

**CHAOS AND TIME REVERSAL
SYMMETRY : AN INTRODUCTION**

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1 Time reversal symmetry and physics

It has been said that physics is the study of symmetries of nature. This statement may be an exaggeration, but it holds some truth. Symmetries hold some of the fundamental properties of physical systems and help both to frame their description mathematically and to guide our way of looking at nature.

Symmetries come in two kinds, *continuous* and *discrete*. Usually the symmetries that we deal with are continuous; examples are rotational symmetry, described by the groups $O(2)$, $SU(2)$ and so on. In a sense, discrete symmetries can be the more difficult to treat mathematically, since the concept of infinitesimal symmetry cannot be used.

In these lectures we will be discussing the effect of one particular type of discrete symmetry, time-reversal symmetry, on dynamical systems.

We will refer to a system that possesses time-reversal symmetry as *reversible*. This will be our definition of reversibility. It should be noted that dynamical reversibility is not equivalent to invertibility, or to thermodynamical reversibility, which implies that no heat is applied.

The earliest use of time-reversal symmetry was Loschmidt's demonstration that Boltzmann's proof of the H -theorem was in error because it started from equations of motion that are invariant and ended with a consequence that is not. Boltzmann later went on to prove that Maxwell's equations are reversible.

Time reversal symmetry T plays a role in many branches of physics. Examples are

Electromagnetism	: Maxwell's equations
Gravitation	: Einstein equations
Quantum mechanics	: Schrödinger equation
Field theory	: CPT theorem

Maxwell's equations are invariant under the transformation

$$\begin{cases} t \mapsto -t \\ \mathbf{B} \mapsto -\mathbf{B}, \end{cases} \quad (1)$$

and Schrödinger's equation is invariant under

$$\begin{cases} t \mapsto -t \\ \psi \mapsto \psi^*. \end{cases} \quad (2)$$

The subject of these lectures will not be quantum mechanics or relativity, but T invariance in classical dynamical systems.

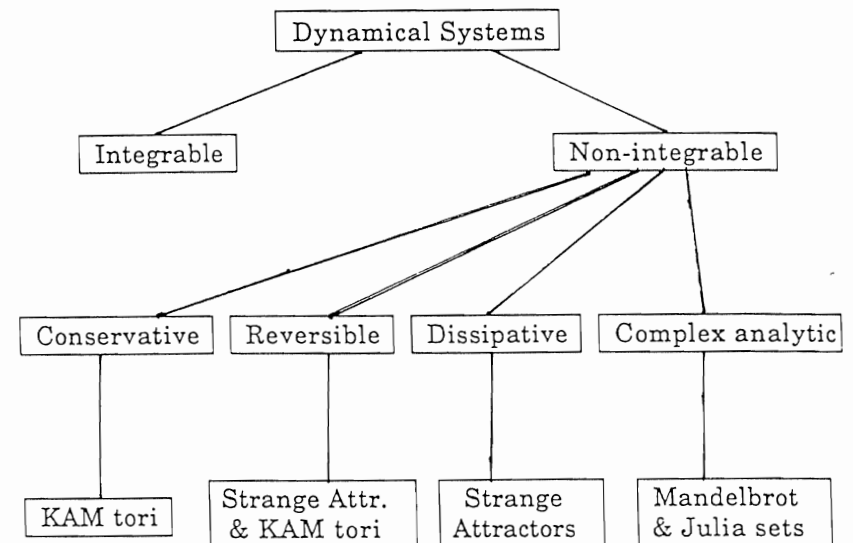


Figure 1: Schematic diagram of a classification of dynamical systems.

2 Time-reversal symmetry and chaos

Where do reversible systems fit into the overall picture of dynamical systems? The diagram in Fig. 1 gives a break-down of the areas of dynamical systems. Integrable systems are conservative; there exist a 'sufficient' number of integrals or constants of the motion. This class makes up a small part of the totality of dynamical systems. The non-integrable systems make up the vast majority, and may be divided as shown into conservative, reversible, dissipative, and complex analytic systems, characterized respectively by KAM tori, strange attractors and Mandelbrot and Julia sets. Reversible systems, upon which we will focus, are intermediate between conservative and dissipative systems, and exhibit both KAM tori and strange attractors.

