TUPLING IN THREE-DIMENSIONAL REVERSIBLE MAPPINGS

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Abstract

Feigenbaum's parameter scaling exponents δ are calculated for $k \cdot 3^m$, $k \cdot 5^m$, $k \cdot 7^m$, $k \cdot 9^m$ sequences of periodic in orbits a class of three-dimensional non-measure preserving reversible mappings. Some preliminary results on orbit scaling exponents for $k \cdot 3^m$ sequences are also included. Scaling exponents found so far are the same as those found for two-dimensional reversible mappings.

The presence of time-reversal symmetry (reversibility) in a dynamical system has important consequences for the dynamic behavior (for a review see [1]). Reversible mappings of the plane have been shown to exhibit a number of qualitative and quantitative similarities with their hamiltonian (symplectic) counterparts [1]-[5]. Analogous to hamiltonian systems, repeated application of local versions of the (reversible) KAM theorems and Poincaré-Birkhoff theorem lead to a complex picture of phase space in which structures are repeated on all scales. A particular aspect of this self-similarity known as class renormalisation [6] concerns the scaling properties of periodic orbits. The infinite hierarchy of periodic orbits observed is known to play an essential role in transport models of reversible and hamiltonian systems [7].

Period-doubling has been studied extensively in reversible area preserving mappings [8]-[14], non-area preserving reversible mappings of the plane [1]-[2] and polynomial mappings of the complex plane [15]. Sequences of higher order bifurcations (n-tupling with n>2) have also been studied in these mappings [6],[16]-[17]. The scaling behavior of higher dimensional mappings and their associated universality classes are less well understood and study has been restricted to period-doubling in four and six-dimensional symplectic reversible mappings [18]-[21]. Period doubling associated with curves of orbits in three-dimensional volume preserving reversible mappings with one integral have recently been examined [22]. In this study we construct a three-dimensional non-measure preserving reversible mapping and study the scaling properties of sequences of odd-length periodic orbits in some examples of this mapping.

A mapping $T : R^m \mapsto R^m$ is called *reversible* if it can be written as the composition of two involution mappings I_1 and I_2,

$$T = I_1 \circ I_2 \tag{1}$$

and

$$I_1 \circ I_1 = I_2 \circ I_2 = Id. \tag{2}$$

Define

$$\begin{aligned} A_{n,i} &= Fix(I_i) \cap Fix(T^{2n} I_i), i = 1,2 \\ B_n &= Fix(I_2) \cap Fix(T^{2n} I_1), \end{aligned} \tag{3}$$

where Fix denotes the set of invariant points of a mapping and T^n denotes the mapping T composed n-times. If $\mathbf{r} \in A_{n,i}$ (i=1,2), then $\mathbf{r}$ is a point on a periodic orbit of length 2n. If $\mathbf{r} \in B_n$, then r is a point on a periodic orbit of length 2n+1 [1].

Now consider a class of non-measure and orientation preserving two-dimensional (m=2) reversible mappings given by

$$T_{2D} : x' = \frac{f_1(y) + x}{1 - x f_3(y)}, y' = \frac{g_1(x') + y}{1 - y g_3(x')}, \tag{4}$$

with involutions

$$I_1 : x' = x, y' = \frac{g_1(x) - y}{1 + y g_3(x)}, I_2 : x' = \frac{f_1(y) + x}{1 - x f_3(y)}, y' = -y, \tag{5}$$

where functions g_1 and g_3 are arbitrary while f_1 and f_3 are arbitrary odd functions (this corresponds to Class IV of [1]). The fixed points of the involutions ($I_i\mathbf{r} = \mathbf{r}$, i=1,2) form lines in the plane called symmetry lines. As mentioned above, symmetry lines have special signifance because on these lines are points belonging to periodic orbits of T. Not all periodic orbits of T have a point on one or both of the lines but those that do are referred to as *symmetric* periodic orbits. The task of finding symmetric periodic orbits thus consists of searching along a line rather the a plane. Symmetric orbits of even period have two points on one of the symmetry lines while symmetric orbits of odd period have one point on both of the lines. Symmetric periodic orbits have return Jacobian determinant equal to ± 1 depending on whether

T is orientation preserving (+1) or reversing (−1) and on the odd or even length of the periodic orbit.

