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The lattice Gel'fand–Dikii hierarchy*

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Abstract. Using the direct linearization method we construct lattice versions of the hierarchy of Gel'fand–Dikii equations, the first members being the lattice KdV equation and the lattice Boussinesq equation. The (local) initial value problem on the lattice of this family of equations is formulated, giving rise to integrable multi-dimensional mappings. The involutivity of the invariants of these mappings is established on the basis of a novel classical r-matrix structure.

1. Introduction

Integrable lattice equations are exact space and time discretizations of integrable PDEs, [1–3], i.e. integrable partial *difference* equations. A typical example of such an integrable lattice equation is the lattice potential KdV equation, [4, 5]

$$(p - q + u_{n,m+1} - u_{n+1,m})(p + q - u_{n+1,m+1} + u_{n,m}) = p^2 - q^2 \tag{1.1}$$

in which $u = u_{n,m}$ is the dynamical (field) variable at the site (n, m) of a rectangular lattice, $(n, m \in \mathbb{Z})$, and $p, q \in \mathbb{C}$ are the lattice parameters, measuring the lattice spacing in the n- and m-directions respectively. Lattice equations such as equation (1.1) carry many of the characteristics endemic in continuous integrable systems, such as the existence of a Lax or Zakharov–Shabat system, an inverse scattering scheme and a sufficient number of integrals of the (discrete) time-flow in involution with respect to a proper symplectic form. An interesting application is the construction of integrable *mappings*, i.e. systems with a finite number of degrees of freedom in which the evolution is given in terms of discrete time-steps, cf [6–8]. Integrable mappings were investigated also in the recent literature, cf [9–14].

In [6–8] we constructed a family of exactly integrable finite-dimensional mappings from the lattice KdV equation (1.1) by considering 'local' initial value problems on so-called 'staircases' on the lattice. These mappings are Lagrangian as a consequence

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of the existence of an action principle, which can be derived from the following action of the KdV lattice, [7]

$$\mathcal{S} = \sum_{n,m\in\mathbb{Z}} \left[u_{n,m}\left(u_{n+1,m} - u_{n,m+1}\right) + \epsilon\delta \ln\left(\delta + u_{n,m+1} - u_{n+1,m}\right)\right] \tag{1.2}$$

(where $\epsilon = p + q$, $\delta = p - q$). From the Lagrangian property one can derive a symplectic or Poisson structure in terms of which the mapping is a canonical transformation. The complete integrability in the sense of Liouville, cf [10], has been established by obtaining a complete set of invariants of the mapping which are in involution with respect to this Poisson structure, [6–8].

Equation (1.1) has the property that it is local around a plaquette on the lattice, i.e. the variables involved in the equation are the variables u on the four different sites around a simple plaquette on the rectangular lattice. In this paper we are interested in integrable lattices of a slightly more general type, namely examples where the equation involves not only the variables around a simple plaquette, but also next-nearest and in general farther neighbours. An example of such a lattice equation is the following one that we refer to as the lattice Boussinesq (BSQ) equation

$$\begin{aligned}\frac{p^3-q^3}{p-q+u_{n+1,m+1}-u_{n+2,m}} &- \frac{p^3-q^3}{p-q+u_{n,m+2}-u_{n+1,m+1}} - u_{n,m+1}u_{n+1,m+2}\\ &+ u_{n+1,m}u_{n+2,m+1} + u_{n+2,m+2}\left(p-q+u_{n+1,m+2}-u_{n+2,m+1}\right)\\ &+ u_{n,m}\left(p-q+u_{n,m+1}-u_{n+1,m}\right)\\ &= (2p+q)\left(u_{n+1,m}+u_{n+1,m+2}\right) - (p+2q)\left(u_{n,m+1}+u_{n+2,m+1}\right).\end{aligned} \tag{1.3}$$

This equation reduces to the continuum (potential) BSQ equation after a suitable continuum limit. Similar to the lattice KdV, the lattice BSQ (1.3) can be obtained from an action, namely ($\epsilon \equiv p^2 + pq + q^2$, $\delta \equiv p - q$)

$$\begin{aligned}\mathcal{S} = \sum_{n,m\in\mathbb{Z}} \Big[&\epsilon\delta\ln\left(\delta + u_{n,m+1} - u_{n+1,m}\right)\\ &- \left(p+q+u_{n,m}\right)\left(p+q-u_{n+1,m+1}\right)\left(\delta + u_{n,m+1} - u_{n+1,m}\right)\\ &+ q\,u_{n,m}\,u_{n,m+1} - p\,u_{n,m}\,u_{n+1,m}\Big]\end{aligned} \tag{1.4}$$

the Euler–Lagrange equations of which yields the lattice BSQ equation (1.3).