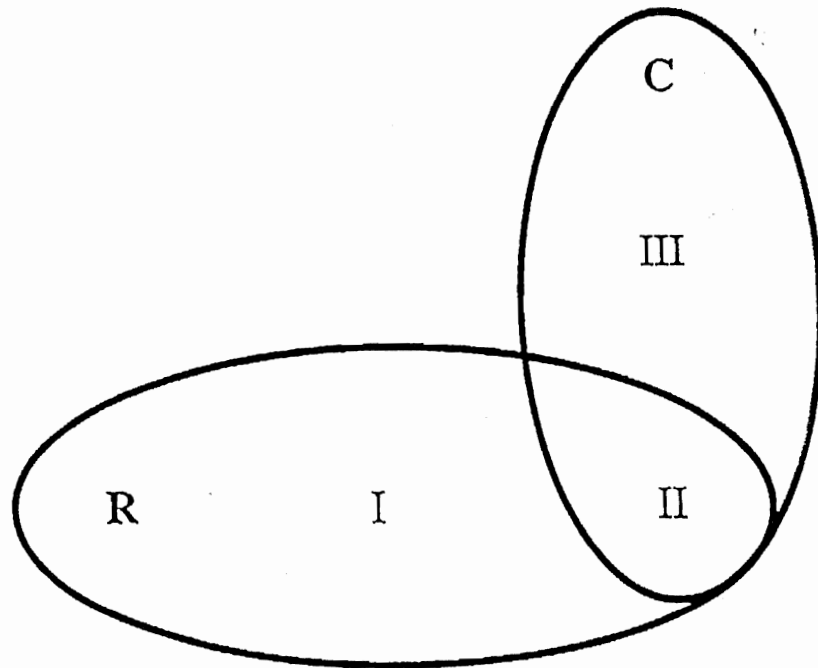


Figure 2: Reversible vs conservative dynamical systems: C = conservative (Hamiltonian) R = reversible.

3 Time-reversal symmetry and ODEs

The class of reversible ODEs is in fact quite large. Some examples are the following, where the dot denotes d/dt :

$$\ddot{x} = F(x) \quad (F: \mathbb{R} \rightarrow \mathbb{R}), \quad (3)$$

$$\ddot{x} = F(x, \dot{x}^2) \quad (F: \mathbb{R}^2 \rightarrow \mathbb{R}), \quad (4)$$

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \quad (\mathbf{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n). \quad (5)$$

These equations are invariant under the transformation $t \mapsto -t$. There is a big difference between Eq. (3) on one hand and Eq. (4) and Eq. (5) on the other:

- the system (3) is conservative: it can be derived from a Hamiltonian

$$H(x, p) = \frac{p^2}{2} + V(x), \quad (6)$$

where $dV/dx = -F$.

• systems (4) and (5) are in general not conservative, although there are some special cases. Thus there are systems which are both reversible *and* conservative, and others which are reversible but *not* conservative. This is represented in Fig. 2.

What is time-reversal symmetry for ordinary differential equations? Take as an example Eq. (3),

$$\ddot{x} = F(x) \quad \text{T invariance: } t \mapsto -t.$$

Rewriting this as a first-order system, we obtain

$$\begin{cases} \dot{x} = y \\ \dot{y} = F(x) \end{cases} \quad \text{T invariance: } \begin{matrix} t \mapsto -t \\ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} \end{matrix}, \quad (7)$$

so that time-reversal symmetry corresponds to reversing the velocities. Applying a non-linear coordinate transformation (or 'conjugacy')

$$\begin{pmatrix} u \\ v \end{pmatrix} = C \begin{pmatrix} x \\ y \end{pmatrix},$$

the first-order derivatives are given by

$$\begin{aligned} \dot{u} &= g(u, v), \\ \dot{v} &= h(u, v). \end{aligned} \quad (8)$$

T-invariance implies that, eq.(8) is invariant under $t \mapsto -t$ and

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto G \begin{pmatrix} u \\ v \end{pmatrix} = CTC^{-1} \begin{pmatrix} u \\ v \end{pmatrix}.$$

The operator G is an *involution*; that is, two applications of G return the system to its original state. A simple example of this is a reflection. G can be seen to be an involution, as

$$G \circ G = C \circ T \circ C^{-1} \circ C \circ T \circ C^{-1} = T \circ T = \text{Id}. \quad (9)$$

Here Id represents the identity mapping. We can now take the following definition.

