\in \text{Fix}(L^n \circ S) \cap \text{Fix}(L^m \circ S) && (2.7)\\
&\Rightarrow L^{n-m} \circ L^m \circ S \circ L^m \circ S\mathbf{x}_0 = \mathbf{x}_0 && (2.8)\\
&\Rightarrow L^{n-m} \circ S^2\mathbf{x}_0 = \mathbf{x}_0 \quad \text{as S is a symmetry} && (2.9)\\
&\Rightarrow L^{n-m}\mathbf{x}_0 = \mathbf{x}_0. \quad \text{via Lemma 2} && (2.10)
\end{aligned}$$

[1]We assume that both the map and the (reversing) symmetries are invertible.

The symmetry of the orbit follows from the premise and Lemma 1. Conversely, if the orbit is symmetric with respect to a reversing symmetry S, then there is an n in $\mathbb{Z}$ such that $L^{-n}\mathrm{x}_0 = S\mathrm{x}_0$. Since the orbit is periodic it follows directly that $L^{p-n}\mathrm{x}_0 = S\mathrm{x}_0$. □

The use of Theorem (3) is made easier by the fact that

$$\begin{aligned} \mathrm{Fix}(L^{2n} \circ S) &= L^n(\mathrm{Fix}(S)) & (2.11) \\ \mathrm{Fix}(L^{2n+1} \circ S) &= L^n(\mathrm{Fix}(L \circ S)) & (2.12) \end{aligned}$$

when S is a reversing symmetry.

In the case that a periodic orbit is symmetric with respect to more that one reversing symmetry other results exist (Lamb and Quispel, 1993).

3 K-SYMMETRIES AND REVERSING K-SYMMETRIES

We come now to the main concern of this paper: the concept of a *k-symmetry.* Let us motivate our discussion with an example.

Let us consider the map L of the plane given by

$$L : \begin{cases} x' = \lambda \sin(\pi x) - y + \kappa \sin\left[\pi x - \pi\kappa \sin\left[\pi y - \pi\lambda \sin(\pi x)\right]\right] \\ y' = -x + \kappa \sin[\pi y - \pi\lambda \sin(\pi x)] - \lambda \sin(\pi x') \end{cases} \tag{3.1}$$

where κ, $\lambda \in \mathbb{R}$ are parameters. Admittedly this map is contrived but it provides a convenient example with which to illustrate our discussion. An example from a physical application is discussed in the next section.

A cursory view of the phase portrait of the map L, Figure 1, shows clear evidence of discrete symmetry.[2] The phase portrait is symmetric under reflection in both the x and the y axes. One can readily show that these reflections are neither symmetries nor reversing symmetries. In fact one can readily show that the mirrors in the x-axis, M_x, and y-axis, M_y, satisfy

$$\begin{aligned} M_x &= L^2 \circ M_x \circ L^2 \\ M_y &= L^2 \circ M_y \circ L^2 \end{aligned}$$

This study motivates us to look for maps M and S satisfying either

$$M \circ L^k = L^k \circ M \tag{3.2}$$

or

$$S \circ L^k = L^{-k} \circ S \tag{3.3}$$

We denote the set of all symmetries of L^k by $\mathcal{G}_k$. Similarly the set of all symmetries of L^k *together with* its reversing symmetries is denoted by $\mathcal{E}_k$.

In order to make a precise definition of the concept of both k-symmetry and k-symmetric orbits we define a map ϕ_L as follows

$$\begin{aligned} \phi_L(M) &= L \circ M \circ L^{-1} & (3.4) \\ \phi_L(S) &= L \circ S \circ L & \end{aligned}$$

[2]There is also translational invariance in this example. In fact, the phase portrait has crystallographic symmetry and possesses a stochastic web. For some discussions of the last phenomenon see e.g. Zaslavsky et al. (1991), Hoveijn (1992), and Lamb (1993a, 1993b).

Note that if M is a symmetry of L^k then so is $\phi_L(M)$. Similarly if S is a reversing symmetry of L^k then so is $\phi_L(S)$. In the above example $\phi_L(M_x) = M_y$.

The map ϕ_L enjoys the following properties

$$\begin{array}{lrcl} \text{(i)} & \phi_L^n(M \circ \tilde{M}) & = & \phi_L^n(M) \circ \phi_L^n(\tilde{M}), \\ \text{(ii)} & \phi_L^n(S \circ \tilde{S}) & = & \phi_L^n(S) \circ \phi_L^{-n}(\tilde{S}), \\ \text{(iii)} & \phi_L^n(M \circ S) & = & \phi_L^n(M) \circ \phi_L^n(S), \\ \text{(iv)} & \phi_L^n(S \circ M) & = & \phi_L^n(S) \circ \phi_L^{-n}(M). \end{array} \quad (3.5)$$

$\forall n \in \mathbb{Z}$. Note that ϕ_L is a group automorphism of $\mathcal{G}_k$ but not of $\mathcal{E}_k$.

We first present the

Definition 4 *$U \in \mathcal{E}_{\tilde{k}}$ is called a (reversing) k-symmetry of L if k is the smallest positive integer such that*

$$\phi_L^k(U) = U. \quad (3.6)$$

We denote this as $\#_L(U) = k$.