Equations (1.1) and (1.3) are the lowest order members of a hierarchy of lattice equations of what we will call the lattice Gel'fand–Dikii (GD) hierarchy, whose higher-order members are generally formed by coupled systems of partial difference equations. It is this class of lattice systems that we study in this paper. These lattice equations go by suitable continuum limits to the corresponding PDEs of the usual continuous Gel'fand–Dikii hierarchy, cf e.g. [15–17]. The GD class of equations have recently been shown to arise in connection with string theory in zero dimension and two-dimensional quantum gravity, [18], cf also [19, 20]. Furthermore, we investigate in this paper the mapping reductions of the lattice GD hierarchy, leading to a large class of integrable rational mappings. We prove the integrability of these mappings in the sense of Liouville on the basis of the associated r-matrix structure.

The outline of this paper is as follows. In section 2 we introduce the lattice GD hierarchy starting from the direct linearization method (DLM). The structure of the Lax operators of the members of the hierarchy is explained. In section 3 we show how to reduce the lattice in order to obtain integrable mappings, and present their symplectic structure. In section 4 we introduce a novel (so-called non-ultralocal) r-matrix structure which is suitable for the canonical structure of the mappings in the GD class. Finally, in section 5 we go over to a description in terms of a global Lax representation (the 'big' Lax) which leads to the proper variables in which the mappings and their invariants are most conveniently expressed. The involutivity of the invariants of the mappings is proven on the basis of the r-matrix structure.

2. Lattice Gel'fand–Dikii hierarchy

Many integrable lattice equations have been found and studied by means of the direct linearization method (DLM), cf e.g. [4, 5]. The DLM, which is based on the use of linear singular integral equations (generalizing the ones that occur in the Riemann–Hilbert formulation of the inverse scattering transform), has been introduced by Fokas and Ablowitz in [21] order to treat the initial value problem of the second Painlevé equation PII, cf also [22]. In [23], we generalized this approach to include equations of Boussinesq (BSQ) type, and along this line one can extend the DLM to the equations in the GD hierarchy. Furthermore, the DLM provides a very convenient tool for introducing Bäcklund transformations, cf [24], whose Bianchi identities lead to the lattice equations by a suitable reinterpretation of the variables, [4, 5]. Therefore, this approach is very suitable for obtaining a lattice analogue of the GD hierarchy. For this purpose we introduce the integral equation, [23]

$$\boldsymbol{u}_k + \rho_k \int_C \mathrm{d}\lambda(\ell)\, \frac{\boldsymbol{u}_\ell}{k - \omega \ell} = \rho_k \boldsymbol{c}_k . \tag{2.1}$$

The integral in equation (2.1) is performed over an arbitrary contour in the complex ℓ-plane, k and ℓ playing the role of spectral parameters. The 'wavefunction' $\boldsymbol{u}_k$ is to be solved from the linear integral equation under suitable conditions on the measure $\mathrm{d}\lambda(\ell)$ and contour C, the main restriction being that they must be chosen such that the solution of the integral equation is unique given the inhomogeneous term. The object $\boldsymbol{c}_k$ on the RHS in (2.1) is an infinite-component vector whose components are powers of k, i.e. $\boldsymbol{c}_k = (c_k^{(i)})$, $c_k^{(i)} = k^i$, $i \in \mathbb{Z}$, and as a consequence, the solutions of (2.1) are also infinite-component vectors $\boldsymbol{u}_k = (u_k^{(i)})$, $i \in \mathbb{Z}$, where each $u_k^{(i)}$ is the solution of (2.1) with the corresponding factor k^i in the inhomogeneous term. The parameter ω, which is a root of unity $\omega \equiv \exp(2\pi \mathrm{i}/N)$, $(N = 2, 3, \ldots)$, specifies the various members of the GD hierarchy. For $N = 2$ we obtain KdV-type equations, whereas for $N = 3$ the equations are of Boussinesq-type.

The inhomogeneous term contains a 'plane-wave factor' ρ_k that can be chosen to depend on additional variables, namely the space and time variables of the system that one is interested in. One may take e.g. continuous variables x and t, in which case ρ_k is a plane-wave factor leading, for general N, to the Gel'fand–Dikii (GD) hierarchy, [15], cf also [16, 17]. In this article, we are interested in two-dimensional lattice equations, for which the ρ_k, and therefore also the $\boldsymbol{u}_k$, depend on two integers (n, m) labelling the lattice sites. For the investigation of the lattice equations associated with the GD hierarchy it is convenient to impose special transformation properties (corresponding to Bäcklund transformations) for the factors ρ_k, which can be interpreted as translations on a two-dimensional lattice. For this purpose, let us consider the transformation