Definition 1 *The ODE*

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \quad (\mathbf{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n) \quad (10)$$

is reversible if there is an involution G in phase space such that Eq. (10) is invariant under $t \mapsto -t$, $\mathbf{x} \mapsto G(\mathbf{x})$.

This definition is invariant under coordinate transformations. Note that not all involutions are conjugate to T . To illustrate this point, take as an example a particle in three dimensions, described by a six-dimensional vector of position and momentum coordinates. The action of time reversal then appears as

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ -p_1 \\ -p_2 \\ -p_3 \end{pmatrix}, \quad (11)$$

so that

$$T = \text{diag}\{1, 1, 1, -1, -1, -1\}$$

and $T^2 = 1$. Now the form

$$G = \text{diag}\{1, -1, -1, -1, -1, -1\}$$

is also an involution, but is not conjugate to T , as can be seen from the Jordan canonical form.

3.1 Examples of reversible ODEs

3.1.1 Some Hamiltonian systems

All Hamiltonian systems with a Hamiltonian even in the momenta are reversible. Then the reversing involution is

$$G \begin{pmatrix} \mathbf{x} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ -\mathbf{p} \end{pmatrix}. \quad (12)$$

3.1.2 The logistic differential equation

The logistic ODE

$$\dot{x} = x(1-x), \quad (13)$$

which is the continuous analogue of the logistic map. The reversing involution for this equation is

$$G(x) = 1-x, \quad (14)$$

which can be easily verified. The system has two equilibrium points, at $x = 0$ and $x = 1$. The linearization of the ODE at the fixed points determines their stability. At $x = 0$ the linearized equation is $\dot{x} = x$; at $x = 1$ the linearization is $\dot{x} = -x$, so that $x = 0$ is a repeller and $x = 1$ is an attractor. This is therefore an example of a reversible system that has both an attractor and a repeller. Under time-reversal, the attractors and repellers reverse their nature.

3.1.3 ABC flow in hydrodynamics

The equations for this are given by

$$\begin{aligned} \dot{x} &= A \sin z + C \cos y, \\ \dot{y} &= B \sin x + A \cos z, \\ \dot{z} &= C \sin y + B \cos x. \end{aligned} \quad (15)$$

Here A, B, C are parameters. A reversing involution is defined by

$$G \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \pi - x \\ -y \\ z \end{pmatrix}. \quad (16)$$

3.1.4 A non-linear example

Involutions need not be linear; for instance, $\int \frac{1}{x} + \frac{1}{2} \frac{(y-x^2)^2}{x^2}$

$$\begin{aligned} \dot{x} &= xy - x^3, \\ \dot{y} &= x + y^2 - x^4, \end{aligned} \quad (17)$$

has the reversing involution

$$G \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y + 2x^2 \end{pmatrix}. \quad (18)$$

3.1.5 An example in higher dimensions

Take a first-order system in $n + m$ dimensions:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{y}), & \mathbf{x} &\in \mathbb{R}^n, \\ \dot{\mathbf{y}} &= \mathbf{g}(\mathbf{x}, \mathbf{y}), & \mathbf{y} &\in \mathbb{R}^m, \end{aligned} \quad (19)$$

where \mathbf{f} is even in \mathbf{x} and \mathbf{g} is odd in \mathbf{x} . The reversing involution is

$$G \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} -\mathbf{x} \\ \mathbf{y} \end{pmatrix}. \quad (20)$$

4 Applications of reversible ODEs

Reversible ordinary differential equations occur in a variety of physical circumstances. Some examples include the following.

• **Externally injected ruby and CO₂ lasers** The equations describing the dynamics inside these systems are.

$$\begin{aligned}\dot{\phi} &= C_1 - \frac{1}{E} \sin \phi, \\ \dot{E} &= WE + \cos \phi, \\ \dot{W} &= C_2 - E^2,\end{aligned}\quad (21)$$

where E is the cavity field amplitude (scaled by the external frequency), ϕ is the phase, and W is the (scaled) population inversion. C_1 and C_2 are parameters. Here we have

$$G \begin{pmatrix} \phi \\ E \\ W \end{pmatrix} = \begin{pmatrix} \pi - \phi \\ E \\ -W \end{pmatrix}. \quad (22)$$

• **Isothermal oscillator** A model for a harmonic oscillator which is coupled to a heat bath is

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -x - zy, \\ \dot{z} &= \alpha(y^2 - kT).\end{aligned}\quad (23)$$

