Physics Letters A 184 (1993) 64–70
North-Holland

PHYSICS LETTERS A

# Lie symmetries and the integration of difference equations

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Received 16 June 1993; revised manuscript received 5 October 1993; accepted for publication 3 November 1993
Communicated by A.P. Fordy

The Lie method is generalised to ordinary *difference* equations. We prove that the order of ordinary difference equations can be reduced by one, provided the equation under consideration possesses an evolutionary Lie point symmetry. This is illustrated by solving a first-order difference equation. We also give a sufficient condition for reduction of the order by two. This is illustrated by solving a second-order difference equation.

## 1. Introduction

In the second half of the nineteenth century there existed a variety of special techniques designed to solve certain particular types of ordinary differential equations (ODEs) such as separable, homogeneous or exact equations. Sophus Lie unified and extended these techniques, through his discovery that they were all special cases of a general integration procedure based on the invariance of the differential equation under a continuous group of symmetries [1].

At present, the state of the theory of difference equations is somewhat similar to the state of the theory of differential equations one hundred years ago. A variety of special techniques is used to try to solve difference equations, e.g. substitution, nonlinear functional relation, Schröder's generating function, Maeda's method, or the theory of integrable maps [2–7]. In this Letter, and in forthcoming publications, our aim will be to show that once again all these different methods can be unified and extended in a general integration procedure based on the invariance of the difference equation under a continuous group of symmetries [#1].

In this Letter, our approach will be twofold. Firstly, we show how Lie's algorithm for finding symmetries can be extended to difference equations. Secondly, we show how, using these Lie symmetries, the order of a system of difference equations can be reduced by one or two. Reduction of the order by one is illustrated by solving a first-order difference equation explicitly. Reduction of the order by two is illustrated by solving a second-order difference equation explicitly.

## 2. Reduction of the order of difference equations from $q$ to $q-1$

Consider a system of $q$ coupled "first-order" difference equations given by

$$u_i(x+1) = F_i(x, u_1(x), u_2(x), \dots, u_q(x)),\quad i = 1, 2, \dots, q, \tag{1}$$

where the $F_i$ are given functions. (Note that the independent variable $x$ is assumed to take on all real values, and is not restricted to the integers.) Assume that this system is invariant under the one-parameter infinitesimal evolutionary Lie point symmetry [#2]

$$x^* = x, \tag{2a}$$

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[#1] For a somewhat different approach see ref. [8].

[#2] An evolutionary symmetry is defined as a symmetry that does not change the independent variable. Also note that the essential and crucial difference with Maeda's method is that we do *not* (and need *not*) assume that the symmetry is a function of the dependent variable only.

$$u_i^*(x^*) = u_i(x) + \epsilon v_i(x, \boldsymbol{u}(x)),$$
$$i = 1, 2, \ldots, q, \tag{2b}$$

where $\boldsymbol{u}(x) = (u_1(x), u_2(x), \ldots, u_q(x))$. *Then system (1) reduces from $q$ to $q-1$ ordinary difference equations.*

The proof (which is an analog of the proof of reduction of order by canonical coordinates for ODEs) proceeds in two parts:

(1) Any infinitesimal point symmetry

$$x^* = x + \epsilon\xi(x, \boldsymbol{u}(x)),$$
$$u_i^*(x^*) = u_i(x) + \epsilon v_i(x, \boldsymbol{u}(x)),$$
$$i = 1, 2, \ldots, q, \tag{3}$$

can be transformed to

$$y^* = y,$$
$$w_i^*(y^*) = w_i(y), \quad i = 1, 2, \ldots, q-1,$$
$$w_q^*(y^*) = w_q(y) + \epsilon, \tag{4}$$

by introducing canonical coordinates

$$y = \eta(x, \boldsymbol{u}), \quad \boldsymbol{w} = \boldsymbol{\zeta}(x, \boldsymbol{u}). \tag{5}$$

For a proof, see ref. [1].

(2) For ordinary *differential* equations (ODEs) the above is sufficient to prove the reduction of order by one (see ref. [1]), as any transformation (5) takes an ODE into another ODE. An ordinary *difference* equation, however, is not always transformed into another difference equation by a general transformation as given in eq. (5). A sufficient condition for this to happen, is that the transformation be of the form

$$y = x, \quad \boldsymbol{w} = \boldsymbol{\zeta}(x, \boldsymbol{u}), \tag{6}$$

which is the case for evolutionary point symmetries (since we can rectify (2b) separately).

