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CONTINUOUS SYMMETRIES OF DIFFERENCE EQUATIONS

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Abstract. A method is given to find the continuous symmetries of ordinary difference equations. It is shown how the Lie symmetries can be used to find the general solution of difference equations. The method can also be applied to mappings.

In the second half of the nineteenth century Sophus Lie unified and extended the solution methods for ordinary differential equations. He discovered that the existing special techniques used for solving ordinary differential equations were all special cases of a general integration procedure based on the invariance of the differential equation under a continuous group of symmetries. In partial differential equations a continuous group can be used to find so-called "group invariant" or "similarity" solutions [1].

It has recently become clear that the same ideas work for difference equations. In refs. [2-4] Lie's method for finding symmetries was extended to differential-difference equations. It was also shown how Lie symmetries can be used to obtain similarity solutions of *partial* differential-difference equations [3,4]. In this paper we will show that the same method can be applied to *ordinary* difference equations.[1] To simplify the presentation we give an intuitive treatment based on an example, rather than presenting the method in full mathematical detail and generality. We note here that the method of [6], though superficially somewhat similar, in general does not yield Lie symmetries, and, in contrast to the method presented here, cannot be applied to nonautonomous difference equations.

The example we will treat in this paper is the difference equation

$$u(x+1) = 4u(x)^3 + 3u(x), \tag{1}$$

where u is a function: $\mathbb{R} \to \mathbb{R}$. We will find the Lie symmetries of this equation, and use them to derive its general solution. Consider the one parameter (ϵ) group of infinitesimal point transformations in $(x, u(x))$ given by

$$x^* = x + \epsilon\xi(x, u(x)) + O(\epsilon^2), \tag{2a}$$

$$u^*(x^*) = u(x) + \epsilon v(x, u(x)) + O(\epsilon^2). \tag{2b}$$

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[1] For an alternative approach see [5].

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Equation (1) is invariant under the transformation (2) if

$$u^*(x^*+1) = 4u^*(x^*)^3 + 3u^*(x^*). \tag{3}$$

Therefore, we need to know $u^*(x^*+1)$. The expression for this quantity was given in [2-4]

$$u^*(x^*+1) = u(x+1) + \epsilon v(x+1, u(x+1)) \\ + \epsilon[\xi(x,u(x)) - \xi(x+1, u(x+1))]\frac{du}{dx}(x+1). \tag{4}$$

Using (2) and (4) in (3) we obtain the equation to be satisfied by the symmetries:

$$v(x+1, u(x+1)) + [\xi(x,u(x)) - \xi(x+1,u(x+1))]\frac{du}{dx}(x+1) \\ = [12u^2(x)+3]v(x,u(x)). \tag{5}$$

Equation (5) implies

$$\xi(x,u(x)) - \xi(x+1,u(x+1)) = 0, \tag{6a}$$

$$v(x+1,u(x+1)) = 3[4u(x)^2+1]v(x,u(x)). \tag{6b}$$

Looking at equation (6a), it is clear that any unit periodic function $\alpha(x)$, i.e. a function defined by $\alpha(x) = \alpha(x+1)$ [3, 4, 7], is a solution, and substituting the equation of motion (1) in (6b) we obtain

$$\xi(x,u(x)) = \alpha(x) \tag{7}$$

$$v(x+1, 4u(x)^3 + 3u(x)) = 3[4u^2(x)+1]v(x,u(x)). \tag{8}$$

Equation (8) is a functional equation, that we solve by expanding $v(x,u(x))$ in a Laurent series in $\frac{1}{u(x)}$

$$v(x,u(x)) = a_0(x)u^b(x) + a_1(x)u^{b-1}(x) + \dots\ , \tag{9}$$

where $a_i(x)$, $i = 0,1,2,\dots$ are unknown functions of x and b is an unknown constant, all to be determined. Inserting equation (9) in (8), and equating the leading terms, it follows that

$$b = 1 \text{ and } a_0(x) = 3^x\beta(x), \tag{10}$$

where $\beta(x)$ is another arbitrary unit periodic function. Equating coefficients in the expansion of the right hand and left hand sides of (8), we get

$$v(x,u) = 3^x\beta(x)u[1 + \frac{1}{2}(\frac{1}{u})^2 - \frac{1}{8}(\frac{1}{u})^4 + \frac{1}{16}(\frac{1}{u})^6 - \frac{5}{128}(\frac{1}{u})^8 + \frac{7}{256}(\frac{1}{u})^{10} - \dots]. \tag{11}$$

With a little experience we recognize the terms in the square bracket as the first terms in the expansion of $(1+\frac{1}{u^2})^{\frac{1}{2}}$. Indeed, substituting this in (11) it is easy to verify that

$$v(x,u(x)) = 3^x\beta(x)(1+u^2(x))^{\frac{1}{2}} \tag{12}$$

is a solution of (8) and hence an infinitesimal symmetry of (1). [Note that even if a difference equation is autonomous, its symmetries are in general non-autonomous.]

The finite symmetry transformation associated with (12) is given by

$$\overline{u} = [\exp\{\epsilon\ v(x, u(x))\frac{\partial}{\partial u}\}]u. \tag{13}$$

In this case, the right hand side of (13) equals

$$[\exp\{\epsilon\ 3^x \beta(x)(1+u^2)^{\frac{1}{2}}\frac{\partial}{\partial u}\}]u. \tag{14}$$

We now look for a homogenizing change of variables $u \to z$ such that exponential operator in (14) becomes

$$\exp\{\epsilon\ 3^x \beta(x)\frac{\partial}{\partial z}\}. \tag{15}$$

To this end we must choose

$$z = \int \frac{du}{\sqrt{1+u^2}} = \operatorname{arcsinh} u. \tag{16}$$

Thus (14) becomes

$$[\exp \epsilon\ 3^x \beta(x)\frac{\partial}{\partial z}] \sinh z = \sinh(z + \epsilon\ 3^x \beta(x)). \tag{17}$$

Hence a finite symmetry group leaving equation (1) invariant is

$$\overline{u}(x) = \sinh[\operatorname{arcsinh} u(x) + \epsilon\ 3^x \beta(x)]. \tag{18}$$

The finite symmetry group (18) can be used to generate infinitely many solutions from a given solution. For example, to get the general solution of the difference equation (1), we apply the group (18) to a particular solution of (1), say $u \equiv 0$. Thus, taking $\epsilon = 1$ without loss of generality, the general solution of (1) is

$$\overline{u}(x) = \sinh(3^x \beta(x)), \tag{19}$$

where, as was mentioned, $\beta(x)$ is an arbitrary unit periodic function determined by the initial conditions.[2] For applications of this method to other autonomous and non-autonomous difference equations (with one or more spans) we refer the interested reader to [8].

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[2] Those readers who are interested in mappings rather than difference equations can take $\beta(x)$ equal to a constant.

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