KAM theorems for the product of two involutions of different types

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(Received 15 June 1993; accepted for publication 18 August 1993)

The properties of quasiperiodic motions in Hamiltonian, volume preserving, and reversible systems are summarized. KAM theorems concerning lower-dimensional invariant tori are announced for G-reversible mappings A such that the fixed point manifolds $\mathrm{Fix}(G)$ and $\mathrm{Fix}(AG)$ of the reversing involutions G and AG are of different dimensions. The case where the manifold $\mathrm{Fix}(G)$ itself consists of several connected components of different dimensions is also briefly discussed.

I. INTRODUCTION

In the studies of the behavior of dynamical systems, invariant tori filled up with quasiperiodic motions continue to receive much attention. Their importance stems, in the long run, from the fact that any finite-dimensional connected compact abelian Lie group is a torus.[1,2] Equilibrium points and closed trajectories of vector fields as well as fixed points of mappings can be considered as extreme particular cases of invariant tori. The properties of invariant tori are very sensitive to what structures (symmetries) the dynamical system in question is assumed to preserve. Invariant tori of a system that possesses no symmetry and satisfies no conservation law (such systems are often said to be dissipative) are generically isolated in the phase space, and one cannot expect the motion on those tori to be quasiperiodic.[3] On the contrary, dissipation-free or symmetric (Hamiltonian, volume preserving, reversible, etc.) systems often exhibit smooth or Cantor families of invariant tori with quasiperiodic dynamics. The latter case is the subject of the Kolmogorov–Arnold–Moser (KAM) theory.

The KAM theory is always and justifiably referred to as a branch of the perturbation theory. Nevertheless, the central KAM results can be reformulated in an entirely nonperturbative manner. The most abstract form of the KAM theorems is that for some integers $d_1,...,d_\nu$, k, m and some structure (characterized by integers $d_1,...,d_\nu$) in the phase space, a *typical* dynamical system preserving this structure admits k-parameter Cantor families of *reducible* invariant m-tori supporting *Diophantine quasiperiodic* motions, the $(k+m)$-dimensional Lebesgue measure of the union of these tori being *positive.*

Here and henceforth, the word "typical" means the following: there is a manifold M of class C^R equipped with the structure under consideration such that in the space of all C^r systems on M preserving this structure, systems possessing families of invariant tori described above constitute a set with nonempty interior in the C^r topology. We will not be interested in the values of R and r and shall only suppose that R and r are sufficiently large. In many cases, KAM theorems have in fact been proven in the real analytic category only. As a rule, the KAM technique provides invariant tori which are analytic (C^∞, finitely smooth) if the system in question is analytic (C^∞, finitely smooth, respectively).

Recall also the meaning of the words "reducible" and "Diophantine." Given a vector field V on an N-dimensional manifold M, a V-invariant m-torus in M with quasiperiodic motions is said to be reducible (to be more precise, the variational equations along the torus are said to be reducible) if near this torus, one can introduce coordinates

$$(\varphi_1 \bmod 2\pi,...,\varphi_m \bmod 2\pi,\ x_1,...,x_{N-m}) \tag{1}$$

in which the torus takes the form $(x=0)$ and the field V determines the system of differential equations of the Floquet type

$$\frac{d\varphi}{dt}=\omega+O(x),\quad \frac{dx}{dt}=\Omega x+O(|x|^2), \tag{2}$$

where ω is the frequency vector and Ω a constant matrix. Quasiperiodic motion $d\varphi/dt=\omega$ on the torus $(x=0)$ is called Diophantine if the frequencies $\omega_1,...,\omega_m$ are strongly incommensurable, i.e., they satisfy the Diophantine condition

$$|q_1\omega_1+\cdots+q_m\omega_m|>c(|q_1|+\cdots+|q_m|)^{-\gamma}$$

for all $q\in\mathbb{Z}^m\setminus\{0\}$ where c and γ are some positive constants. Analogously, given a mapping A: $M\to M$, an A-invariant m-torus in M with quasiperiodic dynamics is said to be reducible if near this torus, there exist coordinates (1) in which the torus takes the form $(x=0)$ and the mapping A, the form

$$A:\quad \varphi'=\varphi+\omega+O(x),\quad x'=\Omega x+O(|x|^2), \tag{3}$$

where ω and Ω are constants. The Diophantine condition on the frequency vector ω in the discrete-time case is

$$|2\pi q_0+q_1\omega_1+\cdots+q_m\omega_m|>c(|q_1|+\cdots+|q_m|)^{-\gamma}$$

for all $(q_1,...,q_m)\in\mathbb{Z}^m\setminus\{0\}$, $q_0\in\mathbb{Z}$ where c and γ are some positive constants.

1054-1500/93/3(4)/757/13/$6.00

Finally, the words "k-parameter Cantor family" mean a family of objects parametrized over a Cantor set in $\mathbb{R}^k$ (i.e., a closed set in $\mathbb{R}^k$ whose complement is everywhere dense) of positive measure (cf. Ref. 4).

Remark. The term "reducible" is sometimes used in a completely different sense, namely, an invariant manifold is said to be reducible if it contains an orbit not everywhere dense. An m-torus supporting a parallel motion with frequencies $\omega_1,...,\omega_m$ is reducible in that sense if it is resonant, i.e., the numbers $\omega_1,...,\omega_m$ (for flows) or π, $\omega_1,...,\omega_m$ (for mappings) are rationally dependent.

Almost all the results in the KAM theory pertain to Hamiltonian vector fields, exact symplectic diffeomorphisms, globally volume preserving vector fields and diffeomorphisms, and reversible vector fields and diffeomorphisms. The present paper continues the series of articles[5-10] in which lower-dimensional invariant tori in reversible systems were studied.

Recall that a vector field V on a manifold M is said to be *reversible* with respect to an involution G of this manifold ($G: M\rightarrow M$, $G^2=id$) if G transforms V into the opposite field $-V$, i.e., $G(x(-t))$ is a solution of the equation $dx/dt=V(x)$ whenever $x(t)$ is a solution. Analogously, an invertible mapping $A: M\rightarrow M$ is said to be *reversible* with respect to an involution G of M if G conjugates A with A^{-1}, i.e., $GAG=A^{-1}$, in which case the involution G casts the forward orbit of a point x under the mapping A into the backward orbit of the point $G(x)$. References 11–15 present a general survey on reversible systems as well as various examples and physical applications, Refs. 14 and 15 containing also an extensive bibliography.

Reversible dynamical systems combine the properties of Hamiltonian and dissipative systems, in particular, the phase space of a reversible system is often divided into regions with conservative dynamics and regions displaying entirely dissipative motion.[15-19] Moreover, the conservative and dissipative features of reversible systems may interact via symmetry-breaking bifurcations.[15,19]

Although the behavior exhibited by reversible systems in "conservative" regions of the phase space is very similar to that of Hamiltonian systems,[14] there is an important difference between reversible and Hamiltonian realms that manifests itself just in the conservative properties of reversible vector fields and mappings. According to the Darboux theorem,[20] all the symplectic structures in a given even phase space dimension are locally equivalent (note that all the volume elements in a given dimension are also locally equivalent). On the other hand, involutions differ in their *types*.[7-10,12-14] Namely, a smooth involution G is said to be of type (q,p) at a fixed point 0 if the linear part of G at 0 has the eigenvalue 1 of multiplicity p and the eigenvalue -1 of multiplicity q. Thus, whereas symplectic structures and volume elements have the only local invariant (the phase space dimension), involutions are described by two invariants (the numbers p and q). The situation is slightly relieved by the fact that these two integers constitute a complete collection of local invariants of the involution. Indeed, any involution is conjugate around a fixed point to its linear part (this is in fact true for a smooth action of any compact group and is usually referred to as the Bochner theorem[1,15]). If an involution G is of type (q,p) at a fixed point 0, one can introduce local coordinates $x\in\mathbb{R}^p$ and $y\in\mathbb{R}^q$ near 0 in which G takes the form G: $x'=x$, $y'=-y$. As a consequence, through 0 there passes the smooth p-dimensional manifold ($y=0$) consisting of fixed points of G. At each of these fixed points, the involution G is of type (q,p).

It is easy to prove that the fixed point set of any continuous involution of $\mathbb{R}^N$ is nonempty (cf. Ref. 21 where the case $N=2$ has been analyzed in detail). This is not necessarily true for involutions of other manifolds, e.g., the rotation $\varphi'=\varphi+\pi$ of a circle is an involution without fixed points. The type of such an involution cannot be well defined (cf. Sec. V below).

The existence of involutions G of different types [i.e., with different dimensions of the fixed point manifold $\mathrm{Fix}(G)$] for one and the same value of the phase space dimension is of great importance for the reversible KAM theory because the regions with conservative features of the motion in reversible systems are located around the fixed point manifolds of the reversing involutions.

Thus, speaking of a reversible vector field one should always specify the type of the reversing involution. The situation for reversible diffeomorphisms is twice as delicate. The reason is that any reversible mapping has in fact infinitely many reversing involutions (provided that no power of this mapping is the identity).[15,22] If a mapping A is reversible with respect to an involution G, then one can easily verify that all the mappings $A^jG=GA^{-j}$ for j an integer are also involutions which reverse A. The equality $A=(A^{j+1}G)(A^jG)$ shows that the mapping A can be decomposed into the product of two involutions in infinitely many ways. The set $\{A^jG\}$ is sometimes called a *family of symmetries* of A, all the involutions in this family enjoying equal rights. At first sight, a G-reversible mapping A is to be characterized by pointing out the types of all its symmetries A^jG. Fortunately, the types of only two of these reversing involutions, namely G and AG, are to be taken into account.