We thus know that, in terms of $x$ and $w$, system (1) takes the form

$$w_i(x+1) = H_i(x, w_1(x), \ldots, w_{q-1}(x)),$$
$$i = 1, 2, \ldots, q-1, \tag{7a}$$

$$w_q(x+1) = w_q(x) + H_q(x, w_1(x), \ldots, w_{q-1}(x)). \tag{7b}$$

Note that eq. (7b) is decoupled from eq. (7a) and can be solved trivially for $w_q$ in terms of $x, w_1(x), \ldots, w_{q-1}(x)$. As a corollary, we see that a system that possesses one evolutionary symmetry possesses infinitely many. This is so because eq. (7) is actually invariant under the infinite-dimensional symmetry group

$$x^* = x,$$
$$w_i^*(x) = w_i(x), \quad i = 1, 2, \ldots, q-1,$$
$$w_i^*(x) = w_q(x) + \epsilon\lambda(x), \tag{8}$$

where $\lambda(x)$ is an arbitrary unit periodic function. This explains why the above reduction will not necessarily work for ordinary differential–difference equations or for difference equations with incommensurate spans.

## 3. An example: solution of first-order difference equation

To derive Lie symmetries for difference equations[#3], we need the following "prolongation" [9,10],

$$u^*(x^* + \omega) = u(x+\omega) + \epsilon v(x+\omega, u(x+\omega))$$
$$+ \epsilon[\xi(x, u(x)) - \xi(x+\omega, u(x+\omega))]$$
$$\times \frac{\mathrm{d}}{\mathrm{d}x}u(x+\omega), \tag{9}$$

where in this Letter the spans $\omega$ will be either 1 or 2.

A general form for an autonomous "first-order" difference equation (i.e. an equation with a single span) is

$$u(x+1) = F(u(x)), \tag{10}$$

where $F$ is a given function. Equation (10) is invariant under transformation (3) if

$$u^*(x^*+1) = F(u^*(x^*)), \tag{11}$$

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[#3] We first find *all* point symmetries, and then choose one particular evolutionary symmetry to perform the reduction of order.

provided $u$ satisfies (10). Using (3) and (9) in eq. (11) we obtain

$$\begin{aligned} &v(x+1, u(x+1)) \\ &\quad + [\xi(x, u(x)) - \xi(x+1, u(x+1))] \frac{\mathrm{d}u(x+1)}{\mathrm{d}x} \\ &\quad = v(x, u(x)) \frac{\partial F}{\partial u}(u(x)). \end{aligned} \tag{12}$$

Equation (12) implies

$$\xi(x, u(x)) - \xi(x+1, u(x+1)) = 0, \tag{13a}$$

$$\begin{aligned} &v(x+1, u(x+1)) \\ &\quad = v(x, u(x)) \frac{\partial F}{\partial u}(u(x)). \end{aligned} \tag{13b}$$

Any unit periodic function $\alpha(x)$, i.e. a function defined by $\alpha(x) = \alpha(x+1)$, is a solution of (13a) [10,11], and substituting the equation of motion (10) in (13b), we obtain

$$\xi(x, u(x)) = \alpha(x), \tag{14a}$$

$$\begin{aligned} &v(x+1, F(u(x))) \\ &\quad = v(x, u(x)) \frac{\partial F}{\partial u}(u(x)). \end{aligned} \tag{14b}$$

Our main task now is to find a solution of the functional equation (14b) [12]. This may not always be easy. Here we will indicate a possible way by which we can arrive at such a solution. To this end, we assume the equation of motion (10) has a "fixed point" (in this Letter we restrict ourselves to fixed points at $u = \infty$) and that $F(u(x))$ and $v(x, u(x))$ can be expanded near $u = \infty$ as a Laurent series in $1/u(x)$ (we could equivalently apply a transformation $U(x) := 1/u(x)$ and expand near the fixed point $U(x) = 0$ in a Taylor series in $U(x)$),

$$\begin{aligned} v(x, u(x)) &= a_0(x) u^b(x) + a_1(x) u^{b-1}(x) \\ &\quad + a_2(x) u^{b-2}(x) + \ldots, \end{aligned} \tag{15}$$