$$\rho_k \mapsto \tilde{\rho}_k = \frac{p+k}{p+\omega k}\rho_k \tag{2.2}$$

with $p \in \mathbb{C}$, cf [4, 5]. From the integral equation (2.1) with ρ_k and $\boldsymbol{u}_k$ replaced by $\tilde{\rho}_k$ and $\tilde{\boldsymbol{u}}_k$, one obtains a linear (spectral) problem relating the infinite-component vectors $\tilde{\boldsymbol{u}}_k$ and $\boldsymbol{u}_k$. The coefficients in these relations contain an infinite-sized ($\mathbb{Z}\times\mathbb{Z}$) potential matrix $\mathbf{U}$, with entries $u^{(i,j)}$, ($i, j \in \mathbb{Z}$), associated with the solution $\boldsymbol{u}_\ell$ of (2.1) by an integration over the same contour with the same measure, i.e.

$$\mathbf{U} = \int_C \mathrm{d}\lambda(\ell)\, \boldsymbol{u}_\ell \rho_\ell\, {}^{\mathrm{t}}\boldsymbol{c}_\ell \tag{2.3}$$

where the superscript t denotes transposition, i.e. the integrand in (2.3) is a dyadic expression. Making use of the above mentioned uniqueness condition on the measure and contour, the following linear relations involving $\boldsymbol{u}_k$ and $\tilde{\boldsymbol{u}}_k$ can be derived

$$(p+\omega k)\tilde{\boldsymbol{u}}_k = (p+\boldsymbol{\Lambda} - \tilde{\mathbf{U}}\cdot\mathbf{O})\cdot\boldsymbol{u}_k \tag{2.4a}$$

$$\begin{aligned}\left[\prod_{j=2}^{N}(p+\omega^j k)\right]\boldsymbol{u}_k = {} & \left[\prod_{j=1}^{N-1}(p+\omega^j\boldsymbol{\Lambda})\right]\cdot\tilde{\boldsymbol{u}}_k \\ & -\sum_{i=1}^{N-1}\omega^i\,\mathbf{U}\cdot\left[\prod_{j=2}^{i}(p+\omega^j\,{}^{\mathrm{t}}\boldsymbol{\Lambda})\right]\cdot\mathbf{O}\cdot\left[\prod_{j=i+1}^{N-1}(p+\omega^j\boldsymbol{\Lambda})\right]\cdot\tilde{\boldsymbol{u}}_k.\end{aligned} \tag{2.4b}$$

The $\boldsymbol{\Lambda}$ and ${}^{\mathrm{t}}\boldsymbol{\Lambda}$ in equation (2.4b) are index-raising operators, acting on the left of $\boldsymbol{u}_k$ and $\mathbf{U}$ by $(\boldsymbol{\Lambda}\cdot\boldsymbol{u}_k)^{(i)} = u_k^{(i+1)}$ and $(\boldsymbol{\Lambda}\cdot\mathbf{U})^{(i,j)} = u^{(i+1,j)}$ respectively, and on the right by $(\mathbf{U}\cdot{}^{\mathrm{t}}\boldsymbol{\Lambda})^{(i,j)} = u^{(i,j+1)}$, and $\mathbf{O}$ is a projection matrix singling out the central (zero) component in the infinite-component description, i.e. $(\mathbf{O}\cdot\boldsymbol{u}_k)^{(i)} = u_k^{(0)}\delta_{i,0}$. From (2.1) one can also derive an algebraic equation given by

$$k^N\boldsymbol{u}_k = \boldsymbol{\Lambda}^N\cdot\boldsymbol{u}_k - \sum_{j=0}^{N-1}\mathbf{U}\cdot(\omega\,{}^{\mathrm{t}}\boldsymbol{\Lambda})^j\cdot\mathbf{O}\cdot\boldsymbol{\Lambda}^{N-1-j}\cdot\boldsymbol{u}_k. \tag{2.5}$$

Integrating equations (2.4a), (2.4b) over the contour C, using (2.3), one immediately obtains a coupled system for the infinite-component matrix $\mathbf{U}$, namely

$$\tilde{\mathbf{U}}\cdot(p+\omega\,{}^{\mathrm{t}}\boldsymbol{\Lambda}) = (p+\boldsymbol{\Lambda})\cdot\mathbf{U} - \tilde{\mathbf{U}}\cdot\mathbf{O}\cdot\mathbf{U} \tag{2.6a}$$

$$\begin{aligned}\mathbf{U}\cdot\left[\prod_{j=2}^{N}(p+\omega^j\,{}^{\mathrm{t}}\boldsymbol{\Lambda})\right] = {} & \left[\prod_{j=1}^{N-1}(p+\omega^j\boldsymbol{\Lambda})\right]\cdot\tilde{\mathbf{U}} \\ & -\sum_{i=1}^{N-1}\omega^i\,\mathbf{U}\cdot\left[\prod_{j=2}^{i}(p+\omega^j\,{}^{\mathrm{t}}\boldsymbol{\Lambda})\right]\cdot\mathbf{O}\cdot\left[\prod_{j=i+1}^{N-1}(p+\omega^j\boldsymbol{\Lambda})\right]\cdot\tilde{\mathbf{U}}\end{aligned} \tag{2.6b}$$