Suppose, for instance, that a mapping A and its reversing involution G have a common fixed point 0. Denote by A_1 and G_1, respectively, the linearizations of A and G at 0. Let $(q_\varepsilon,p_\varepsilon)$ be the type of the restriction of G_1 to the ε-root space of A_1, where $\varepsilon=\pm1$ [the (±1)-root spaces of A_1 are invariant under G_1, see Ref. 23]. Then the type of the involution A^jG at 0 is equal to

$$(q_1+q_{-1}+l,p_1+p_{-1}+l)$$

for j even and to

$$(q_1+p_{-1}+l,p_1+q_{-1}+l)$$

for j odd[23] where $2l$ is the difference between the phase space dimension and the sum $p_1+p_{-1}+q_1+q_{-1}$. More generally, the fixed point manifolds $\mathrm{Fix}(A^jG)$ of the involutions A^jG can be obtained from those of G and AG by taking their images under powers of A (see Refs. 15 and 22):

$$\mathrm{Fix}(A^{2j}G)=A^j[\mathrm{Fix}(G)],$$

$$\mathrm{Fix}(A^{2j+1}G)=A^j[\mathrm{Fix}(AG)].$$

Thus, if the involution G is of type (q,p), so is A^jG for any even j, and if the involution AG is of type (Q,P), so is A^jG for any odd j.

To summarize, G-reversible vector fields V can be classified according to the types of their reversing involutions G, and G-reversible mappings A according to the types of the two involutions G and AG. We shall not consider the so-called multiply reversible[15] fields V possessing more than one reversing involution G, and mappings A possessing more than one family of symmetries $\{A^jG\}$.

The fixed time shift F^t along the trajectories of a G-reversible vector field is reversible with respect to the same involution G, and vice versa. The involutions F^tG and G are always of the same type. More generally, let F_τ^t be the flow map for the time from τ to t of a nonautonomous differential equation $dx/dt=V(t,x)$. Let G be a phase space involution and assume that $G(x(-t))$ is a solution of the equation whenever $x(t)$ is a solution [the latter condition is equivalent to that for each t the involution G transforms vector field $V(t,\cdot)$ into the field $-V(-t,\cdot)$]. Then $GF_\tau^tG=F_{-\tau}^{-t}$ (see Refs. 12 and 13). Now suppose that the function $V(t,x)$ is periodic in time with period $2T$. Then

$$F_0^{2T}=F_T^{2T}F_0^T=GF_{-T}^{-2T}GF_0^T=GF_T^0GF_0^T=G(F_0^T)^{-1}GF_0^T.$$

Consequently, the succession map F_0^{2T} is G-reversible,[12,13,23] the involutions $F_0^{2T}G$ and G being of the same type.

Nevertheless, it is very easy to construct examples of involutions G and G-reversible mappings A such that the involutions AG and G have different types. Indeed, it suffices to set $A=\tilde{G}G$ where $\tilde{G}$ is an arbitrary involution of the type which is not equal to that of G.

Most studies of reversible systems to date have been restricted to the case where the reversing involution is of the "isosceles" type (p,p). There are also some papers on the reversible KAM theory which deal with vector fields[7–9,12–14,24,25] and diffeomorphisms[8,10,12,13,26] reversible with respect to involutions of types (q,p) with $p\neq q$.

However, until now all the theorems concerning quasiperiodic motions in reversible diffeomorphisms have been confined to G-reversible mappings A such that *the involutions G and AG are of the same type.* In the present paper, we remedy this situation by announcing a theorem on the existence of invariant tori for reversible mappings A which are the products of two involutions AG and G *of different types.*

II. GENERIC FAMILIES OF INVARIANT TORI

A. Hamiltonian systems

The essence of the Hamiltonian KAM theory for vector fields on finite-dimensional symplectic manifolds is the following statement.

Principle 1. For each $0\leqslant m\leqslant 2n-1$, a typical Hamiltonian vector field with n degrees of freedom possesses k-parameter Cantor families of reducible invariant m-tori filled up with Diophantine quasiperiodic motions, where $k=\min(m,2n-m)$. These tori are isotropic for $m\leqslant n$ and coisotropic for $m\geqslant n$ (Lagrangian for $m=n$). The $(k+m)$-dimensional Lebesgue measure of the union of the tori is positive.

Recall that a submanifold L of a symplectic manifold M is said to be *isotropic* if the restriction to L of the symplectic structure vanishes (i.e., at any point $x\in L$, the tangent space T_xL to L lies in its skew-orthogonal complement), and is said to be *coisotropic* if at any point $x\in L$, the tangent space T_xL contains its skew-orthogonal complement.

The literature devoted to the "classical" dimension $m=n$ in Principle 1 is now immense, see Ref. 27 for a review and a large bibliography. Lower-dimensional tori $(2\leqslant m<n)$ are considered, e.g., in Refs. 4, 25, and 28–43 in various contexts; Moson,[36] Huitema,[25] and Broer, Huitema, and Takens[42] dealing with families of vector fields rather than individual fields. The case $m>n$ has been analyzed in Refs. 44 and 45. The cases $m=0$ (equilibria) and $m=1$ (periodic trajectories) are trivial since they involve neither small divisors $q_1\omega_1+\cdots+q_m\omega_m$ nor arithmetical conditions on the frequencies $\omega_1,\ldots,\omega_m$. One-parameter families of invariant m-tori for $m=1$ and $m=2n-1$ are smooth rather than Cantor (see Ref. 45 for the nontrivial case $m=2n-1$).

The matrix Ω in Eq. (2) has eigenvalue 0 of multiplicity k at each torus (for $m>0$). For lower-dimensional tori $(m<n)$, the remaining $2(n-m)$ eigenvalues of the matrix Ω occur in pairs $\lambda, -\lambda$. Both the hyperbolic case ($\mathrm{Re}\,\lambda\neq 0$ for all the nonzero eigenvalues λ) and the nonhyperbolic case (some of these eigenvalues are purely imaginary) are possible. From each hyperbolic torus, there issue two n-dimensional Lagrangian invariant manifolds (whiskers)[34,35,39,40] consisting of trajectories approaching the torus as $t\to+\infty$ or $t\to-\infty$.

The Hamiltonian KAM principle for mappings is of the same form as that for vector fields.

Principle 2. For each $0\leqslant m\leqslant 2n-1$, a typical exact symplectic diffeomorphism of an exact symplectic $2n$-dimensional manifold possesses k-parameter Cantor families of reducible invariant m-tori with Diophantine quasiperiodic dynamics, where $k=\min(m,2n-m)$. These tori have the same isotropicity and measure properties as in the case of Hamiltonian vector fields (see Principle 1).

Recall that a symplectic manifold M is said to be *exact* if the symplectic structure ω is exact: $\omega=d\xi$, and a symplectic mapping A: $M\to M$ is called *exact* (or *globally*) symplectic if the 1-form $A^*\xi-\xi$ is not only closed, but even exact.

The "classical" case $m=n$ in Principle 2 has been well studied.[20,32,46,47] Lower-dimensional tori $(1\leqslant m<n)$ were considered by Zehnder.[35,48] The theorems on quasiperiodic motions with small numbers of frequencies in Hamiltonian differential equations periodic in time[43,49,50] can be immediately reduced to statements on lower-dimensional tori of symplectic diffeomorphisms. The case $m>n$ has not been explored yet in the literature (as far as the authors know) except for $(2n-1)$-tori for which the condition of exact symplecticity can in fact be replaced by a much weaker condition of global volume preservation[51–53] (see below). Principle 2 for $n < m < 2n-1$ should therefore be treated as a conjecture. The case $m=0$ (fixed points) is trivial.

The matrix Ω in Eq. (3) has eigenvalue 1 of multiplicity k at each torus (for $m>0$). For lower-dimensional tori ($m<n$), the remaining $2(n-m)$ eigenvalues of the matrix Ω come in pairs λ, λ^{-1}.

The existence of invariant m-tori with $m>n$ seems to require only $(m-n+1)$-exactness of the symplectic structure and the mapping (see the Appendix for the definition of s-exactness for $s>1$). To be more precise, we conjecture the following.

Principle 2′. Let $2 \leqslant s \leqslant n$. Then for each $n+s-1 \leqslant m \leqslant 2n-1$, a typical s-exact symplectic diffeomorphism of an s-exact symplectic $2n$-dimensional manifold possesses $(2n-m)$-parameter Cantor families of coisotropic reducible invariant m-tori with Diophantine quasiperiodic motions. The $2n$-dimensional Lebesgue measure of the union of the tori is positive.

This conjecture has been proven for $m=2n-1$ (see Refs. 51–53) in which case the condition of n-exact symplecticity can in fact be relaxed to that of global volume preservation.

B. Volume preserving systems

The KAM principles for volume preserving vector fields and mappings will be formulated here simultaneously.