The term kT is the product of Boltzmann's constant and the temperature and α is a parameter. The reversing involution is

$$G \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ -z \end{pmatrix}. \quad (24)$$

• **Gas flame** An equation used to model the dynamics of the perturbations in a gas flame is the partial differential equation

$$\xi_t + \xi_{xxxx} + 2\alpha\xi_{xx} + \xi + \xi_x^2 = 0, \quad (25)$$

where the function $\xi = \xi(x, t)$ is the displacement of the flame point from the unperturbed position, and α is a parameter. Consider stationary solutions of the above equation, which then satisfy

$$\xi_{xxxx} + 2\alpha\xi_{xx} + \xi + \xi_x^2 = 0, \quad (26)$$

and so have left-right 'reversibility' or symmetry as $x \mapsto -x$. Mathematically, the treatment of left-right symmetry and the treatment of time-reversal symmetry in ODEs is the same.

• **Integrable cases** Many soliton equations and integrable dynamical systems are reversible.

• Applications to particle sedimentation and to circuits containing Josephson junctions are given in reference 1.

5 Time-reversal symmetry and mappings

We will now extend the ideas of the previous sections to discrete mappings. An example of a reversible mapping is

$$x_{i+1} + x_{i-1} = F(x_i). \quad (27)$$

Eq. (27) is reversible under the transformation $i \mapsto -i$, which is the discrete equivalent of $t \mapsto -t$. To generalize this,

Definition 2 *The mapping*

$$\mathbf{x}_{i+1} = \mathbf{F}(\mathbf{x}_i), \quad \mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (28)$$

is reversible if there is an involution \mathbf{G} in phase space such that Eq. 28 is invariant under $i \mapsto -i$, $\mathbf{x} \mapsto \mathbf{G}(\mathbf{x})$.

Writing this out explicitly (from now on vector nature is understood),

$$\left. \begin{aligned} G(x_{-i-1}) &= F \circ G(x_{-i}) \\ x_{-i} &= F(x_{-i-1}) \end{aligned} \right\} G(x_{-i-1}) = F \circ G \circ F(x_{-i-1})$$

which implies the equivalent definition of reversibility for the map F :

$$\begin{aligned} F \circ G \circ F &= G \\ G \circ G &= \text{Id} \end{aligned} \quad (29)$$

which is a purely algebraic definition. Further, (29) implies

$$F \circ G \circ F \circ G = G \circ G = \text{Id},$$

which means that $F \circ G$ is an involution, and so F is the product of two involutions. This results in another form of the above definition which is often the easiest to work with:

$$\begin{aligned} F &= H \circ G, \\ H \circ H &= G \circ G = \text{Id}. \end{aligned} \quad (30)$$

[This is proven simply by $H = F \circ G \mapsto H \circ G = F \circ G \circ G = F$] This gives then a straight-forward method of constructing reversible maps: simply choose two involutions and take their composition. Two examples are

• **The area-preserving Hénon map** Beginning with the transformation H and the involution G defined by

$$H : \begin{cases} x' = x \\ y' = -y + \alpha x + x^2 \end{cases} \quad G : \begin{cases} x' = y \\ y' = x, \end{cases} \quad (31)$$

the composition $F = H \circ G$ is the area-preserving Hénon map,

$$F : \begin{cases} x' = y \\ y' = -x + \alpha y + y^2. \end{cases} \quad (32)$$

• **The standard map** The composition of

$$H : \begin{cases} x' = x \\ y' = -y + K \sin x \end{cases} \quad \text{and} \quad G : \begin{cases} x' = y \\ y' = x \end{cases} \quad (33)$$

results in the standard map, given by

$$F : \begin{cases} x' = y \\ y' = -x + K \sin y. \end{cases} \quad (34)$$

Both of these examples are area-preserving.

There are two important questions to be answered concerning the construction of such reversible maps:

- (1) How does one obtain the involutions?
- (2) Are reversible two-dimensional maps always area-preserving?