If U is an element of $\mathcal{E}_k$ then it is certainly an element of $\mathcal{E}_{\tilde{k}}$ if $\tilde{k}$ is a multiple of k. This definition avoids the ambiguity which may possibly arise from this fact.

Let us now look at how k-symmetry manifests itself. Suppose Γ is an invariant set of L. If U was a (reversing) symmetry then it would follow that $U\Gamma$ would also be an invariant set. If $U\Gamma = \Gamma$ we would say that the invariant set was symmetric. The most natural definition of a k-symmetric invariant set would mirror this definition excepting that U would now be a k-symmetry. For the k-symmetric case however the situation is more complicated and this definition is inappropriate. As an example of what can happen suppose $\mathbf{x}_0$ is a fixed point of L, then if U is a (reversing) k-symmetry iof L it follows that $U\mathbf{x}_0$ in general belongs to a periodic orbit of period k. This orbit is given by $\{\phi_L^i(U)\mathbf{x}_0 : i = 1, \ldots, k-1\}$. More importantly suppose Γ is an invariant set and U is a (reversing) k-symmetry, then $U\Gamma$ need not be an invariant set. In the general case one can see that $\bigcup_{i=0}^{k-1} \phi_L^i(U)\Gamma$ is an invariant set but it is in general a union of orbits. Because of these difficulties the "natural" definition of a k-symmetric orbit is too strong. Of course a choice of definition is dictated by its utility. It turns out that useful results can be demonstrated for k-symmetric orbits defined in the following sense

Definition 5 *An orbit $\Gamma(\mathbf{x}_0)$ is k-symmetric with respect to a (reversing) k-symmetry U if for all $\mathbf{x}_i \in \Gamma(\mathbf{x}_0)$ there is an integer p such that $\phi_L^p(U)\mathbf{x}_i \in \Gamma(\mathbf{x}_0)$.*

Thus, for example, a fixed point of the map L is k-symmetric with respect to U if $\phi_L^i(U)\mathbf{x}_0 = \mathbf{x}_0$ for some i. In general it means that for each point $\mathbf{x}_i \in \Gamma$ one of the elements of the set $\{\phi_L^i(U)\mathbf{x}_i\}_{i=0}^{k-1}$ must be in the original orbit.

As mentioned in the introduction, the presence of k-symmetries is reflected in the symmetries observed in the global phase portrait of a dynamical system. In fact, if $\mathcal{P} \in \Omega$ is the set of all (points on) periodic orbits of L, and $\mathcal{C} \in \Omega$ is the set of all (points on) chaotic orbits of L, then for all $U \in \mathcal{E}_{\mathbb{N}}$,

$$U\mathcal{P} = \mathcal{P}, \quad (3.7)$$

$$U\mathcal{C} = \mathcal{C}. \quad (3.8)$$

In fact, in Lamb (1993a), symmetry properties of stochastic webs were understood in the spirit of (3.8).[3]

[3]For discussions on the way (reversing) symmetries manifest themselves in the global phase portrait, see e.g. also Chossat and Golubitsky (1988), Kimball and Dumas (1990), and Lamb (1993b).

Let us now present the analog of the FSI method. First we present the analog of the lemmas of the last section.

Lemma 6 $\Gamma(\mathbf{x}_0)$ *is k-symmetric with respect to the reversing k-symmetry S if and only if* $\exists m, n \in \mathbb{Z}$ *such that* $\mathbf{x}_0 \in \mathrm{Fix}(L^m \circ \phi_L^n(S))$.

Proof.

$$\mathbf{x}_0 = L^m \circ \phi_L^n(S)\mathbf{x}_0 \Leftrightarrow \forall p \in \mathbb{Z} \quad \phi_L^{n-p}(S) \circ L^p \mathbf{x}_0 = L^{-m-p}\mathbf{x}_0 \Box \tag{3.9}$$

Once again the fixed sets of interest here can be obtained via iteration of a few fixed sets

$$\mathrm{Fix}(L^{2n} \circ \phi_L^q(S)) = L^n(\mathrm{Fix}(\phi_L^{q+n}(S)) \tag{3.10}$$

$$\mathrm{Fix}(L^{2n+1} \circ \phi_L^q(S)) = L^n(\mathrm{Fix}(L \circ \phi_L^{q+n}(S))) \tag{3.11}$$

Lemma 7 $\mathbf{x}_0 \in \mathrm{Fix}(L^m \circ S) \Rightarrow \mathbf{x}_0 \in \mathrm{Fix}(\phi_L^m(S) \circ S)$.

Proof.
$L^m \circ S\mathbf{x}_0 = \mathbf{x}_0 \Rightarrow L^m \circ S \circ L^m \circ S\mathbf{x}_0 = \mathbf{x}_0 \Box$

Now we state a generalisation of the FSI method for systems with reversing k-symmetries.