$$\begin{aligned} F(u(x)) &= c_0 u^d(x) + c_1 u^{d-1}(x) \\ &\quad + c_2 u^{d-2}(x) + \ldots, \quad d > 0, \end{aligned} \tag{16}$$

where the $a_i$, $i = 0, 1, 2, \ldots$, are unknown functions of $x$, and $b$ is an unknown constant, all to be determined. The constants $c_i$, $i = 0, 1, 2, \ldots$, and $d$ follow from the given function $F(u(x))$. Inserting expansions (15) and (16) in (14b), and equating the coefficients of leading-order terms it follows that

$$b = 1 \quad (\text{unless } d = 1), \tag{17a}$$

$$a_0(x+1) = d a_0(x). \tag{17b}$$

If $d = 1$ the Laurent series expansion fails to determine the coefficient $b$. The solution of eq. (17b) is

$$a_0(x) = d^x \beta(x), \tag{18}$$

where $\beta(x)$ is another unit periodic function. We then calculate the coefficients $a_1(x), a_2(x), \ldots$, until the series terminates or (hopefully) a pattern emerges allowing us to determine $v(x, u(x))$. Once we have obtained the infinitesimal evolutionary symmetry $v(x, u(x))$, the homogenizing variable $w(x)$ will be given by

$$w(x) = \int^{u(x)} \frac{\mathrm{d}u'}{v(x, u')}. \tag{19}$$

To make the above procedure more specific, let us study the following example,

$$u(x+1) = F(u(x)) = 4u^3(x) + 3u(x), \tag{20}$$

where $u$ is a function: $\mathbb{R} \to \mathbb{R}$. Thus eq. (14b) becomes

$$\begin{aligned} &v(x+1, 4u^3(x) + 3u(x)) \\ &\quad = v(x, u(x)) [12u^2(x) + 3]. \end{aligned} \tag{21}$$

Equation (20) is a cubic equation, hence it follows from eq. (16) that $d = 3$, and therefore

$$a_0(x) = 3^x \beta(x), \tag{22}$$

with $\beta(x) = \beta(x+1)$. Inserting eq. (15) in (21) and equating the coefficients in the expansions on the right-hand side and the left-hand side, we obtain the following expansion for the infinitesimal symmetry $v(x, u(x))$,

$$\begin{aligned} v(x, u) &= 3^x \beta(x) u (1 + \tfrac{1}{2} u^{-2} - \tfrac{1}{8} u^{-4} \\ &\quad + \tfrac{1}{16} u^{-6} - \tfrac{5}{128} u^{-8} + \ldots). \end{aligned} \tag{23}$$

We recognize the terms in the square bracket as the first terms in the expansion of $[1 + u^{-2}(x)]^{1/2}$. Indeed, it is easy to verify that

$$v(x, u(x)) = 3^x \beta(x)[1 + u^2(x)]^{1/2} \qquad (24)$$

is a solution of (21), and hence an infinitesimal symmetry of (20). (*Note that, even for autonomous difference equations, the symmetries are generally nonautonomous.* Note also that, in agreement with what was said before, eq. (20) possesses infinitely many evolutionary symmetries.)

The homogenizing variable $w(x)$ given by (19) is then [#4]

$$w(x) = 3^{-x} \int^{u(x)} \frac{du'}{\sqrt{1 + u'^2}} = 3^{-x} \operatorname{arcsinh}[u(x)]. \qquad (25)$$

Inverting this transformation, and substituting in the equation of motion (20), we obtain the equation satisfied by $w(x)$,

$$w(x + 1) = w(x). \qquad (26)$$

Obviously, the solution of (26) is

$$w(x) = \gamma(x), \qquad (27)$$

where $\gamma(x)$ is an arbitrary unit periodic function. Using (25) again, we obtain the general solution of our eq. (20),

$$u(x) = \sinh[3^x \gamma(x)], \qquad (28)$$

where, as was mentioned,

$$\gamma(x) = \gamma(x + 1), \quad \forall x \in \mathbb{R}.$$

(Those readers who are primarily interested in mappings rather than in difference equations, can in the final result restrict $x$ to be integer and $\gamma(x)$ to be a constant.)