and in addition the algebraic equation

$$\mathsf{U}\cdot{}^{t}\Lambda^{N}=\Lambda^{N}\cdot\mathsf{U}-\sum_{j=0}^{N-1}\mathsf{U}\cdot(\omega\,{}^{t}\Lambda)^{j}\cdot\mathsf{O}\cdot\Lambda^{N-1-j}\cdot\mathsf{U}. \tag{2.7}$$

Equations (2.4)–(2.7) have been derived from (2.1) by using the transformation (2.2). Using a second transformation similar to (2.2) but with p replaced by q and $\tilde{}$ by $\hat{}$, one obtains a second set of linear relations involving infinite-component vectors $\hat{u}_k$ instead of $\tilde{u}_k$ and an infinite-component potential matrix $\hat{\mathsf{U}}$ instead of $\tilde{\mathsf{U}}$. Using these two transformations one can eliminate the index-raising matrices Λ and ${}^t\Lambda$ from the equations for U to find a coupled set of nonlinear equations involving a finite number of components $u^{(i,j)}$, with $i=0$, $j=0,1,\ldots,N-1$ and $j=0$, $i=0,1,\ldots,N-1$. Equivalently one can work out the compatibility conditions for the set of linear relations for $\tilde{u}_k$ and $\hat{u}_k$ to find the same set of relations for the potentials $u^{(i,j)}$. Thus we obtain a coupled system of partial difference equations, after reinterpreting the transformed variables as follows, this system can be interpreted as a coupled system of partial difference equations by making the identifications

$$u^{(i,j)}=u^{(i,j)}_{n,m}\qquad \tilde{u}^{(i,j)}=u^{(i,j)}_{n+1,m}\qquad \hat{u}^{(i,j)}=u^{(i,j)}_{n,m+1}\qquad \hat{\tilde{u}}^{(i,j)}=u^{(i,j)}_{n+1,m+1} \tag{2.8}$$

in which the potentials are assumed to live on a two-dimensional lattice whose sites are labelled by (n,m), $n,m\in\mathbb{Z}$, and the lattice translations in the n- and m-direction correspond to the transformations $u\mapsto\tilde{u}$ and $u\mapsto\hat{u}$ resepectively. The parameters p and q play the role of the lattice parameters measuring the lattice width between neighbouring sites in the n- and m-direction respectively.

From the linear relations (2.4), (2.5) one can derive a Zakharov–Shabat type of linear problem for this system of partial difference equations, namely

$$(p+\omega k)\tilde{\varphi}_k=\mathcal{L}_k\cdot\varphi_k\qquad (q+\omega k)\hat{\varphi}_k=\mathcal{M}_k\cdot\varphi_k \tag{2.9}$$

in which

$$\varphi_k=\begin{pmatrix}u_k^{(0)}\\ u_k^{(1)}\\ \vdots\\ u_k^{(N-1)}\end{pmatrix}$$
$$\mathcal{L}_k=\begin{pmatrix}p-\tilde{u}^{(0,0)} & 1 & & & \\ -\tilde{u}^{(1,0)} & p & 1 & & \\ \vdots & & \ddots & \ddots & \\ -\tilde{u}^{(N-2,0)} & 0 & \cdots & p & 1\\ k^N+* & \omega^{N-2}u^{(0,N-2)} & \cdots & \omega u^{(0,1)} & p+u^{(0,0)}\end{pmatrix} \tag{2.10}$$

and where the matrix $\mathcal{M}_k$ is a similar matrix obtained after the replacements $p\mapsto q$ and $\tilde{}\mapsto\hat{}$. The term $*\equiv\omega^{N-1}u^{(0,N-1)}-\tilde{u}^{(N-1,0)}$ in the left-lower corner of the matrix $\mathcal{L}_k$ is such that the determinant $\det(\mathcal{L}_k) = p^N - (-k)^N$, as may be checked from equations (2.9), (2.10), i.e. we have the expression

$$* = \sum_{j=0}^{N-2} (-p)^{N-1-j} \left(\tilde{u}^{(j,0)} - \omega^j u^{(0,j)}\right) - \sum_{j=1}^{N-1} \sum_{i=0}^{N-1-j} (-p)^{N-1-j-i} \omega^i u^{(0,i)} \tilde{u}^{(j-1,0)}. \tag{2.11}$$