Principles 3 and 4. Typical globally volume preserving vector fields and diffeomorphisms on an N-dimensional manifold ($N \geqslant 3$) possess one-parameter Cantor families of reducible invariant $(N-1)$-tori with Diophantine quasiperiodic dynamics. The N-dimensional Lebesgue measure of the union of these tori is positive.

Recall that if σ is a volume element on an N-dimensional manifold M then a divergence-free vector field V on M is said to be *globally* volume preserving if the $(N-1)$-form $i_V\sigma=\sigma(V,\cdot)$ is not only closed but even exact. If the form σ is exact: $\sigma=d\tau$, then a mapping A: $M\to M$ is called *globally* volume preserving if the form $A^*\tau-\tau$ is exact.

The number Ω in Eq. (2) for volume preserving flows is equal to zero. The number Ω in Eq. (3) for volume preserving mappings is equal to unity.

Invariant tori of codimension one of globally volume preserving vector fields are considered in Refs. 25 and 42 (however, these works deal with families of vector fields rather than individual fields). The case of mappings has been studied in Ref. 51 for $N=3$ and in Refs. 52 and 53 for N arbitrary. As is stated in Ref. 52, the global volume preservation property of the mapping can be weakened to the so-called intersection property (the analogous fact concerning diffeomorphisms of the plane is well known[46]).

C. Reversible systems

Speaking of invariant tori of reversible systems, we will always suppose that the tori in question are also invariant under the reversing involution G. Coordinate frames (1) near such tori will be assumed to possess the additional property that the involution G in these coordinates takes the form

$$G:\ \varphi'=-\varphi, \quad x'=Rx,$$

where R is an involutive matrix.

Principle 5. Let p and q be non-negative integers. For each m in the range

$$0 \leqslant m \leqslant q \quad \text{for } p \geqslant q,$$

$$1 \leqslant m \leqslant q \quad \text{for } p=q-1,$$

$$q-p+1 \leqslant m \leqslant q \quad \text{for } 1 \leqslant p \leqslant q-2,$$

in the phase space of a typical vector field V reversible with respect to an involution G of type (q,p), there are $(m+p-q)$-parameter Cantor families of reducible m-tori invariant under the involution G and the flow of the field V. These tori are filled up with Diophantine quasiperiodic motions. The $(2m+p-q)$-dimensional Lebesgue measure of the union of the tori is positive.

The cases $m=0$ (equilibria) and $m=1$ (periodic trajectories) here are trivial and $(m+p-q)$-parameter families of invariant m-tori for $m \leqslant 1$ are in fact smooth rather than Cantor. As far as m-tori with $m \geqslant 2$ are concerned, the reversible KAM theorems for vector fields were first formulated and proven in the case $m=q=p$, see Refs. 29, 30, 32, 49, and 54. This case attracted much attention later on as well.[26,39,55,56] Invariant q-tori for $p>q$ have been constructed and analyzed in Refs. 12–14, 25, and for $p<q$ in Refs. 24 and 25. *Lower-dimensional* invariant tori (of dimensions $m<q$) of reversible flows in the $p=q$ framework were studied in many papers (see Refs. 5, 6, 14, 36, 43, 57, and 58). The existence of invariant m-tori with $m<q$ has been proven for $p>q$ as well[7–9] (see also Ref. 43). To the authors' knowledge, the case where $m<q$ and $p<q$ has not been considered yet except for some remarks in Ref. 43 and the very recent research of Huitema.[59]

The matrix Ω in Eq. (2) has eigenvalue 0 of multiplicity $m+p-q$ at each torus (for $m>q-p$). The remaining $2(q-m)$ eigenvalues of the matrix Ω (for $m<q$) occur in pairs λ, $-\lambda$.

The reversible KAM principle for mappings is more complicated than that for vector fields.

Principle 6. Let p, q, P, Q be non-negative integers and $p+q=P+Q$. For each m in the range

$$0 \leqslant m \leqslant \min(q,Q) \quad \text{for } p+P \geqslant q+Q, \tag{4}$$

$$\tfrac{1}{2}(q+Q-p-P)+1 \leqslant m \leqslant \min(q,Q) \quad \text{for } p+P<q+Q, \tag{5}$$

a typical diffeomorphism A decomposable into the product of two involutions of types (q,p) and (Q,P) possesses k-parameter Cantor families of reducible invariant m-tori supporting Diophantine quasiperiodic dynamics, where

$$k=\tfrac{1}{2}(p+P-q-Q)+m. \tag{6}$$

These tori are also invariant under both the involutions. The $(k+m)$-dimensional Lebesgue measure of the union of the tori is positive.

In dimension $m=0$ (fixed points) this principle is trivial and k-parameter families of fixed points are in fact smooth rather than Cantor.

As was emphasized in Sec. I, until now all the studies of invariant tori in reversible mappings have been restricted to the case $p=P$, $q=Q$. Invariant q-tori for $p\geqslant q$ were found in Refs. 12, 13, 15, and 26. Invariant m-tori with $m<q$ (*lower-dimensional* tori) have been examined in Refs. 6 and 60–63 for $p=q$ and in Refs. 8 and 10 for $p>q$. The existence of invariant tori for $p<q$ remains a conjecture.

The present paper is devoted to the case where p, P, q, Q, and $m\leqslant\min(q,Q)$ are arbitrary. Nevertheless, we confine ourselves with the mappings for which $p+P\geqslant q+Q$.

To describe the spectrum of the matrix Ω in Eq. (3), it is convenient to set $P=p-r$, $Q=q+r$ where we suppose for definiteness that $r\geqslant 0$. Relations (4)–(6) can be rewritten as

$$0\leqslant m\leqslant q \quad \text{for } p\geqslant q+r,$$

$$q-p+r+1\leqslant m\leqslant q \quad \text{for } p<q+r,$$

$$k=m+p-q-r.$$

The matrix Ω in Eq. (3) has eigenvalue 1 of multiplicity k (if $k>0$) and eigenvalue -1 of multiplicity r (if $r>0$) at each torus. The remaining $2(q-m)$ eigenvalues of the matrix Ω (for $m<q$) come in pairs λ, λ^{-1}.

A family of invariant tori will be said to be *strongly hyperbolic* if the spectra of the matrices Ω are real at all the tori. In particular, families of invariant tori of maximal possible dimension q are always strongly hyperbolic. If $p\geqslant q+r$ (recall that $r=p-P=Q-q\geqslant 0$) then constructing strongly hyperbolic families of invariant tori can be easily reduced to the case $P=p$, $Q=q$.

Indeed, consider the square A^2 of the original mapping A. If A is reversible with respect to involution G, the types of involutions G and AG being (q,p) and $(Q,P)=(q+r,p-r)$, respectively, then the mapping A^2 is also reversible with respect to G, both involutions G and A^2G being of type (q,p). Suppose that we have found a $(p-q+m)$-parameter strongly hyperbolic Cantor family of m-tori invariant under the mapping A^2 and the involution G. This Cantor family turns out to consist of $(p-q)$-parameter *smooth* families which, in turn, are parametrized over an m-dimensional Cantor set $\Xi\subset\mathbb{R}^m$ of positive measure (for $m=q$ this property was announced in Refs. 12 and 26 and proven in Ref. 13 but it in fact holds for all $m\leqslant q$, the key assumption being strong hyperbolicity). All the invariant tori constituting a given smooth family have the same frequency vector. Now consider any smooth family of m-tori. These tori T_u^m are parametrized by parameter u over an open domain $U\subset\mathbb{R}^{p-q}$. Each torus T_u^m is invariant under A^2, but the original mapping A puts a torus T_u^m into some other (generally speaking) torus $T_{g(u)}^m$. The next application of the mapping A casts the torus $T_{g(u)}^m$ back to the torus T_u^m (the tori T_u^m are therefore said to be *periodically invariant* under A with period 2, cf. Ref. 64). We have defined an involution $g\colon U\to U$, and it is not hard to realize that this involution is of type $(r,p-q-r)$. Each point u of the $(p-q-r)$-dimensional manifold $\mathrm{Fix}(g)$ corresponds to torus T_u^m invariant under the original mapping A. Thus, we have found a smooth $(p-q-r)$-parameter family of m-tori invariant under the mapping A and the involution G. Taking the union of all such families over the set Ξ we will obtain a desired $(m+p-q-r)$-parameter Cantor family of m-tori invariant under A and G. This family is strongly hyperbolic.

Nevertheless, we wish to construct families of invariant tori with arbitrary reflexive spectra of the matrices Ω. Therefore, in Secs. III and IV we formulate precise theorems on the existence of invariant tori in the case $r>0$ independently of the preceding results[6,8,10,12,13,26] concerning the case $r=0$.

To conclude the discussion of the general KAM principles for symmetric systems, we note that proving Principles 1 and 2 for $m>n$, Principles 3 and 4, Principle 5 for $p<q$, and Principle 6 for $p+P<q+Q$ requires Diophantine approximations on *submanifolds* of the Euclidean space[24,44,52,65] rather than in open domains.