Our strategy for extending the class of mappings given by Eq. (4) into three-dimensions is to first trivially extend both the involutions

$$I_1 : x' = x, y' = \frac{g_1(x) - y}{1 + y g_3(x)}, z' = -z \tag{6}$$

$$I_2 : x' = \frac{f_1(y) + x}{1 - x f_3(y)}, y' = -y, z' = z. \tag{7}$$

Clearly the z is decoupled in these equations and to remedy this we conjugate one of the involutions, I_2, by a nonlinear invertible transformation

$$S : x' = x + \epsilon z h(y), y' = y, z' = z + \epsilon y \tag{8}$$

where h is an arbitrary function and ϵ is a perturbation parameter (when $\epsilon = 0$, Eq. (8) is the identity and the composition of the two involutions is decoupled in z and hence effectively two-dimensional). Note that $I_2' = S^{-1} \circ I_2 \circ S$ is still an involution. We now construct the new mapping $T_{3D} : R^3 \mapsto R^3$

$$T_{3D} = I_1 \circ I_2' = I_1 \circ (S^{-1} \circ I_2 \circ S), \tag{9}$$

which written explicitly gives

$$T_{3D} : \begin{array}{l} x' = \frac{f_1(y) + x + \epsilon z h(y)}{1 - (x + \epsilon z h(y)) f_3(y)} - \epsilon(z + 2\epsilon y) h(-y) \\ y' = \frac{g_1(x') + y}{1 - y g_3(x')} \\ z' = -(z + 2\epsilon y). \end{array} \tag{10}$$

Observe that Fix I_1 is one-dimensional (curve) while Fix I_2' is two-dimensional (plane). Referring to Eq. (3), this means that the set $A_{n,1}$ will in general be empty. The set $A_{n,2}$ will be infinite and the set B_n will be finite. This corresponds respectively to zero, infinitely many or finitely many, symmetric periodic orbits in each of the sets. Since we wish to have a finite set of periodic orbits of a given length we only consider the last case here, i.e. symmetric periodic orbits of odd length (cf. [22]).

For our first three examples we choose (following [1]-[3]) the functions f_1, f_3, g_1 and g_3 as

$$\begin{array}{l} f_1(x) = \frac{-x}{1+c^2x^2}, f_3(x) = \frac{c^2x^3}{c^2+x^2} \\ g_1(x) = 2x(1-k-x), g_3(x) = const. \end{array} \tag{11}$$

The parameter c represents the non-measure preserving perturbation and is fixed while k is a free parameter which we use to follow sequences of periodic orbits. Table I indicates the various choices of g_3, h(y), c and ϵ for our three examples. Our fourth example is also a non-measure preserving reversible three-dimensional mapping constructed from a different class of two-dimensional mappings (Class I of [1]) in the same way

$$\begin{array}{l} x' = (k-y)(1+(y'-1)(y'-1)) \\ y' = \frac{x+\epsilon(2y-k)(z+\epsilon(y-k))}{1+(y+1-k)^2} \\ z' = -z+\epsilon(k-2y). \end{array} \tag{12}$$

Again k is a free parameter while ϵ is specified in Table I.

The linear stability of a periodic orbit is determined by the eigenvalues of the Jacobian matrix which satisfy

$$\begin{array}{c} \lambda^3 - A\lambda^2 + B\lambda - C = 0 \\ A = tr(J), B = [tr^2(J) - tr(J^2)]/2, C = det(J) \end{array} \tag{13}$$

where tr(J) and det(J) denote the trace and determinant of the 3x3 return Jacobian matrix. The three coefficients A, B and C are not independent and can be reduced: for symmetric odd periodic orbits in orientation reversing mappings C=det(J)= -1 and we have one eigenvalue λ=-1 [22] which implies A=-B, so effectively there is just the one free coefficient (as is the case in two dimensions). The residue [23] can be defined conveniently as

$$R = \frac{1}{4}[2 - (tr(J) + 1)]. \tag{14}$$

For R<0 or R>1 the orbit is unstable (hyperbolic) while for 0<R<1 the orbit is stable (elliptic). For elliptic orbits we have

$$R = sin^2(\pi\omega) \tag{15}$$

where ω is the rotation number of a given orbit. Every periodic orbit is characterised by a rational rotation number $\omega = \frac{p}{q}$ (i.e $\lambda = \exp(2\pi ip/q)$) so that setting ω equal to some rational frequency in Eq. (15) gives the critical value of the residue for which periodic orbits of the same frequency are born [1]. The values of k corresponding to the critical value of the residue for each orbit with the appropriate frequency have been observed to form a convergent sequence for various classes of mappings [1], [15]. The convergence of the sequence is asymptotically geometric with the scaling factor δ given by

$$\delta_n = \frac{k_{n-1} - k_n}{k_n - k_{n+1}}, \tag{16}$$

and the limiting value of δ depends on the frequency i.e. $\delta(p,q)$ [6],[16]-[17].