The first of these questions can be answered by considering a general construction of an involution. Take an involution I , and apply an arbitrary coordinate transformation C . Then the composition

$$H = C \circ I \circ C^{-1}$$

is also an involution. Therefore choosing I to be trivial, involutions can be constructed by application of the appropriate coordinate transformation. As an example, take

$$I : \begin{cases} x' = x \\ y' = -y \end{cases} \quad (35)$$

and the coordinate transformation

$$C : \begin{cases} x' = x \\ y' = y + \frac{1}{2}\alpha x + \frac{1}{2}x^2 \end{cases} \quad (36)$$

which results in the transformation we used in the first of the previous examples to obtain the Hénon map,

$$H : \begin{cases} x' = x \\ y' = -y + \alpha x + x^2. \end{cases} \quad (37)$$

To answer the second of the above questions, consider

$$F = H \circ G.$$

If G and H are both area-preserving, then F is area-preserving.

If G is area-preserving and H is not, then F is *not* area-preserving. How do we get an involution H that is not area-preserving? This is done simply by taking $H = C \circ I \circ C^{-1}$, where I is area-preserving and C is not. Then

$$F = H \circ G = C \circ I \circ C^{-1} \circ G$$

is a non-area-preserving invertible map (in general). Here we may choose I and G to be trivial. To take an example, with

$$C : \begin{cases} x' = x[1 + (y - 1)^2] \\ y' = y \end{cases} \quad (38)$$

and

$$I : \begin{cases} x' = y \\ y' = x, \end{cases} \quad (39)$$

we construct H according to

$$H = C \circ I \circ C^{-1} : \begin{cases} x' = y[1 + (y' - 1)^2] \\ y' = x/[1 + (y - 1)^2]. \end{cases} \quad (40)$$

Take the involution

$$G : \begin{cases} x' = x \\ y' = \alpha - y, \end{cases} \quad (41)$$

then F is the composition of H and G . The resulting non-area-preserving map is

$$F : \begin{cases} x' = (\alpha - y)[1 + (y' - 1)^2] \\ y' = x/[1 + (\alpha - y - 1)^2]. \end{cases} \quad (42)$$

6 Applications of reversible maps

We give two examples of physical situations described by discrete models which are reversible.

• **Chemical basis of morphogenesis** The diffusion of two different chemicals through a medium can be described by

$$\begin{aligned} \frac{d}{dt}x_i &= D_1(x_{i+1} - 2x_i + x_{i-1}) + x_i y_i - y_i - \alpha, \\ \frac{d}{dt}y_i &= D_2(y_{i+1} - 2y_i + y_{i-1}) + \beta - x_i y_i, \end{aligned} \quad (43)$$

where x_i is the concentration of chemical A in cell i , and y_i is the concentration of chemical B in cell i ; D_1 and D_2 are diffusion constants and α and β are parameters. Stationary solutions of these equations are

$$x_{i+1} - 2x_i + x_{i-1} = (\alpha + y_i - x_i y_i)/D_1 \quad (44)$$

$$y_{i+1} - 2y_i + y_{i-1} = (x_i y_i - \beta)/D_2 \quad (45)$$

which are symmetric under $i \rightarrow -i$.
 • **Quasicrystals** A one-dimensional quasi-crystal is represented by the equation

$$x_{i+1} + x_{i-2} = x_i x_{i-1}, \quad (46)$$

where x_i is the trace of the transfer matrix at site i of the quasi-crystal. Eq. 46 is invariant under $i \rightarrow -i$.

7 Properties of reversible systems

Reversible systems have some interesting properties which show that the symmetric orbits of the system look very similar to orbits of area-preserving maps. In this section we look at how reversible systems relate to purely conservative or purely dissipative systems.

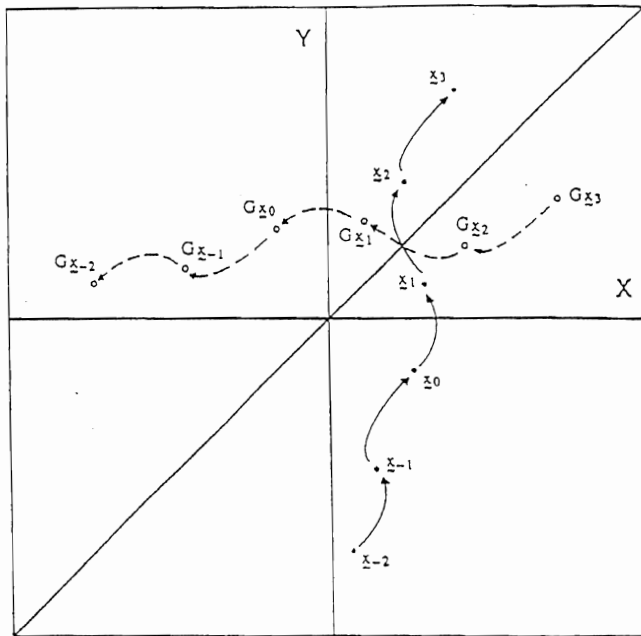


Figure 3: Trajectories of an orbit and its transformation under G when the orbit contains no point on the axis of symmetry.