For later use, we need a factorization of the Lax matrix (2.10) which is as follows

$$\mathcal{L}_k = \mathbf{A}(p)\, \Lambda(\lambda_p)\, \tilde{\mathbf{B}}(p) \qquad (\lambda_p \equiv k^N - (-p)^N) \tag{2.12}$$

in which $\Lambda(\lambda_p)$ depends on the spectral parameter

$$\Lambda(\lambda) \equiv \lambda \mathbf{E}_{N,1} + \sum_{i=1}^{N-1} \mathbf{E}_{i,i+1} \tag{2.13}$$

where the matrices $\mathbf{E}_{i,j}$ are the standard generators of GL_N, i.e. $(\mathbf{E}_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}$. The matrices $\mathbf{A}(p)$ and $\mathbf{B}(p)$ depend on the potentials $u^{(i,j)}$, through

$$\mathbf{A}(p) = 1 + \sum_{j=1}^{N-1} \left(\omega^{j-1} u^{(0,j-1)} \mathbf{E}_{N,N-j} + p \mathbf{E}_{j+1,j}\right) \tag{2.14a}$$

$$\mathbf{B}(p) = 1 - \sum_{j=1}^{N-1} \left((-p)^j + \sum_{i=0}^{j-1} (-p)^{j-1-i} u^{(i,0)}\right) \mathbf{E}_{j+1,1}. \tag{2.14b}$$

In equation (2.12) we have the freedom to multiply the matrix $\mathbf{A}$ from the right by an arbitrary (spectral parameter independent) matrix

$$\mathbf{C} = 1 + \sum_{i>j=1}^{N-1} c_{ij} \mathbf{E}_{ij}$$

and the matrix $\mathbf{B}$ from the left by $\Lambda(\lambda_p)^{-1}\mathbf{C}^{-1}\Lambda(\lambda_p)$, (where the dependence on the spectral parameter drops out).

For $N = 2$, the compatibility of the system (2.9) leads to a lattice version of the (potential) KdV equation, which was studied in a number of papers, cf [6–8]. The compatibility relations of the system (2.9) for $N \geqslant 3$ leads to the coupled system of equations

$$\hat{u}^{(j+1,0)} - \tilde{u}^{(j+1,0)} = (p - q + \hat{u} - \tilde{u})\, \hat{\tilde{u}}^{(j,0)} - p\, \hat{u}^{(j,0)} + q\, \tilde{u}^{(j,0)} \tag{2.15a}$$

$$\omega\left(\hat{u}^{(0,j+1)} - \tilde{u}^{(0,j+1)}\right) = -(p - q + \hat{u} - \tilde{u})\, u^{(0,j)} - q\, \hat{u}^{(0,j)} + p\, \tilde{u}^{(0,j)} \tag{2.15b}$$

$$j = 0, \ldots, N-3$$

together with $u \equiv u^{(0,0)}$

$$
\begin{aligned}
(p - q + \hat{u} - \tilde{u})&\left(\hat{\tilde{u}}^{(N-2,0)} - \omega^{N-2} u^{(0,N-2)}\right) \\
&= (p + q + u)\left[(p - q + \hat{u} - \tilde{u})\hat{\tilde{u}}^{(N-3,0)} - p\hat{u}^{(N-3,0)} + q\tilde{u}^{(N-3,0)}\right] \\
&\quad + \sum_{j=0}^{N-3} \left[(-p)^{N-1-j}\left(\tilde{u}^{(j,0)} - \omega^j u^{(0,j)}\right)\right. \\
&\quad - (-q)^{N-1-j}\left(\hat{u}^{(j,0)} - \omega^j u^{(0,j)}\right) \\
&\quad \left. - \omega^j u^{(0,j)}\left((-p)^{N-2-j}\tilde{u} - (-q)^{N-2-j}\hat{u}\right)\right] \\
&\quad - \sum_{j=2}^{N-2}\sum_{i=0}^{N-1-j} \omega^i u^{(0,i)}\left[(-p)^{N-1-j-i}\tilde{u}^{(j-1,0)} - (-q)^{N-1-j-i}\hat{u}^{(j-1,0)}\right].
\end{aligned}
\tag{2.15c}
$$