## 4. Reduction of the order of difference equations from $q$ to $q - 2$

We have seen that a system of $q$ difference equations can be reduced to $q - 1$ difference equations if the system possesses an infinite family P of evolutionary symmetries. We now show that a further reduction to $q-2$ difference equations is possible, provided the system possesses an extra evolutionary symmetry $X_2$, in addition to the infinite family P, and provided that $X_2$ commutes with some member ($X_1$, say) of P. In order to prove this, we may assume that a transformation has taken place such that

$$X_1 = \frac{\partial}{\partial w_q}, \qquad (29a)$$

$$X_2 = \sum_{i=1}^{q} b_i(x, w_1, \ldots, w_q) \frac{\partial}{\partial w_i}, \qquad (29b)$$

and the system has form (7). From

$$[X_1, X_2] = 0 \qquad (30)$$

we obtain

$$\frac{\partial}{\partial w_q} b_i(x, w_1, \ldots, w_q) = 0, \quad i = 1, 2, \ldots, q. \qquad (31)$$

Hence

$$b_i = b_i(x, w_1, w_2, \ldots, w_{q-1}), \quad i = 1, 2, \ldots, q. \qquad (32)$$

We distinguish two cases:

$$\text{(I)}\ b_i(x, w_1, w_2, \ldots, w_{q-1}) \neq 0 \quad \text{for some } i \in \{1, 2, \ldots, q - 1\}, \qquad (33a)$$

$$\text{(II)}\ b_i(x, w_1, w_2, \ldots, w_{q-1}) = 0, \quad \forall i \in \{1, 2, \ldots, q - 1\}. \qquad (33b)$$

In case (I) we are done, because we thus have the system of $q - 1$ difference equations (7a) invariant under the evolutionary symmetry,

$$\sum_{i=1}^{q-1} b_i(x, w_1, w_2, \ldots, w_{q-1}) \frac{\partial}{\partial w_i}. \qquad (34)$$

Hence the system of equations (7a) can be further reduced to a system of $q - 2$ difference equations.

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[#4] We have chosen $\alpha(x) \equiv 0$, $\beta(x) \equiv 1$.

In case (II) system (7) is invariant under

$$b_q(x, w_1, w_2, \ldots, w_{q-1})\frac{\partial}{\partial w_q}. \tag{35}$$

This implies that

$$b_q(x+1, w_1(x+1), w_2(x+1), \ldots, w_{q-1}(x+1))$$
$$= b_q(x, w_1(x), w_2(x), \ldots, w_{q-1}(x)) \tag{36}$$

and therefore

$$b_q(x, w_1(x), w_2(x), \ldots, w_{q-1}(x)) = \mu(x), \tag{37}$$

where $\mu(x)$ is an arbitrary unit periodic function. Since $X_2 \notin \mathrm{P}$, this equation is not a trivial identity, i.e.

$$\frac{\partial}{\partial w_i} b_q(x, w_1, \ldots, w_{q-1}) \neq 0$$

for some $i \in \{1, \ldots, q-1\}$. Eliminating $w_i(x)$ between (7a) and (37), (7a) is reduced from $q-1$ to $q-2$ difference equations.

## 5. An example: solution of a second-order difference equation

For convenience, our example will be a second-order difference equation, rather than a system of two coupled first-order systems. Our example is the equation[#5]

$$u(x+2) = \frac{2u(x+1) - u(x)[1 - u^2(x+1)]}{1 - u^2(x+1) + 2u(x)u(x+1)}, \tag{38}$$

where $u$ is a function: $\mathbb{R} \to \mathbb{R}$. This is a second-order difference equation (i.e. an equation with $\omega_2 = 2\omega_1$). Invariance under the one-parameter infinitesimal point transformation (3) gives that $\xi$ is again a unit periodic function in $x$ alone,

$$\xi(x, u(x)) = \alpha(x) \tag{39a}$$

and that $v(x, u(x))$ must satisfy

$$v(x+2, u(x+2))[1 - u^2(x+1) + 2u(x+1)u(x)]$$
$$-2v(x+1, u(x+1))\{u(x+1)[u(x+2) + u(x)]$$
$$- u(x+2)u(x) + 1\} + v(x, u(x))$$
$$\times [1 - u^2(x+1) + 2u(x+1)u(x+2)] = 0, \tag{39b}$$

provided $u(x)$ is a solution of (38). To determine the infinitesimal symmetry $v(x, u(x))$, we again expand $v(x, u(x))$ in a Laurent series in $1/u(x)$,