D. Dissipative systems

To look for invariant tori with quasiperiodic dynamics in dissipative systems, one has to introduce *external* parameters. Consider a smooth s-parameter family of vector fields V_a, $a\in\mathbb{R}^s$. We will say that this family is (m,μ)-regular ($m\geqslant 2$, $\mu\geqslant 0$) if for values of a in some Cantor set $\Xi\subset\mathbb{R}^s$ of positive measure, the field V_a possesses an invariant isolated m-torus T_a^m which supports Diophantine quasiperiodic motions and is reducible, the matrix Ω in Eq. (2) having μ pairs of nonreal eigenvalues. For $s\geqslant m+\mu-1$ the (m,μ)-regular families V_a turn out to constitute an open set in the space of all the families.[25,42] The same statement holds for invariant tori of codimension greater than one of volume preserving vector fields. The situation for invariant m-tori ($m\geqslant 1$) of diffeomorphisms is entirely analogous with the estimate $s\geqslant m+\mu$.

Let an s-parameter family of vector fields V_a be (m,μ)-regular. One may distinguish the hyperbolic behavior [none of the eigenvalues of the matrices Ω in Eq. (2) for the tori T_a^m, $a\in\Xi$, lie on the imaginary axis] and the nonhyperbolic behavior (some of the eigenvalues of the matrices Ω are purely imaginary). In the hyperbolic case, the field V_a possesses an invariant isolated m-torus T_a^m for all the values of a, but this torus is only finitely differentiable for $a\in\mathbb{R}^s\setminus\Xi$ (even if the field itself is C^∞ or analytic) and does not carry quasiperiodic motions. In the nonhyperbolic case, one cannot expect the field V_a to admit an invariant m-torus for $a\in\mathbb{R}^s\setminus\Xi$.

We enumerate some important aspects of the KAM theory that have not been touched upon in our very brief survey but are of great interest in the genericity approach to KAM phenomena. These aspects are Aubry–Mather sets (in particular, cantori), the converse KAM theory which guarantees the nonexistence of invariant tori under certain conditions (for a recent review on these two topics in the case of a symplectic mapping of the plane see Ref. 66), the infinite-dimensional KAM theory developed by now, with minor exceptions, for Hamiltonian flows only,[38,67–70] general Bruno's results on invariant submanifolds and families of invariant tori of analytic differential equations,[39,71] and the quasiperiodic bifurcation theory.[72,73] One should also mention the quite general formulation of KAM statements in terms of Lie algebras of vector fields, Lie groups of diffeomorphisms and artificial parameters,[25,29,42] a cohomological interpretation of the solvability of the main equations appearing in the proofs of KAM results,[13] and KAM theorems for the so-called weakly reversible systems[12,13,74] (for which the reversing diffeomorphism is not assumed to be an involution).

The Aubry–Mather theory[66,75–77] developed mainly for two-dimensional mappings is of special interest for us because it pertains to neither small perturbations of integrable systems nor neighborhoods of fixed points. Unfortunately, this theory usually provides invariant sets which are not necessarily smooth (and, moreover, it is cantoral invariant sets that attract major attention). To obtain smooth invariant tori one should require the closeness to an integrable mapping or critical element (e.g., a fixed point, invariant circle, or sufficiently long finite orbit with "good" behavior[78]).

III. THE MAIN THEOREM

In this section we give a precise formulation of the theorem on invariant tori for the product of two involutions of different types. We will work in the real analytic category although this seems to be just an insignificant technical restriction (nevertheless, the authors are aware of only two papers[56,58] containing C^∞ and finitely differentiable KAM theorems for reversible systems).

We will follow the general lines of Refs. 5–10. The notations in this section are somewhat different from those in Secs. I and II. In the sequel, the letters n, N, m, r, and k denote fixed non-negative integers, $n \leqslant N$, $2m+r \leqslant k$, and the sum $k+r$ is even. For $a,\ b \in \mathbb{C}^\mu$ we shall write $|a| = |a_1| + \cdots + |a_\mu|$, $\|a\|^2 = |a_1|^2 + \cdots + |a_\mu|^2$, $a \cdot b = a_1 b_1 + \cdots + a_\mu b_\mu$. The letters p and q will denote vectors in $\mathbb{Z}^m$ and $\mathbb{Z}^n$, respectively.

Let $\varepsilon \geqslant 0$ be a small parameter and B an N-dimensional ball in $\mathbb{R}^N$. Consider the mapping

$$A_\varepsilon:\ \begin{aligned} &\varphi' = \varphi + \omega(I) + f_1(\varphi,I,\xi) + \varepsilon g_1(\varphi,I,\xi,\varepsilon),\\ &I' = I + f_2(\varphi,I,\xi) + \varepsilon g_2(\varphi,I,\xi,\varepsilon),\\ &\xi' = \Omega(I)\xi + f_3(\varphi,I,\xi) + \varepsilon g_3(\varphi,I,\xi,\varepsilon), \end{aligned} \tag{7}$$

holomorphic in some complex neighborhood Ψ (independent of ε) of the set

$$\{\varphi \in (\mathbb{R}/2\pi\mathbb{Z})^n,$$

$$I \in B \subset \mathbb{R}^N,$$

$$\xi = 0 \in \mathbb{R}^k\} \subset (\mathbb{C}/2\pi\mathbb{Z})^n \times \mathbb{C}^{N+k}.$$

Thus, variable φ ranges in the domain

$$\{\varphi \in \mathbb{C}^n:\ |\mathrm{Im}\,\varphi_j| < \zeta,\ 1 \leqslant j \leqslant n\}$$

for some $\zeta > 0$, the vector-valued functions f_s, g_s in Eq. (7) being 2π-periodic in φ_j $(1 \leqslant s \leqslant 3)$, variable I ranges in some neighborhood of B in $\mathbb{C}^N$, and variable ξ ranges in some neighborhood of the origin in $\mathbb{C}^k$.

Assume mapping (7) to satisfy the following conditions:

(a) φ', I', ξ' are real whenever φ, I, ξ are real;

(b) $f_1 = O(\xi)$, $f_2 = O(\xi)$, $f_3 = O(|\xi|^2)$;

(c) the quantities $|g_s|$, $1 \leqslant s \leqslant 3$, do not exceed in Ψ some constant independent of ε;

(d) mapping A_ε is reversible with respect to the involution

$$G:\ \varphi' = -\varphi,\quad I' = I,\quad \xi' = Q\xi, \tag{8}$$

where Q is a real involutive $k \times k$ matrix [in particular, $\Omega(I)Q\Omega(I) = Q$ for all I];

(e) matrix $\Omega(I)$ has eigenvalue -1 of multiplicity r for all I, the (-1)-root space of $\Omega(I)$ lying in the 1-eigenspace of matrix Q;

(f) the remaining $k-r$ eigenvalues of $\Omega(I)$ are simple and different from 1.

Here two comments should follow. The first one concerns the theory of linear reversible systems developed in Ref. 23. Recall that if λ is an eigenvalue of a matrix M of multiplicity μ then the λ-root space of M is the 0-eigenspace of the operator $(M-\lambda E)^\mu$ where E denotes the identity matrix. The dimension of the λ-root space is always equal to μ whereas the dimension of the λ-eigenspace of the matrix M ranges from 1 to μ. Now let a matrix M be reversible with respect to an involutive matrix Q (i.e., $MQM = Q$). Suppose that $\sigma \in \{1; -1\}$ is an eigenvalue of M. The σ-root space of M is invariant[23] under Q. We will say that the eigenvalue σ of M is *perfect* if the σ-root space of M wholly lies in either the 1-eigenspace of Q or the (-1)-eigenspace of Q. If this is the case then the σ-root space of M coincides with the σ-eigenspace,[23] i.e., each σ-root vector is an eigenvector. Now let M have the eigenvalue 1 of multiplicity a, eigenvalue -1 of multiplicity b, and assume both these eigenvalues to be perfect. If all the remaining eigenvalues of M are simple then M is diagonalizable over $\mathbb{C}$. Moreover, all the Q-reversible matrices sufficiently close to M are diagonalizable and have the eigenvalues 1 and -1 of multiplicities a and b, respectively, both these eigenvalues being perfect.[23]

Consequently, the structure of the spectra of the matrices $\Omega(I)$ described in conditions (e) and (f) above is one of the typical possibilities. A small perturbation of the family of matrices $\Omega(I)$ would lead to a new family with the same structure of the spectra (provided that the perturbed matrices are still reversible with respect to Q). Moreover, all the matrices $\Omega(I)$ are diagonalizable over $\mathbb{C}$.

The second comment is how to create reversible mappings of the form (7). Consider the involution

$$P:\ \varphi' = -\varphi + \omega(I),\quad I' = I,\quad \xi' = \Omega(I)Q\xi$$

and an *arbitrary* mapping of the form

$$K_\varepsilon:\ \begin{aligned} \varphi' &= \varphi+\tilde{f}_1(\varphi,I,\xi)+\varepsilon\tilde{g}_1(\varphi,I,\xi,\varepsilon),\\ I' &= I+\tilde{f}_2(\varphi,I,\xi)+\varepsilon\tilde{g}_2(\varphi,I,\xi,\varepsilon),\\ \xi' &= \xi+\tilde{f}_3(\varphi,I,\xi)+\varepsilon\tilde{g}_3(\varphi,I,\xi,\varepsilon),\end{aligned}$$

where the holomorphic vector-valued functions $\tilde{f}_s$ and $\tilde{g}_s$ $(1\leqslant s\leqslant 3)$ are real whenever φ, I, ξ are real, $\tilde{f}_1=O(\xi)$, $\tilde{f}_2=O(\xi)$, $\tilde{f}_3=O(|\xi|^2)$, and $\tilde{g}_s$ are bounded from above in Ψ uniformly in ε. Then the mapping

$$A_\varepsilon=K_\varepsilon^{-1}PK_\varepsilon G$$

is reversible with respect to G and of the desired form (7).