7.1 Symmetric orbits

Recall that for a reversible map L , the following property holds:

$$LGL = G,$$

where G is an involution. An orbit of the system is represented by a sequence of points x_n which are the positions at times t_i , and as there is time-reversibility, we can write an orbit of L as $O = (\dots, x_{-1}, x_0, x_1, \dots)$. Consider now the orbit at any time t_n . Applying the above property we obtain

$$LGLx_n = Gx_n,$$

which by definition becomes

$$LGx_{n+1} = Gx_n,$$

so that

$$Gx_{n+1} = L^{-1}Gx_n.$$

This result shows that the set of points $GO = (\dots, Gx_{-1}, Gx_0, Gx_1, \dots)$ is also an orbit of the system (see Fig. 3). If the orbit O contains a point which lies on the axis of symmetry, then the transformed orbit GO is identical to O . This is illustrated in Fig. 4. We will refer to any orbit containing such a point as a symmetric orbit.

7.2 Periodic orbits are easy to find

Let the two points x_0 and x_n lie on the symmetry line. Using time reversibility, it is then easy to show that x_0 is periodic of period $2n$. Therefore to find symmetric periodic orbits, we need only search on the symmetry line.

7.3 Symmetric orbits exhibit conservative behaviour

A map is area-preserving if the determinant of its Jacobian matrix J has the value 1. Now, in two-dimensional reversible maps, J varies throughout the plane. However, at symmetric fixed points in reversible maps, $J = 1$ (or -1). This implies that symmetric fixed points are elliptic, parabolic or hyperbolic. For *elliptic* symmetric fixed points, there is a KAM theorem, just as in the case of conservative systems.

Symmetric period-doubling Period-doubling is the name given to the bifurcation of the stable solution of the map to a new stable solution with a period that is double the previous one. This occurs as a parameter is varied. Denote the parameter by C , and let C_n be the value of C at which the orbit of period 2^n period-doubles. It has been found that the values of C at which the bifurcations occur are asymptotically governed by a simple law,

$$C_n = C_\infty + A\delta_{\text{sym}}^{-n}. \quad (47)$$

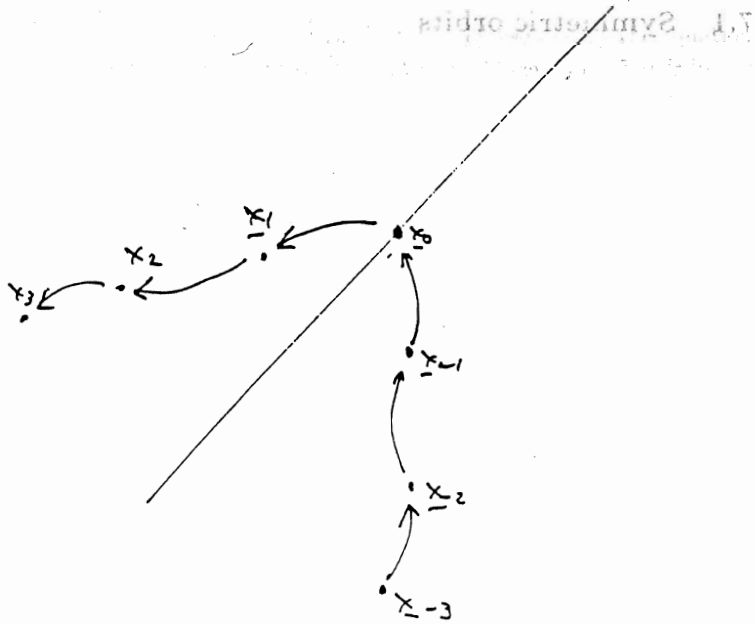


Figure 4: Trajectory of an orbit when the orbit has a point on the axis of symmetry: its transformation under G is identical.