Using equations (2.15*a*), (2.15*b*) in combination with (2.15*c*) one can eliminate $u^{(N-2,0)}$ and $u^{(0,N-2)}$, by using the following identity

$$
\begin{aligned}
&\left(\hat{\tilde{u}}^{(N-2,0)} - \omega^{N-2} u^{(0,N-2)}\right)^{\wedge} - \left(\hat{\tilde{u}}^{(N-2,0)} - \omega^{N-2} u^{(0,N-2)}\right)^{\sim} \\
&\quad = \left(\hat{u}^{(N-2,0)} - \tilde{u}^{(N-2,0)}\right)^{\hat{\sim}} - \omega^{N-2}\left(\hat{u}^{(0,N-2)} - \tilde{u}^{(0,N-2)}\right)
\end{aligned}
\tag{2.16}
$$

and find a coupled system of equations in terms of $u^{(j,0)}$, $u^{(0,j)}$, $j = 0, \cdots, N-3$, $(N \geqslant 3)$. We shall refer to this system of equations as the lattice GD hierarchy. In the simplest case $N = 3$ we get a scalar equation in terms of $u_{n,m} \equiv u_{n,m}^{(0,0)}$ which is precisely equation (1.3). It is this equation that we refer to as the lattice (potential) Boussinesq (BSQ) equation, as by appropriate continuum limits we recover the potential BSQ equation from (1.3). The DL for the continuum BSQ equation was studied in [23].

Remark.. A different ZS system which gives rise to equations that we will identify with the lattice modified GD equations (or 'Toda–GD' equations) is given in appendix A, together with the Miura transformation relating them to the systems in the GD hierarchy. In the special case $N = 3$ this leads to an interesting new lattice equation which is the discrete analogue of the modified BSQ (MBSQ) equation, presented first in [23].

3. Mapping reduction

As an application let us now consider some (finite-dimensional) integrable *mappings* associated with the various members of the lattice GD hierarchy. In the spirit of [6] we can construct integrable mappings from any of the lattice equations by considering appropriate periodic initial value problems on the lattice.

Let us now consider initial value problems for the lattice GD hierarchy of equations that were presented in the previous section. One way of doing this is to give initial data on a horizontal line and consider either decaying, cf [25], or periodic boundary conditions, which in either case leads to a non-local scheme. Another way, cf [6],

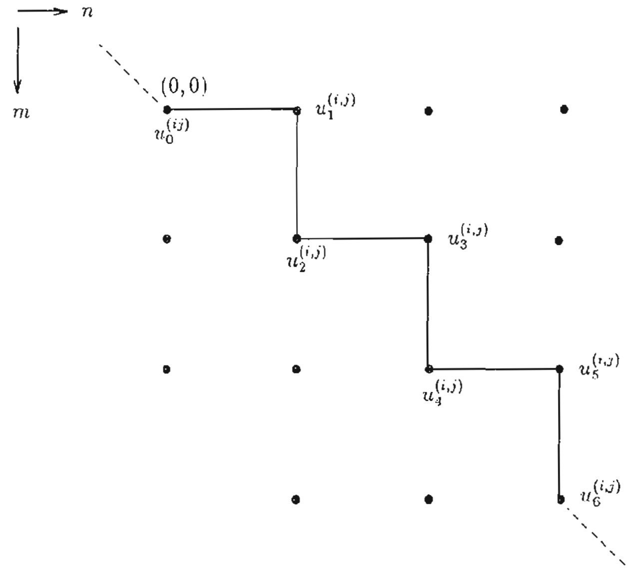

Figure 1. Configuration of initial data on a staircase in the lattice.

which gives rise to a *local* iteration scheme, is to assign initial data on a 'staircase'. By a staircase we mean a sequence of neighbouring lattice sites with m and n non-decreasing, as e.g. illustrated in figure 1. From the fact that the compatibility of the Lax representation (2.9), (2.10) at each site involves only the values of the potentials, $u^{(0,j)}, u^{(i,0)}$, $(i, j = 0, \ldots, N-2)$, at the four lattice sites around a simple plaquette, it follows that the information on these staircases evolves through the lattice along parallel staircases. Furthermore, because of the 'convexity' of the staircase the initial-value problem is well-posed. To be specific, we consider a staircase consisting of P alternating horizontal and vertical steps, associated with periodic initial conditions of the type

$$u^{(i,j)}_{n+P,m+P} = u^{(i,j)}_{n,m}.$$

Starting from such initial conditions it is immediately seen that the same periodicity applies to downward iterates of the same staircase, as a consequence of the form of the equation. Taking such a standard staircase through the origin $(n, m) = (0, 0)$, we assign initial data as follows (cf figure 1)

$$u^{(i,j)}_{n,n} =: u^{(i,j)}_{2n} \qquad u^{(i,j)}_{n+1,n} =: u^{(i,j)}_{2n+1}$$

with the same periodicity property, i.e.

$$u^{(i,j)}_{n} = u^{(i,j)}_{n+2P}$$

and we can use the lattice equations to calculate the 'updated' variables, i.e. the value of the variables $u^{(i,j)}_{2n}$ and $u^{(i,j)}_{2n+1}$ after a one time-step evolution in the m-direction, namely

$$u^{(i,j)\prime}_{2n} = u^{(i,j)}_{n,n+1} \qquad u^{(i,j)\prime}_{2n+1} = u^{(i,j)}_{n+1,n+1}.$$