Conditions (e) and (f) above imply[23] that the type of the linear involution $\xi'=Q\xi$ is $[\frac{1}{2}(k-r),\frac{1}{2}(k+r)]$ whereas the linear involution $\xi'=\Omega(I)Q\xi$ is of type $[\frac{1}{2}(k+r),\frac{1}{2}(k-r)]$. The involutions G (8) and $A_\varepsilon G$ are therefore of types

$$[n+\tfrac{1}{2}(k-r),N+\tfrac{1}{2}(k+r)]$$

and

$$[n+\tfrac{1}{2}(k+r),N+\tfrac{1}{2}(k-r)],$$

respectively. The $(n+N)$-dimensional surface $(\xi=0)$ is foliated into n-tori $(\xi=0,\ I=\text{const})$ invariant under mapping A_0 and involution G. These tori carry parallel dynamics with frequency vectors $\omega(I)$ and are reducible [if $\partial^2 f_2(\varphi,I,0)/\partial\xi\,\partial\varphi\equiv 0$]. Our goal is a persistence result for these tori as ε becomes positive.

The Greek letters α and β in this section will denote integers in the range $1\leqslant\alpha\leqslant k$, $1\leqslant\beta\leqslant k$.

Let the frequency map ω: $B\rightarrow\mathbb{R}^n$, $I\mapsto\omega(I)$ be a submersion, i.e., rank $[\partial\omega(I)/\partial I]=n$ everywhere in B (the nondegeneracy condition). Then one can construct, at least locally near any point in B, a $\mathbb{C}^{N-n}$-valued holomorphic function $\chi(I)$ such that the mapping $I\mapsto[\omega(I),\chi(I)]$ is a local diffeomorphism and $\chi(I)$ is real whenever I is real.

Denote by $\lambda_1(I),\ldots,\lambda_k(I)$ the eigenvalues of the matrix $\Omega(I)$ $[\lambda_1(I)=\cdots=\lambda_r(I)\equiv-1]$ and by $C_\alpha(I)$ the infimum of numbers C such that

$$|\arg\lambda_\alpha(I+J)-\arg\lambda_\alpha(I)|\leqslant C\|\omega(I+J)-\omega(I)\|$$

whenever $\chi(I+J)=\chi(I)$ and $J\in\mathbb{R}^N$ is sufficiently small [here arg designates the phase: $\lambda=|\lambda|\exp(i\cdot\arg\lambda)$]. Of course, the constants $C_\alpha(I)$ depend on the choice of the function χ near the point $I\in B$ under consideration. It is clear that the $C_\alpha(I)$ are bounded uniformly from above on B. If eigenvalue $\lambda_\alpha(I)$ is real then $C_\alpha(I)\equiv 0$ [in particular, $C_1(I)=\cdots=C_r(I)\equiv 0$].

Theorem 1. Suppose that conditions (a)–(f) above are satisfied, the frequency map $I\mapsto\omega(I)$ is a submersion throughout B, and the following nonresonance conditions are fulfilled for any point $I\in B$ (cf. Refs. 5 and 6):

$$\exp[iq\cdot\omega(I)]\neq\lambda_\alpha(I)$$

for $\|q\|\leqslant C_\alpha(I)$,

$$\exp[iq\cdot\omega(I)]\neq\lambda_\alpha(I)/\lambda_\beta(I)$$

for $0<\|q\|\leqslant C_\alpha(I)+C_\beta(I)$ (recall that $q\in\mathbb{Z}^n$).

Then, for all sufficiently small values of ε, mapping (7) and involution (8) possess common invariant analytic real tori T_* of dimension n. These tori are close to the invariant n-tori $(\xi=0,\ I=\text{const})$ of the unperturbed mapping A_0, the motions on the perturbed tori being Diophantine quasiperiodic with frequency vectors close to $\omega(I)$. Moreover, the tori T_* are reducible. Denote by Ξ_ε the union of those varieties $\Pi(T_*)$ lying in $(\mathbb{R}/2\pi\mathbb{Z})^n\times B$ where Π is the projection

$$\Pi:\ (\varphi,I,\xi)\mapsto(\varphi,I)$$

and $(\mathbb{R}/2\pi\mathbb{Z})^n$ is the domain over which real variable φ ranges. Then

$$\lim_{\varepsilon\rightarrow 0}\frac{\operatorname{mes}(\Xi_\varepsilon)}{(2\pi)^n\operatorname{vol}(B)}=1,$$

mes being the $(n+N)$-dimensional Lebesgue measure and vol(B) the N-dimensional volume of the ball B.

Theorem 1 can be proven by the standard Kolmogorov method of constructing a rapidly convergent infinite sequence of coordinate transformations commuting with the reversing involution G, the definition domains of these transformations shrinking down to the perturbed n-torus sought for. Nevertheless, the iterative KAM procedure in the setup of Theorem 1 is very cumbersome and tedious. A detailed proof will be published elsewhere.

We should emphasize that one cannot predict which invariant tori $(\xi=0,\ I=\text{const})$ of A_0 will survive the perturbation, except for the case where the spectra of the matrices $\Omega(I)$ are real (strongly hyperbolic tori, see Sec. II). If all the eigenvalues $\lambda_1(I),\ldots,\lambda_k(I)$ of $\Omega(I)$ are real for all $I\in B$ then each torus $(\xi=0,\ I=\text{const})$ with Diophantine frequency vector $\omega(I)$ persists under the perturbation. Moreover, the perturbed n-tori in the strongly hyperbolic case are organized into analytic $(N-n)$-parameter families, each family consisting of the tori with the same frequency vector.

The persistence of all the tori $(\xi=0,\ I=\text{const})$ with Diophantine frequency vectors $\omega(I)$ seems to take place in the general hyperbolic case [none of the eigenvalues $\lambda_{r+1}(I),\ldots,\lambda_k(I)$ lie on the unit circle] also. However, in the absence of strong hyperbolicity some of these tori may give rise to finitely differentiable perturbed tori rather than analytic ones (cf. Ref. 34).

Theorem 1 admits also a more quantitative formulation (cf. Refs. 6 and 7).

IV. EXCITATION OF ELLIPTIC NORMAL MODES

If some of the "nontrivial" eigenvalues $\lambda_{r+1}(I),\ldots,\lambda_k(I)$ of the matrix $\Omega(I)$ in Eq. (7) lie on the unit circle then one may excite the corresponding normal modes and find, near the surface $(\xi=0)$, invariant tori of A_ε and G of dimensions greater than n. Note that the existence of such tori is not obvious at all even for $\varepsilon=0$. The possibility of the excitation of normal modes around a smooth family of reducible invariant tori of positive dimension in reversible systems was first conjectured in Ref. 79. Precise theorems have been announced in Ref. 9 for reversible vector fields and in Ref. 10 for reversible diffeomorphisms which are the products of two involutions of the same type. In this section, we generalize the construction of Ref. 10 to the products of two involutions of different types.

A technical tool to study the excitation of normal modes is the partial Birkhoff normal form (the word "normal" has been used in this sentence twice in two entirely different meanings!). The normal form theory around fixed points of mappings and equilibrium points or periodic trajectories of vector fields has been expounded in innumerable works, see, e.g., Refs. 3, 13, 20, 32, 39, 43, 55, and 80 (we have listed here some books only). Normal forms around a reducible invariant torus of arbitrary positive dimension are also very useful and have been developed for dissipative,[39,73,81] Hamiltonian,[39,80,82] and reversible[9,10] systems.

The Greek letters α and β in this section will denote integers in the range $1 \leqslant \alpha \leqslant k-2m$, $1 \leqslant \beta \leqslant k-2m$.

Consider again mapping A_ε (7) satisfying the conditions (a)–(f). Suppose that the eigenvalues $\lambda_{k-2m+1}(I),\ldots,\lambda_k(I)$ of the matrix $\Omega(I)$ are of the form

$$\lambda_{k-2m+2j-1}(I)=\exp[i\theta_j(I)],$$

$$\lambda_{k-2m+2j}(I)=\exp[-i\theta_j(I)],$$

where $0<\theta_j(I)<\pi$ $(1\leqslant j\leqslant m)$. Recall that $\lambda_1(I)=\cdots=\lambda_r(I)\equiv -1$. Some of the remaining eigenvalues $\lambda_{r+1}(I),\ldots,\lambda_{k-2m}(I)$ may also be of modulus unity. Let l be an arbitrary positive integer. Assume that for some point $I_0\in B$ the vectors $\omega=\omega(I_0)\in\mathbb{R}^n$, $\theta=\theta(I_0)\in\mathbb{R}^m$, and the numbers $\lambda_\alpha=\lambda_\alpha(I_0)\in\mathbb{C}$ satisfy the following nonresonance conditions (c and γ being positive constants):

$$|\exp[i(q\cdot\omega+p\cdot\theta)]-1|\geqslant c(|q|+1)^{-\gamma} \tag{9}$$

for $|q|+|p|>0$ and $|p|\leqslant 2l+2$,

$$|\exp[i(q\cdot\omega+p\cdot\theta)]-\lambda_\alpha|\geqslant c(|q|+1)^{-\gamma} \tag{10}$$

for $|p|\leqslant 4l+1$,

$$|\exp[i(q\cdot\omega+p\cdot\theta)]-\lambda_\alpha/\lambda_\beta|\geqslant c(|q|+1)^{-\gamma} \tag{11}$$

for $|q|+|p|>0$ and $|p|\leqslant 2l$.