We initially locate the sequences of periodic orbits $l\cdot 3^m$, $l\cdot 5^m$, $l\cdot 7^m$ and $l\cdot 9^m$ in (10) for $\epsilon = 0$ (the two-dimensional orbits) then follow these orbits as ϵ is increased to

[1] The eigenvalues of nonsymmetric periodic orbits in one-parameter families of reversible maps generically do not lie on the on the unit circle. Therefore nonsymmetric periodic orbits in such families of maps do not exhibit q-tupling for $q \neq 2$.

some fixed value (see Table I). For most sequences our starting point is the symmetric (elliptic) fixed point of T_{3D} (i.e. $l = 1$) however $5 \cdot 3^m$ and $3 \cdot 5^m$ sequences were also examined in some of the examples. As k is increased periodic orbits of various frequencies are born at the fixed point and the appropriate orbits are then tracked. Periodic orbits with length of order 10^6 were located using the secant method and calculations were performed in quadruple-precision VAX FORTRAN. In Table II are the results obtained for parameter scalings of our four examples specified by Eq. (10), Eq. (12) and Table I. In Table III we present some preliminary results on orbit scaling for $l \cdot 3^m$ sequences in two examples of the mapping Eq. (10). We calculate the distance scalings associated with three-points of each $l \cdot 3^m$ orbit that have bifurcated from the point of the $l \cdot 3^{m-1}$ orbit that lies in the symmetry plane (see Figure I and [16]; for alternative orbit scaling exponents for two-dimensional maps refer to [6]). The two distances for the $l \cdot 3^m$ orbit are calculated at the birth of the $l \cdot 3^{m+1}$ orbit. Two distance scalings of the $l \cdot 3^m$ sequences of orbits were determined to be $\alpha = \alpha_1\alpha_2 \approx$ -44 and $\beta = \beta_1\beta_2 \approx$ -187. The scaling exponents obtained so far appear to be the same as those found for area preserving two-dimensional mappings. We hope to publish an extended version of this work in the near future.

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Table I Parameter and function settings for the four mappings. Examples 1,2,3 are given by Eq. (10) and Example 4 by Eq. (12).

	Example 1	Example 2	Example 3	Example 4
g_3	0	-c	-c	-
$h(y)$	y	y	y^2	-
c	0.02	0.1	0.1	-
ϵ	0.001	0.001	0.01	0.01

Table II Parameter scaling exponents $\delta(p,q)$ for various $\frac{p}{q}$-tupling sequences in *three-dimensional* reversible maps. For comparison we give the corresponding results for *two-dimensional* area-preserving reversible maps [6],[16] in the last row.

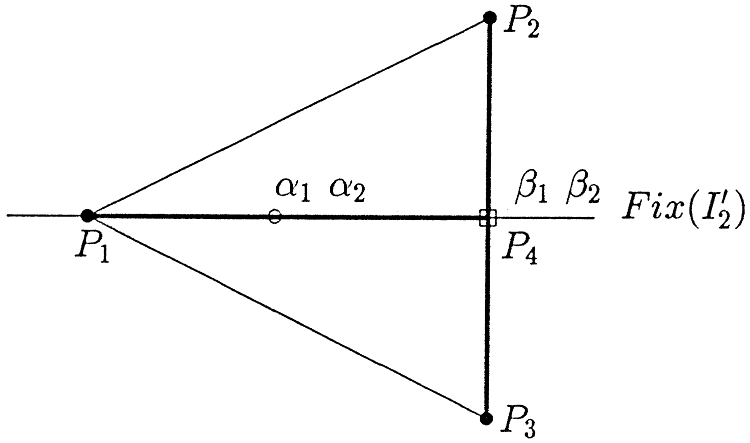
	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{9}$
Example 1	20.184	20.047	–	10.807	–	9.08
Example 2	20.1848	20.047	–	10.80	–	9.0
Example 3	20.184	20.047	–	10.807	–	
Example 4	20.1	–	30.2	–	39.2	
2-D MAPS	20.1848	20.0478	30.257	10.8076	39.279	9.0814

Table III Orbit scaling exponents for $\frac{1}{3}$-tupling sequences in *three-dimensional* reversible maps (see Figure I). For comparison we give the corresponding results for *two-dimensional* area-preserving reversible maps in the last row.

	α_1	α_2	β_1	β_2
Example 1	-17.9	2.45	5.94	-31.4
Example 2	-17.9	2.4	5.94	-31.4
2-D MAPS	-17.9	2.45	5.94	-31.4

Caption for Figure I

Figure I Schematic diagram of the distances used to calculate the orbit scaling exponents α_i and β_i (i=1,2) for a $l \cdot 3^m$ orbit. Drawn are the triangle formed by three points of the $l{\cdot}3^m$ orbit (solid circles denoted P_j, j=1,...3) which has bifurcated from the $l \cdot 3^{m-1}$ orbit (open circle) ; the horizontal line represents the symmetry plane $Fix(I_2')$. The points P_1 and P_4 (hollow square which is not a point in any of the orbits) are in $Fix(I_2')$. The distances calculated are indicated by the heavy lines. The orbit scalings are simply the ratio of each distance of successive orbits: $\alpha_{1,2}$:=$\overline{(P_1 - P_4)}_m / \overline{(P_1 - P_4)}_{m-1}$ and $\beta_{1,2}$:= $\overline{(P_2 - P_3)}_m / \overline{(P_2 - P_3)}_{m-1}$. There is in alternation in the magnitude and sign of the distances so that they scale as the products α=$\alpha_1\alpha_2$ and β=$\beta_1\beta_2$.



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