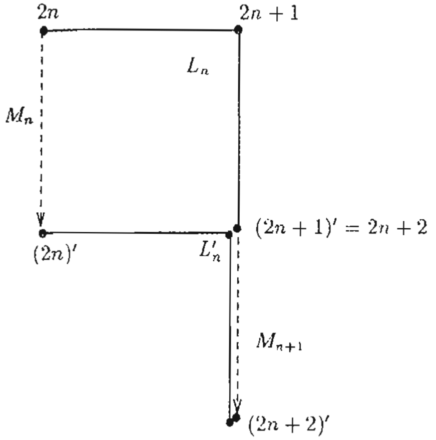

Figure 2. Compatibility condition leading to the ZS system for the mapping.

The factorization of the Lax representation of the lattice, i.e. equation (2.12), together with (2.14), and a similar equation with $p \to q$ and $\tilde{} \to \hat{}$ for $\mathcal{M}_k$, can be shown to lead to a reduction of variables, which will turn out to be the canonical variables of the system. It is in terms of these variables that a mapping reduction is most easily obtained. In fact, considering the configuration of translations on the lattice as indicated in figure 2, we can derive a discrete-time Zakharov–Shabat (ZS) system from the Lax representation of the lattice. Using equation (2.12) we have

$$\begin{aligned}\mathbf{A}_{2n+2}(q)\Lambda_q\mathbf{B}'_{2n+2}(q)\mathbf{A}_{2n+1}(q)\Lambda_q\mathbf{B}_{2n+2}(q)\mathbf{A}_{2n}(p)\Lambda_p\mathbf{B}_{2n+1}(p)\mathbf{A}_{2n-1}(q)\\ = \mathbf{A}'_{2n+1}(q)\Lambda_q\mathbf{B}'_{2n+2}(q)\mathbf{A}'_{2n}(p)\Lambda_p\mathbf{B}'_{2n+1}(p)\mathbf{A}_{2n}(q)\Lambda_q\mathbf{B}'_{2n}(q)\mathbf{A}_{2n-1}(q).\end{aligned} \tag{3.1}$$

This relation, after a recombination of factors, and with the use of $\mathbf{A}'_{2n+1}(q) = \mathbf{A}_{2n+2}(q)$ leads to the ZS system

$$\mathbf{L}'_n(k)\cdot\mathbf{M}_n(k) = \mathbf{M}_{n+1}(k)\cdot\mathbf{L}_n(k) \tag{3.2}$$

with the identifications

$$\mathbf{L}_n = \mathbf{V}_{2n}\cdot\mathbf{V}_{2n-1} \qquad \mathbf{M}_n = \Lambda_{2n}\mathbf{U}_n \tag{3.3}$$

in which $\Lambda_n = \Lambda(\lambda_n)$, introducing also for later use parameters $\lambda_n \equiv \lambda_{p_n}$, $(p_{2n-1} = p, p_{2n} = q)$, and

$$\mathbf{V}_n = \Lambda_n\cdot\mathbf{W}_n \qquad \mathbf{U}_n = \mathbf{B}'_{2n}(q)\mathbf{A}_{2n-1}(q). \tag{3.4}$$

The matrices $\mathbf{W}_n$ which contain the canonical variables of the system, are given by

$$\mathbf{W}_n \equiv \mathbf{B}_{n+2}(p_{n+2})\mathbf{A}_n(p_{n+1}) = \mathbf{1} + \sum_{i>j=1}^{N} v_{i,j}(n)\mathbf{E}_{i,j} \tag{3.5}$$

where

$$v_{N,j}(n) \equiv \omega^{N-j-1} u_n^{(0,N-j-1)} + p_{n+1}\delta_{j,N-1} \tag{3.6a}$$

$$v_{j,1}(n) \equiv -(-p_{n+2})^{j-1} - \sum_{i=0}^{j-2}(-p_{n+2})^{j-2-i} u_{n+2}^{(i,0)} + p_{n+1}\delta_{j,2} \tag{3.6b}$$

for $j = 2, \ldots, N-1$ and

$$v_{N,1}(n) \equiv -(-p_{n+2})^{N-1} + \omega^{N-2} u_n^{(0,N-2)} - \sum_{i=0}^{N-2}(-p_{n+2})^{N-2-i} u_{n+2}^{(i,0)} \tag{3.6c}$$

$$v_{j+1,j} \equiv p_{n+1} \qquad j = 2, \ldots, N-2 \tag{3.6d}$$

$$v_{i,j} \equiv 0 \tag{3.6e}$$

otherwise. As we will show in section 4, it is in these variables v that the Yang–Baxter structure is obtained. For the matrix M_n we have the following expression