Recall that $p\in\mathbb{Z}^m$ and $q\in\mathbb{Z}^n$.

Normal Form Lemma. Under these conditions, near the n-torus $(\xi=0,\ I=I_0)\subset(\mathbb{R}/2\pi\mathbb{Z})^n\times\mathbb{R}^{N+k}$, there exist the coordinates $[\psi\in(\mathbb{R}/2\pi\mathbb{Z})^n,\ u\in\mathbb{R}^N,\ z\in\mathbb{C}^m,\ w\in\mathbb{R}^{k-2m}]$ in which the *unperturbed* mapping A_0 and the involution G (8) take, respectively, the forms

$$\begin{aligned}
&\psi'=\psi+\omega(I_0+u)+F(\rho,u)+\Delta_1(\psi,u,z,w),\\
&u'=u+\Delta_2(\psi,u,z,w),\\
A_0:\ &z_j'=z_j\exp[i\theta_j(I_0+u)+iH_j(\rho,u)]+\Delta_{3j}(\psi,u,z,w),\\
&w'=[\Lambda(I_0+u)+S(\rho,u)]w+\Delta_4(\psi,u,z,w)
\end{aligned} \tag{12}$$

$(1\leqslant j\leqslant m)$ and

$$G:\ \psi'=-\psi,\quad u'=u,\quad z'=\bar{z},\quad w'=Rw. \tag{13}$$

Here

(g) $\rho\in\mathbb{R}^m$ with $\rho_j=|z_j|^2$ $(1\leqslant j\leqslant m)$;

(h) the vector-valued functions F and H and the matrix-valued function S are polynomials in ρ and u of degree not greater than l with real coefficients, these polynomials vanishing at $\rho=0$;

(i) the residual vector-valued functions Δ_ν $(1\leqslant\nu\leqslant 4)$ satisfy the conditions

$$\Delta_1,\Delta_2=O(|z|+|w|),\quad \Delta_3,\Delta_4=O(|z|^2+|w|^2),$$

$$\Delta_\nu(\psi,t^2u,tz,t^{2l+1}w)=O(t^{2l+1+\tau(\nu)})\quad\text{as } t\to 0,$$

where

$$\tau(1)=0,\quad \tau(2)=2,\quad \tau(3)=1,\quad \tau(4)=2l+1;$$

(j) the new coordinates ψ, u, z, w depend analytically on φ and are polynomials in I, ξ;

(k) $\psi=\varphi$, $u=I-I_0$, $z=0$ and $w=0$ for $\xi=0$;

(l) the real $(k-2m)\times(k-2m)$ matrix R in Eq. (13) is involutive and $\Lambda(I)R\Lambda(I)=R$ for all I;

(m) the eigenvalues of the matrix $\Lambda(I)$ are $\lambda_1(I),\ldots,\lambda_{k-2m}(I)$, the (-1)-root space of $\Lambda(I)$ lying in the 1-eigenspace of matrix R.

If the frequency map $I\mapsto\omega(I)$ is a submersion and the left-hand sides of inequalities (9)–(11) do not vanish in B identically for all q, p, α, β in the ranges pointed out, then, for each fixed value of $\gamma>n$, the Lebesgue measure of the set of points $I_0\in B$ subject to these inequalities tends to the volume vol(B) of the ball B as $c\to 0$ (in Ref. 10 we required $\gamma>n-1$ which in fact seems to be insufficient).

Note that the linear involutions $w'=Rw$ and $w'=\Lambda(I)Rw$ are of types

$$[\tfrac{1}{2}(k-r)-m,\tfrac{1}{2}(k+r)-m]$$

and

$$[\tfrac{1}{2}(k+r)-m,\tfrac{1}{2}(k-r)-m],$$

respectively.

Now suppose that in mapping (12), the $(n+m)\times(N+m)$ Jacobi matrix

$$D=\begin{pmatrix}\dfrac{\partial\omega(I_0)}{\partial I} & \dfrac{\partial F(0,0)}{\partial\rho}\\[2ex] \dfrac{\partial\theta(I_0)}{\partial I} & \dfrac{\partial H(0,0)}{\partial\rho}\end{pmatrix} \tag{14}$$

has the maximal possible rank $m+n$ (a nondegeneracy condition guaranteeing the necessary amplitude-frequency modulation). One can augment D by additional $N-n$ lines and obtain a nonsingular $(N+m)\times(N+m)$ matrix. Denote by X the $(N-n)\times(N+m)$ matrix consisting of these $N-n$ lines and define linear operators h: $\mathbb{R}^{m+n}\to\mathbb{R}^N$, η: $\mathbb{R}^{m+n}\to\mathbb{R}^m$ by the relation

$$\begin{pmatrix}D\\X\end{pmatrix}\begin{pmatrix}h(y)\\\eta(y)\end{pmatrix}=\begin{pmatrix}y\\0\end{pmatrix};$$

here $h(y)$ and $\eta(y)$ are column vectors, $y\in\mathbb{R}^{m+n}$, and $0\in\mathbb{R}^{N-n}$. Of course, the operators h and η depend on the choice of the matrix X.

Recall that the matrices $\Lambda(I)$ are diagonalizable over $\mathbb{C}$. Let a matrix $L\in GL(k-2m,\mathbb{C})$ diagonalize $\Lambda(I_0)$, i.e.,

$$L\Lambda(I_0)L^{-1}=\mathrm{diag}[-1,\ldots,-1,\lambda_{r+1}(I_0),\ldots,\lambda_{k-2m}(I_0)].$$

Define the linear operator Y: $\mathbb{R}^{m+n}\to gl(k-2m,\mathbb{C})$ by the equality

$$L[\Lambda(I_0+h(y))+S(\eta(y),0)]L^{-1}$$

$$=L\Lambda(I_0)L^{-1}+Y(y)+O(|y|^2)$$

and denote the elements of the matrix $Y(y)$ by $Y_{\alpha\beta}(y)$. Set

$$C_\alpha=0$$

if $\lambda_\alpha(I_0)$ is real (in particular, for $1\leqslant\alpha\leqslant r$) and

$$C_\alpha=\max_{\|y\|=1}|Y_{\alpha\alpha}(y)|$$

otherwise. Let $\delta>0$ be a small radius.

Theorem 2. Assume that the mapping A_ε (7) satisfies conditions (a)–(f), the hypotheses of the Normal Form Lemma are met for some point $I_0\in B$ with $l=3$, the matrix D (14) is of rank $m+n$, and the following inequalities (additional nonresonance conditions) hold:

$$\exp[i(q\cdot\omega+p\cdot\theta)]\neq\lambda_\alpha$$
$$\text{for } \|q\|^2+\|p\|^2\leqslant C_\alpha^2,$$

$$\exp[i(q\cdot\omega+p\cdot\theta)]\neq\lambda_\alpha/\lambda_\beta$$
$$\text{for } 0<\|q\|^2+\|p\|^2\leqslant(|\lambda_\alpha|^{-1}C_\alpha+|\lambda_\beta|^{-1}C_\beta)^2.$$

Recall that here $\omega=\omega(I_0)$, $\theta=\theta(I_0)$, and $\lambda_\alpha=\lambda_\alpha(I_0)$.

Then, in any real neighborhood of the n-torus

$$(\xi=0,\ I=I_0)=(z=0,\ w=0,\ u=0), \tag{15}$$

for all sufficiently small values of ε, mapping (7) and involution (8) possess common invariant analytic real tori T_* of dimension $m+n$. These tori are close to the tori ($w=0$, $u=\text{const}$, $|z_j|=\text{const}>0$, $1\leqslant j\leqslant m$), the motions on the tori T_* being Diophantine quasiperiodic with frequency vectors close to (ω,θ). Moreover, these tori are reducible. Let U_δ be the δ-neighborhood of the torus (15) in the $(n+N+2m)$-dimensional surface ($w=0$). Denote by $\Xi_{\varepsilon\delta}$ the union of those varieties $\Pi(T_*)$ lying in U_δ where Π is the projection

$$\Pi:\ (\psi,u,z,w)\mapsto(\psi,u,z).$$

Then

$$\lim_{\varepsilon+\delta\to0}\frac{\mathrm{mes}(\Xi_{\varepsilon\delta})}{\mathrm{mes}(U_\delta)}=1,$$

mes being the $(n+N+2m)$-dimensional Lebesgue measure.