Theorem 8 *Suppose* Γ *is a periodic orbit of a map L that is k-symmetric with respect to a reversing k-symmetry S. Then all points of* Γ *lie on intersections of iterates of* $\mathrm{Fix}(\phi_L^p(S))$ *and/or* $\mathrm{Fix}(L \circ \phi_L^q(S))$ *for integer value of p and q.*

Proof. If Γ is k-symmetric then $\exists$ m, n such that

$$\phi_L^n(S)\mathbf{x}_0 = L^{-m}\mathbf{x}_0 \Rightarrow \mathbf{x}_0 \in \mathrm{Fix}(L^m \circ \phi_L^n(S)) \tag{3.12}$$

Suppose the period of the k-symmetric orbit is $\tilde{p}$ then

$$\phi_L^n(S)\mathbf{x}_0 = L^{-m}\mathbf{x}_0 \Rightarrow \phi_L^n(S)\mathbf{x}_0 = L^{-m-\tilde{p}}\mathbf{x}_0 \Rightarrow \mathbf{x}_0 \in \mathrm{Fix}(L^{m+\tilde{p}} \circ \phi_L^n(S)) \tag{3.13}$$

Thus

$$\mathbf{x}_0 \in \mathrm{Fix}(L^m \circ \phi_L^n(S)) \cap \mathrm{Fix}(L^{m+\tilde{p}} \circ \phi_L^n(S)) \tag{3.14}$$

Using (3.10) and (3.11) the result follows. $\Box$

The converse of Theorem 8, i.e. that all intersections of iterates of $\mathrm{Fix}(\phi_L^p(S))$ and/or $\mathrm{Fix}(\phi_L^q(S))$ for all $p, q \in \mathbb{Z}$ are periodic points, can be established with some mild technical conditions (see Lamb and Quispel (1993)).

4 EXAMPLE

There are a number of examples of dynamical systems, arising in physics, which possess (reversing) k-symmetries. Here we shall present the example of a kicked rotator. Other examples are discussed in Lamb and Quispel (1993). A detailed discussion of these examples is beyond the scope of the present paper.

The equations of motion (of the components of the angular momentum of the rotator) for a kicked rotator can be written as (D'Ariano et al., 1992)

$$L : \begin{cases} x' &= -z, \\ y' &= y\cos(kz) - x\sin(kz), \\ z' &= x\cos(kz) + y\sin(kz). \end{cases} \tag{4.1}$$

L possesses the reversing symmetry P_{xz}

$$P_{xz} : \begin{cases} x' &= z, \\ y' &= y, \\ z' &= x. \end{cases} \tag{4.2}$$

Consider the maps

$$I_{xy} : \begin{cases} x' &= -x, \\ y' &= -y, \\ z' &= z, \end{cases} \quad \text{and } I_{yz} : \begin{cases} x' &= x, \\ y' &= -y, \\ z' &= -z. \end{cases} \tag{4.3}$$

and $I_{xz} = I_{xy} \circ I_{yz}$.

L further possesses the symmetry I_{xz} . Moreover, L^2 possesses I_{xy} as a symmetry (D'Ariano et al. 1992) , implying that I_{xy} is a 2-symmetry of L. In fact,

$$\phi_L(I_{xy}) = I_{yz}. \tag{4.4}$$

We find that these symmetries satisfy

$$\phi_L(I_{xz}) = I_{xz}, \tag{4.5}$$
$$\phi_L^2(I_{xy}) = \phi_L(I_{yz}) = I_{xy}, \tag{4.6}$$
$$\phi_L(P_{xz}) = P_{xz}, \tag{4.7}$$
$$\phi_L(P_{xz} \circ I_{xz}) = P_{xz} \circ I_{xz}, \tag{4.8}$$
$$\phi_L^2(P_{xz} \circ I_{xy}) = \phi_L(P_{xz} \circ I_{yz}) = P_{xz} \circ I_{xy}. \tag{4.9}$$

It would be very interesting to study k-symmetric periodic orbits in the spirit of Theorem 8.

5 CONCLUDING REMARKS

In this paper we generalized the concepts of (reversing) symmetries to (reversing) k-symmetries.

In setting up the frame-work for (reversing) k-symmetry groups, we carefully insisted that $k \in \mathbb{Z}$ be finite. We have chosen not to adopt the notion of "commutation of a group $\mathcal{G}$ with a map L", as used by MacKay and others (MacKay, 1984; Llibre and MacKay, 1992; Roberts and Baake, 1993) which permits k to be (formally) infinite. A group $\mathcal{G}$ commutes with a map L if

$$\forall U \in \mathcal{G}, \ \exists \tilde{U} \in \mathcal{G}, \ \text{ such that } U \circ L = L \circ \tilde{U}. \tag{5.1}$$

Although from a group theoretical point of view (5.1) is a nice equivalence relation, this equivalence does not automatically include important dynamical features. In particular, an $U \in \mathcal{G}$ satisfying (5.1) does not have to preserve the set of periodic orbits and the set of chaotic orbits, in contrast to (3.7) and (3.8). For instance, the group Aut(Ω), consisting of all invertible maps of the state space into itself, commutes with every $L \in \text{Aut}(\Omega)$. Of course, this observation does not reveal much about the dynamics of L. The closure of the orbits under ϕ_L is essential for the preservation of dynamical features such as periodic and chaotic orbits.

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