The value of δ for symmetric orbits in reversible 2D maps is a universal constant and has been determined to be $\delta_{\text{sym}} = 8.7210972 \dots$ (Compare this with the known value for conservative maps, $\delta_{\text{cons}} = 8.721097200 \dots$)

Breaking of (symmetric) KAM circles A KAM circle can be approximated by orbits of period p_n . At $C = C_n$ the orbit of period p_n turns unstable. There is an analogous equation to the one above for the value of C_n ,

$$C_n = C_\infty + B\gamma_{\text{sym}}^{-n} \tag{48}$$

The quantity γ_{sym} is universal, and for a generic reversible system has the value $\gamma_{\text{sym}} = -2.665 \dots$ (Compare: $\gamma_{\text{cons}} = -2.6651429 \dots$)

• **Elliptic and hyperbolic points** Elliptic and hyperbolic fixed points in reversible and area-preserving maps are indistinguishable up to and including third order terms in their Taylor expansions.

Parabolic points The only bifurcations between fixed points in area preserving maps are symmetric fixed points in reversible maps (the first order bifurcations are parabolic)

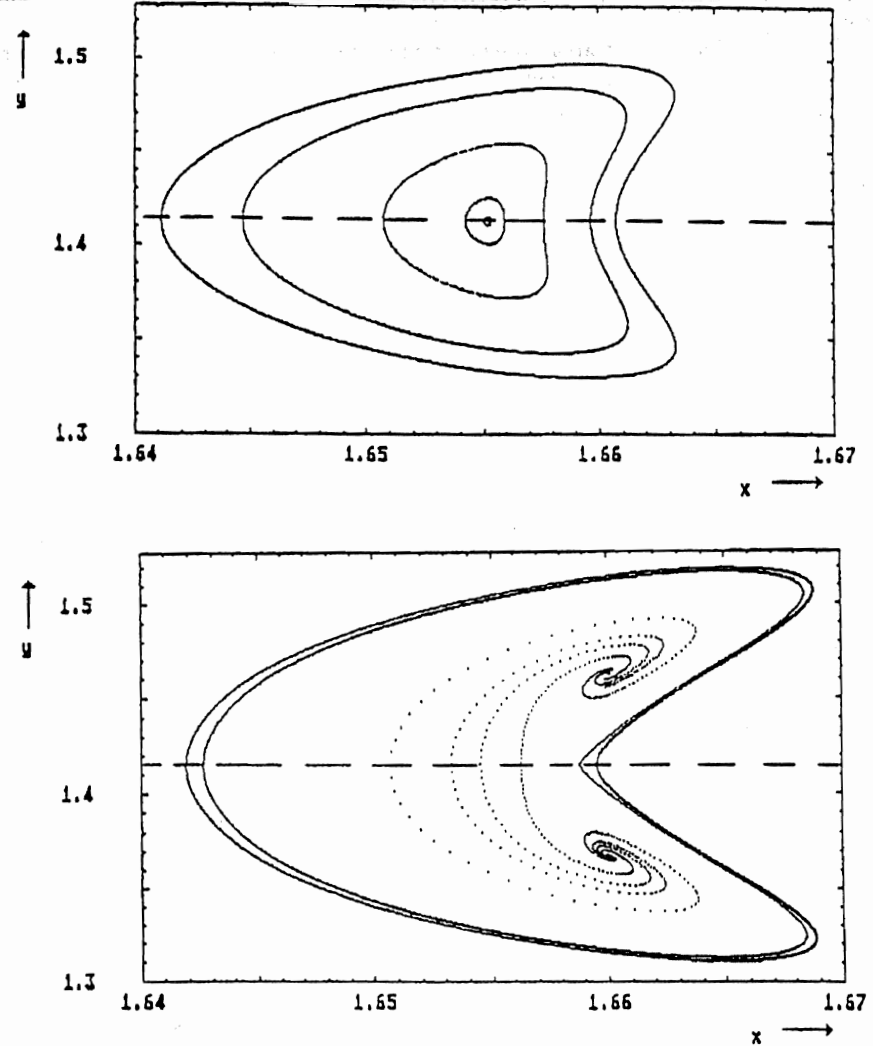


Figure 5: Before (above) and after (below) the bifurcation of an attractor and a repeller from a symmetric fixed point in a reversible map. This Rimmer bifurcation occurs in the map (42) at $\alpha = 2^{3/2}$.

• **Parabolic points** The only difference between fixed points in area-preserving maps and symmetric fixed points in reversible maps (to third order) is in the parabolic case.