Most likely, analogous theorems hold for exact symplectic mappings and Hamiltonian vector fields, but some partial results only are known. Namely, Bruno[39] examined the following problem. Consider an analytic Hamiltonian differential equation $dx/dt=V_a(x)$ with n degrees of freedom, the Hamilton function depending analytically on a parameter $a\in\mathbb{R}^s$. Suppose that for $a=a_0$ this equation possesses a reducible invariant m-torus T_0 with Diophantine quasiperiodic dynamics ($1\leqslant m\leqslant n$), the matrix Ω in Eq. (2) for this torus having precisely $l\leqslant n-m$ pairs of nonzero purely imaginary eigenvalues. Assume that $s\geqslant l-1$. Bruno has proven[39] that under suitable nonresonance and nondegeneracy conditions, for each integer r in the range $0\leqslant r\leqslant l$ the system

$$\frac{dx}{dt}=V_a(x),\quad \frac{da}{dt}=0$$

possesses an $(s+1-l+r)$-parameter analytic family of analytic reducible invariant $(m+r)$-tori which adjoins the torus T_0. The tori in this family support Diophantine quasiperiodic motions, the corresponding matrices Ω in Eq. (2) having $l-r$ pairs of nonzero purely imaginary eigenvalues.

Actually, the motion near T_0 is much more complicated than Bruno's theorem[39] describes, because the $(s+1-l+r)$-parameter analytic families of invariant $(m+r)$-tori mentioned above are in fact generically embedded in $(s+m+r)$-parameter Cantor families. To the best of the authors' knowledge, this statement has not been proven yet although it gives rise to no doubt.

V. INDEFINITE INVOLUTIONS

The type of an involution is a *local* characteristic of that involution near a fixed point. The fixed point manifold $\mathrm{Fix}(G)$ of an involution G: $M\to M$ may be disconnected and consist of several connected components of different dimensions, even in the case where the manifold M itself is connected. Throughout Secs. I and II, we tacitly supposed that the fixed point manifold of any involution involved has the same dimension at all of its points. However, this is not necessarily true for an arbitrary involution.

An appropriate example can be constructed very easily. Let $r>0$, m, and n be non-negative integers. Consider the $(r+1)$-dimensional manifold M_0 obtained from the layer

$$\{(x,y)\in\mathbb{R}^{r+1}:\ x\in\mathbb{R},-1\leqslant x\leqslant1,\ y\in\mathbb{R}^r\}$$

by identifying $(-1,y)$ and $(1,-y)$ for all $y\in\mathbb{R}^r$ (for $r=1$, manifold M_0 is the standard Möbius strip) and the manifold M of dimension $2m+n+r+1$ defined as

$$M=M_0\times\mathbb{C}^m\times\mathbb{R}^n. \tag{16}$$

Let z and w be the coordinates in $\mathbb{C}^m$ and $\mathbb{R}^n$, respectively. The mapping

$$G:\ x'=-x,\quad y'=-y,\quad z'=\bar{z},\quad w'=\varepsilon w \tag{17}$$

($\varepsilon=\pm1$) is a well-defined involution on M. The corresponding fixed point manifold $\mathrm{Fix}(G)$ consists of two connected components $\mathrm{Fix}_1(G)$ and $\mathrm{Fix}_2(G)$.

If $\varepsilon=1$ then

$$\mathrm{Fix}_1(G)=(x=0,y=0,z\in\mathbb{R}^m),$$

$$\mathrm{Fix}_2(G)=(x=\pm1,z\in\mathbb{R}^m),$$

$$\dim\mathrm{Fix}_1(G)=m+n,\quad \dim\mathrm{Fix}_2(G)=m+n+r.$$

The involution G has the type $(m+r+1,m+n)$ at any point of $\mathrm{Fix}_1(G)$ and the type $(m+1,m+n+r)$ at any point of $\mathrm{Fix}_2(G)$.

If $\varepsilon=-1$ then

$$\mathrm{Fix}_1(G)=(x=0,y=0,z\in\mathbb{R}^m,w=0),$$

$$\mathrm{Fix}_2(G)=(x=\pm 1,z\in\mathbb{R}^m,w=0),$$

$$\dim \mathrm{Fix}_1(G)=m, \quad \dim \mathrm{Fix}_2(G)=m+r.$$

The involution G has the type $(m+n+r+1,m)$ at any point of $\mathrm{Fix}_1(G)$ and the type $(m+n+1,m+r)$ at any point of $\mathrm{Fix}_2(G)$.

Now let M be an arbitrary *connected* manifold and G: $M\to M$ an arbitrary smooth involution. We will call G *indefinite* if $\mathrm{Fix}(G)$ is empty or consists of connected components of different dimensions.

Lemma 1. Let a and b be two fixed points of involution G and λ: $[0;1]\to M$ an arbitrary path such that $\lambda(0)=a$ and $\lambda(1)=b$. Let the types of involution G at the points a and b be equal to (q,p) and $(q+r,p-r)$, respectively. Define the loop Λ: $[-1;1]\to M$ by the equalities $\Lambda(t)=\lambda(t)$ for $0\leqslant t\leqslant 1$ and $\Lambda(t)=G(\lambda(-t))$ for $-1\leqslant t\leqslant 0$. Then the loop Λ is orientation preserving if and only if r is even.

Proof. For $0\leqslant t\leqslant 1$, choose the orientations ξ_t of the tangent spaces $T_{\lambda(t)}M=T_{\Lambda(t)}M$ to depend continuously on t. The orientations ξ_t of the tangent spaces $T_{G(\lambda(t))}M = T_{\Lambda(-t)}M$ induced from ξ_t via G also depend continuously on t. The two orientations ξ_0 and $\tilde{\xi}_0$ of T_aM coincide if q is even, and the two orientations ξ_1 and $\tilde{\xi}_1$ of T_bM coincide if $q+r$ is even.

Corollary. If the difference in the dimensions of two connected components of $\mathrm{Fix}(G)$ is odd then the manifold M is not orientable.

In particular, it is obvious that manifold (16) is orientable for r even and nonorientable for r odd.

Let V be a G-reversible vector field on M. It is well known[11–13,26,83] that any symmetric (i.e., invariant under G) closed trajectory of V intersects $\mathrm{Fix}(G)$ at two points, and any trajectory of V which intersects $\mathrm{Fix}(G)$ at two points is closed and symmetric. In the sequel, we shall use the words "symmetric cycle" instead of "symmetric closed trajectory." If Γ is a symmetric cycle of V of period $2T$ then on Γ, one can introduce an angular coordinate φ mod 2π such that the field V determines on Γ the differential equation $d\varphi/dt=\pi/T$, the restriction to Γ of the involution G takes the form $\varphi'=-\varphi$, and Γ intersects $\mathrm{Fix}(G)$ at the points $\varphi=0$ and $\varphi=\pi$.

The monodromy operator[3] of a symmetric cycle of a reversible vector field is reversible,[11–13,26,83] and its determinant is therefore equal to ± 1.

Let a symmetric cycle Γ of a G-reversible vector field V intersect $\mathrm{Fix}(G)$ at two points a and b. We will say that Γ is of type $(q,p)(Q,P)$ if the types of G at the points a and b are (q,p) and (Q,P).

Lemma 2. The monodromy operator of a symmetric cycle of type $(q,p)(q+r,p-r)$ has determinant $(-1)^r$.

This lemma first obtained in Ref. 83 immediately follows from Lemma 1.

Theorem 3. Symmetric cycles of type $(q,p)(Q,P)$ of a G-reversible vector field V are generically organized into smooth k-parameter families structurally stable with respect to small G-reversible perturbations of V, where

$$k=\tfrac{1}{2}(p+P-q-Q)+1. \tag{18}$$

If the number k (18) is negative then a symmetric cycle of V of type $(q,p)(Q,P)$ can be removed by an arbitrarily slight G-reversible perturbation of the field V.

Proof (cf. Refs. 12 and 13). Consider a symmetric cycle Γ of V of type $(q,p)(Q,P)$. Let Γ intersect $\mathrm{Fix}(G)$ at two points a and b, the dimension of $\mathrm{Fix}(G)$ being equal to p near a and P near b. The trajectories of V passing through the points of $\mathrm{Fix}(G)$ close to a form a $(p+1)$-dimensional "tube" Φ that generically intersects $\mathrm{Fix}(G)$ at b transversally along a smooth surface of dimension

$$(p+1)+P-(p+q)$$

$$=P-q+1=\tfrac{1}{2}(P+p-Q-q)+1=k$$

(for $k\geqslant 0$). The trajectory of V passing through any point of this surface intersects $\mathrm{Fix}(G)$ at two points and is therefore a symmetric cycle [of type $(q,p)(Q,P)$]. For nearby G-reversible fields the tube Φ will still intersect $\mathrm{Fix}(G)$ near b transversally, and the k-parameter family of cycles will just be slightly distorted.

Expressions (6) and (18) are very much alike although they refer to entirely different situations.

In the definite case ($p=P$, $q=Q$) Theorem 3 is well known[12,13,26,83,84] (see also Ref. 11 for the simplest case $p=q=P=Q$).

The monodromy operator of a symmetric cycle of type $(q,p)(q+r,p-r)$ has generically eigenvalue 1 of multiplicity k (if $k=p-q-r+1>0$) and eigenvalue -1 of multiplicity $|r|$ (if $k\geqslant 0$ and $r\neq 0$).