$$v(x, u(x)) = a_0(x)u^b(x) + a_1(x)u^{b-1}(x) + \ldots. \tag{15}$$

To find the leading-order exponent $b$, we assume $u(x+1) \gg 1$, $u(x) \gg 1$. Hence

$$u(x+2) \approx \frac{u(x+1)u(x)}{2u(x) - u(x+1)}. \tag{40}$$

Inserting (15) and (40) in (39b), and comparing the leading-order terms, we find

$$b = 2, \tag{41a}$$

$$a_0(x+2) - 2a_0(x+1) + a_0(x) = 0. \tag{41b}$$

The solution of (41b) is

$$a_0(x) = x\beta(x) + \gamma(x), \tag{42}$$

where $\beta(x)$ and $\gamma(x)$ are arbitrary unit periodic functions.

To determine the other coefficients $a_1(x)$, $a_2(x)$, $a_3(x)$, ..., we now go to the regime where $u(x+1) \gg u(x) \gg 1$. Denoting

$$\frac{1}{u(x)} = p, \quad \frac{u(x)}{u(x+1)} = q,$$
$$\frac{1}{u(x+1)} = pq, \tag{43}$$

where $p$ and $q$ are small parameters and expanding (39b) in a Laurent series in $p$ and $q$, we obtain

$$a_1(x) = 0, \quad a_2(x) = a_0(x),$$
$$a_3(x) = a_4(x) = a_5(x) = \ldots = 0. \tag{44}$$

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[#5] This equation corresponds to the special case $\alpha_1 = -1$, $\gamma_0 = 1$, $\epsilon_0 = -2$, $\epsilon_1 = -2$, $\mu = -1$ (all other parameters zero) of the 12-parameter family of symmetric integrable mapping given in ref. [6].

This leads us to

$$v(x, u(x)) = [x\beta(x) + \gamma(x)][u^2(x) + 1], \quad (45)$$

which, as is easily verified, is indeed a solution of eq. (39b) and hence a local point symmetry of (38).

It follows that, in this example, the two commuting symmetries can (without loss of generality) be taken to be

$$X_1 = (u^2 + 1)\partial/\partial u, \quad (46a)$$

$$X_2 = x(u^2 + 1)\partial/\partial u. \quad (46b)$$

The homogenizing variable $w$ corresponding to (46a) is

$$w(x) = \int^{u(x)} \frac{du'}{u'^2 + 1} = \arctan[u(x)]. \quad (47)$$

Inverting and substituting in (38) we find $w(x)$ satisfies

$$w(x+2) - 2w(x+1) + w(x) = \kappa\pi, \quad (48)$$

for some $\kappa \in \mathbb{Z}$. The solution of this equation is given by

$$w(x) = x\delta(x) + \lambda(x) + \tfrac{1}{2}\kappa\pi x^2, \quad (49)$$

where $\delta(x)$ and $\lambda(x)$ are arbitrary unit periodic functions. Using (47) again we obtain the general solution of our original equation of motion (38),

$$u(x) = \tan[x\delta(x) + \lambda(x) + \tfrac{1}{2}\kappa\pi x^2] \quad (50)$$

where, as was mentioned,

$$\delta(x) = \delta(x+1), \quad \lambda(x) = \lambda(x+1),$$

$$\forall(x) \in \mathbb{R}; \ \kappa \in \mathbb{Z}.$$

(If eq. (38) is considered as a mapping, $\delta(x)$ and $\lambda(x)$ should be taken constant.)

## 6. Summary and discussion

We have extended Lie's method for solving differential equations to the case of difference equations. As examples we have treated a first-order and a second-order autonomous ordinary difference equation, our method however can also be applied to nonautonomous equations, and to solve or reduce the order of higher-order equations. Note that for ordinary *differential* equations a sufficient condition for reduction of order is that the symmetry algebra is solvable. In this Letter we have basically proved that for *difference* equations a sufficient condition for reduction of order is the possession of a (one- or two-dimensional) Abelian symmetry algebra. To date we have not been able to extend our theorems for difference equations to the non-Abelian case. More applications of the above approach to autonomous and nonautonomous difference equations (e.g. to the logistic equation) will be given in ref. [13]. For applications of this method to partial differential–difference equations see refs. [10,14].

## Acknowledgement

We are grateful to G. Prince and F. Haggar for very useful comments, and to K.M. Briggs for bringing ref. [3] to our attention. This investigation is part of the research programme of the Australian Research Council.

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