1. A reversible but not area-preserving phenomenon is the birth of an attractor-repeller pair from a parabolic fixed point. We call this a Rimmer bifurcation. (Fig. 5.)

2. An area-preserving but non-reversible map with a parabolic fixed point at the origin is

$$\begin{aligned}x' &= x + ay^3 + by^4, \\y' &= y + cx'^3 + dx'^4.\end{aligned}\quad (49)$$

7.4 Asymmetric orbits

Asymmetric orbits exhibit 'dissipative' behaviour; J is in general unequal to 1 and there are some features which are the same as in purely dissipative maps.

• **Asymmetric period doubling**

As before, the period-doubling bifurcation sequence occurs according to

$$C_n = C_\infty + D\delta_{\text{asym}}^{-n}. \quad (50)$$

The quantity δ_{asym} is universal for reversible systems and takes the value $\delta_{\text{asym}} = 4.66920\dots$ (Compare: $\delta_{\text{diss}} = 4.6692016091\dots$)

A mapping exhibiting both (symmetric) KAM tori and an (asymmetric) attractor/repeller pair is shown in Fig. 6.

8 Some open problems

We conclude by mentioning some of the many problems of interest still to be solved. Some examples are

• **Is Rannou's map reversible?** Rannou's map is

$$x' = x + y + 1 - \cos y, \quad (51)$$

$$y' = y + C(1 - \cos x' - \sin x'). \quad (52)$$

It is not yet known whether or not this map is reversible.

• **Conservative vs reversible systems** It would be very interesting to make a complete comparison between all properties of conservative systems and those of symmetric orbits in reversible systems. In these lectures we have seen some of the aspects of this comparison but there are still many properties which have not been shown.

• **PDE's.** Very little of the above has as yet been generalized to partial differential equations.

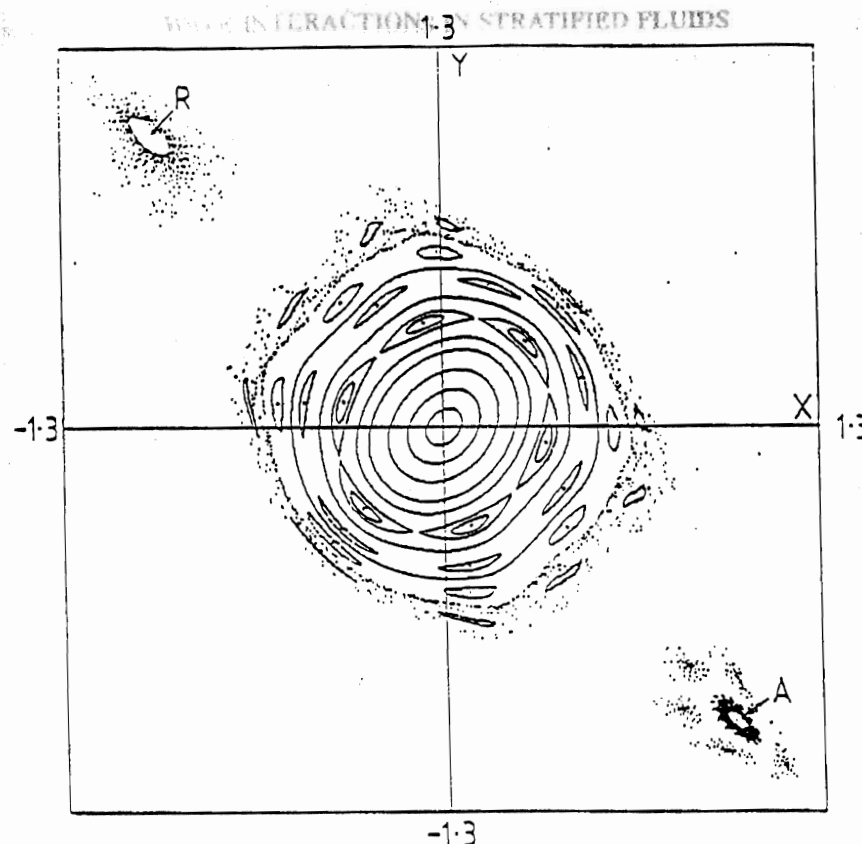


Figure 6: Phase plane of a reversible mapping, exhibiting both conservative and dissipative/expansive features (A = attractor; R = repeller.)

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