Return to our example [Eqs. (16) and (17)] and consider on manifold M (16) the vector field V determining the system of differential equations

$$\frac{dx}{dt}=1, \quad \frac{dy}{dt}=0, \quad \frac{dz}{dt}=i\omega z, \quad \frac{dw}{dt}=0$$

(ω/π being irrational). This field is well defined and reversible with respect to involution G (17). The circles

$$\Gamma_W=\{(x,y,z,w)\colon\ y=0,z=0,w=\mathrm{const}=W\} \tag{19}$$

are periodic trajectories of V of period 2 whereas the circles

$$\Gamma_{Y,W}=\{(x,y,z,w)\colon\ y=\pm Y,z=0,w=\mathrm{const}=W\} \tag{20}$$

($Y=\mathrm{const}\in\mathbb{R}^r$, $Y\neq 0$, the first nonzero component of Y is positive) are periodic trajectories of V of period 4.

If $\varepsilon=1$ then all the circles (19) are symmetric cycles of V. They have the type

$$(m+1,m+n+r)(m+r+1,m+n)$$

and constitute an n-parameter family which agrees with Eq. (18). All the circles (20) are also symmetric cycles. They have the type

$$(m+1,m+n+r)(m+1,m+n+r)$$

and constitute an $(n+r)$-parameter family which also agrees with Eq. (18). Both families are structurally stable with respect to small G-reversible perturbations of V.

Let $n>0$ and $\varepsilon=-1$. Then the only symmetric cycle among circles (19) is Γ_0. This cycle has the type

$$(m+n+1,m+r)(m+n+r+1,m)$$

and is not embedded in any smooth family of symmetric cycles. In other words, Γ_0 constitutes a 0-parameter family [the number (18) being equal to $-n$]. The symmetry condition for circles (20) is also $W=0$. The symmetric cycles $\Gamma_{Y,0}$ have the type

$$(m+n+1,m+r)(m+n+1,m+r)$$

and constitute an r-parameter family [the number (18) being equal to $r-n$]. Both families are structurally unstable. Indeed, the vector field determining the system of differential equations

$$\frac{dx}{dt}=1, \quad \frac{dy}{dt}=0, \quad \frac{dz}{dt}=i\omega z, \quad \frac{dw}{dt}=\delta$$

is reversible with respect to G for any $\delta\in\mathbb{R}^n$ and possesses no periodic trajectories for any $\delta\neq0$.

As we emphasized in Sec. II, the statements concerning smooth families of closed trajectories of vector fields are in fact trivial, extreme particular cases of the KAM principles. Theorem 3 is therefore a trivial case of the indefinite analog of Principle 5. The nontrivial generalizations of Principles 5 and 6 of Sec. II to indefinite involutions are far beyond the present study. We shall confine ourselves with the following remark.

Let G: $M\rightarrow M$ be a smooth involution and $L\subset M$ a G-invariant m-torus such that the restriction of G to L has the form $\varphi'=-\varphi$ for some coordinates $(\varphi_1 \bmod 2\pi,...,\varphi_m \bmod 2\pi)$ on L. The surface Fix(G) intersects L at 2^m points

$$(\varphi_j\in\{0;\pi\}, \ 1\leqslant j\leqslant m). \tag{21}$$

It turns out that the types of G at these fixed points cannot be prescribed arbitrarily.

If a and b are two fixed points of G of the form (21) then their componentwise sum modulo 2π and componentwise product divided by π are also points of the form (21). The set of fixed points of G on L may therefore be treated as the ring $(\mathbb{Z}_2)^m$. Denote by (q_a,p_a) the type of involution G at a fixed point $a\in L$.

Lemma 3. For any two points a and b of the form (21), the integer $p_0+p_{a+b}-p_a-p_b$ is even [0 being the point $(\varphi_j=0, \ 1\leqslant j\leqslant m)$].

Proof. For any point $a=(\varphi_1,...,\varphi_m)$ of the form (21), consider the loop

$$\gamma_a\colon [0;2]\rightarrow L, \quad \gamma_a(t)=(t\varphi_1,...,t\varphi_m);$$

$$\gamma_a(1)=a, \quad \gamma_a(0)=\gamma_a(2)=0.$$

Denote by $[\gamma_a]$ the corresponding element of the fundamental group $\pi_1(L,0)=\mathbb{Z}^m$ (see Ref. 85). It is clear that

$$[\gamma_{a+b}]=[\gamma_a]+[\gamma_b]-2[\gamma_{ab}].$$

On the other hand, γ_a is orientation preserving (as a loop in M) if and only if p_a-p_0 is even (according to Lemma 1). Thus, $p_{a+b}-p_0$ is even if and only if $(p_a-p_0)+(p_b-p_0)$ is even which completes the proof.

ACKNOWLEDGMENTS

The authors are grateful to V. I. Arnold and A. I. Neishtadt for numerous stimulating discussions on reversible systems and quasiperiodic motions during many years. M. B. S. thanks also G. B. Huitema, D. S. Ornstein, J. Pöschel, and H. Rüssmann for some clarifications and acknowledges financial support from the American Mathematical Society (the fSU Aid Fund) and the Russia Academy of Natural Sciences. G. R. W. Q. is supported by the Australian Research Council.

APPENDIX: ON *s*-EXACTNESS

Definition 1. A symplectic manifold (M^{2n},ω) is said to be s-exact $(1\leqslant s\leqslant n)$ if the sth exterior power ω^s of the symplectic structure ω is exact: $\omega^s=d\xi$ where ξ is a $(2s-1)$ form.

It is clear that an s-exact symplectic manifold is also s_1-exact for any $s+1\leqslant s_1\leqslant n$.

Definition 2. A symplectic diffeomorphism A of an s-exact symplectic manifold $(M^{2n},\omega,\omega^s=d\xi)$ is said to be S-exact $(s\leqslant S\leqslant n)$ if the $(2S-1)$-form $(A^*\xi-\xi)\wedge\omega^{S-s}$ is not only closed but even exact.

It is clear that an S-exact symplectic mapping is also S_1-exact for any $S+1\leqslant S_1\leqslant n$.

1-exact symplectic manifolds and diffeomorphisms are usually called merely exact.

An S-exact symplectic mapping A: $M\rightarrow M$ of an s-exact symplectic manifold $(M^{2n},\omega,\omega^s=d\xi)$ is globally volume preserving with respect to the exact volume element $\sigma=\omega^n=d(\xi\wedge\omega^{n-s})$. Indeed, if $A^*\omega=\omega$ and $(A^*\xi-\xi)\wedge\omega^{S-s}=d\zeta$ then

$$A^*(\xi\wedge\omega^{n-s})-\xi\wedge\omega^{n-s}=d(\zeta\wedge\omega^{n-S}).$$

Since a volume element on a compact manifold without boundary cannot be exact, a symplectic $2n$-dimensional compact manifold without boundary cannot be s-exact for any $s\leqslant n$.

Example. Let $0\leqslant m\leqslant n$ and $M^{2n}=\mathbb{R}^{n-m}\times(\mathbb{R}/2\pi\mathbb{Z})^{n+m}$. Denote by x and φ modd 2π the coordinates in $\mathbb{R}^{n-m}$ and $(\mathbb{R}/2\pi\mathbb{Z})^{n+m}$, respectively. The 2-form

$$\omega=\sum_{j=1}^{n-m} dx_j\wedge d\varphi_j+\sum_{k=1}^{m} d\varphi_{n-m+2k-1}\wedge d\varphi_{n-m+2k}$$

is a symplectic structure on M which is not m-exact (for $m>0$) but is $(m+1)$-exact (for $m<n$). Indeed, the integral of ω^m over the $2m$-dimensional cycle (x=const, φ_1=const,...,φ_{n-m}=const) does not vanish, and ω^m is therefore not exact. On the other hand, ω^{m+1} has the form

$$\omega^{m+1}=\sum_{r=1}^{R}\sum_{\alpha,\beta} C_{r\alpha\beta}\, dx_{\alpha_1}\wedge\cdots\wedge dx_{\alpha_r}\wedge\Phi_{r\alpha\beta},$$

$$\Phi_{r\alpha\beta}=d\varphi_{\alpha_1}\wedge\cdots\wedge d\varphi_{\alpha_r}\wedge d\varphi_{\beta_1}\wedge\cdots\wedge d\varphi_{\beta_l},$$

where $R=\min(n-m,m+1)$, $l=2(m+1-r)$, $1\leqslant\alpha_1<\cdots<\alpha_r\leqslant n-m$, $n-m+1\leqslant\beta_1<\cdots<\beta_l\leqslant n+m$, and $C_{r\alpha\beta}$ are certain coefficients. Thus, $\omega^{m+1}=d\xi$ with

$$\xi=\sum_{r=1}^{R}\sum_{\alpha,\beta}C_{r\alpha\beta}x_{\alpha_1}dx_{\alpha_2}\wedge\cdots\wedge dx_{\alpha_r}\wedge\Phi_{r\alpha\beta}.$$

A translation

$$A\colon M\to M,\quad A\colon x'=x+a,\quad \varphi'=\varphi+b$$

(a and b modd 2π being constants) is a symplectic diffeomorphism of the $(m+1)$-exact symplectic manifold M. It is not hard to verify that this mapping is $(m+2)$-exact (for $m\leqslant n-2$) for any a and b. On the other hand, the diffeomorphism A is $(m+1)$-exact if and only if $a=0$.

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