$$\mathsf{M}_n = \Lambda_{2n}\left[\mathbf{1} + \sum_{j=2}^{N-1}\left(v'_{j,1}(2n-2)\mathsf{E}_{j,1} + v_{N,j}(2n-1)\mathsf{E}_{N,j}\right) + w(n)\mathsf{E}_{N,1} + q\sum_{j=2}^{N-2}\mathsf{E}_{j+1,j} + (q-p)\mathsf{E}_{2,1}\right] \tag{3.7}$$

where the 'corner' field $w(n)$ is determined by the compatibility conditions of the ZS system, leading to

$$w(n) = v'_{N,1}(2n-2) + qv'_{N,2}(2n-2) - pv_{N,2}(2n-1) + \left(v'_{N,N-1}(2n-2) - v_{N,N-1}(2n-1)\right)\left(v_{N,2}(2n-2) - p\delta_{N,3}\right). \tag{3.8}$$

The ZS system can be considered to be the consistency condition for the following linear system

$$\phi_{n+1} = \mathsf{V}_n \cdot \phi_n \qquad \phi'_{2n-1} = \mathsf{M}_n \cdot \phi_{2n-1} \tag{3.9}$$

where $\phi_n = (\phi_n^{(1)}, \phi_n^{(2)}, \ldots, \phi_n^{(N)})$, and using the identification (3.3). We impose the following commutation relations between the fields $v_{i,j}$

$$\{v_{i,j}(n)\,,\,v_{k,l}(m)\} = \delta_{n,m+1}\delta_{k,j+1}\delta_{i,N}\delta_{l,1} - \delta_{m,n+1}\delta_{i,l+1}\delta_{k,N}\delta_{j,1}. \tag{3.10}$$

As we shall show in section 5, these commutation relations are precisely such that the mapping coming from the ZS system is symplectic. In order to do that we have to work out the compatibility equations from the ZS system (leading to the mapping), i.e. equation (3.2). However, not all the v-variables are relevant: the mapping is most naturally expressed in terms of special combinations of the variables $v_{i,j}$ which we will introduce in section 5. On the other hand, the variables $v_{i,j}$ are the most natural ones in order to establish the integrability of the mappings, as a consequence of a Yang–Baxter structure that we will develop in the next section.

4. Non-ultralocal Yang–Baxter structure

In this section we will focus on the integrability of the mappings constructed in the previous section.

A first step is the construction of invariants of the mapping. This is done by introducing the monodromy matrix

$$\mathsf{T} = \prod_{n=1}^{\stackrel{\curvearrowleft}{P}} \mathsf{L}_n = \mathsf{T}(\mu) \tag{4.1}$$

in which $\curvearrowleft$ indicates that the factors in the product are arranged from the right to the left, and where $\mu \equiv \lambda_{2n} = k^N - (-p_{2n})^N$. Due to the periodicity of the initial data over P steps along the staircase of figure 1, it is immediately established that T transforms as

$$\mathsf{T}' = \mathsf{M}_{P+1} \cdot \mathsf{T} \cdot \mathsf{M}_1^{-1} \tag{4.2}$$

where $\mathsf{M}_{P+1} = \mathsf{M}_1$ due to the periodicity requirement. Clearly, the trace of powers of the monodromy matrix is invariant under the mapping, and by expanding in powers of the spectral parameter k^N we obtain a sufficient number of functionally independent invariants.

A next step is to establish involutivity of the invariants with respect to a properly chosen symplectic structure. We show in this section that the Poisson structure (3.10) leads to a novel Yang–Baxter structure, from which it is manifest that the traces of the monodromy matrix are in involution. The construction of such a Yang–Baxter structure proceeds as follows.

From equation (3.10) we immediately derive the following Poisson brackets between the matrices W_n†

$$\{\mathsf{W}_{n+1} \overset{\otimes}{,} \mathsf{W}_n\} = (\mathsf{W}_{n+1} \otimes 1) s (1 \otimes \mathsf{W}_n) \tag{4.3}$$

where

$$s = \sum_{i=1}^{N-1} \mathsf{E}_{N,i} \otimes \mathsf{E}_{i+1,1} \tag{4.4}$$

and W_n was given in (3.5). The Poisson brackets (4.3) between the matrices W_n at adjacent sites on the staircase are the only non-trivial ones. We now have the following identity

$$(\Lambda(\mu) \otimes \Lambda(\mu'))\, P\, (\Lambda(\mu)^{-1} \otimes \Lambda(\mu')^{-1}) = P + (\mu - \mu')(s^+ - s^-) \tag{4.5}$$

in which

$$P = \sum_{i,j=1}^{N} \mathsf{E}_{i,j} \otimes \mathsf{E}_{j,i} \tag{4.6}$$

† In this section $\otimes$ denotes the matricial tensor product, i.e. $(\mathsf{A} \otimes \mathsf{B})_{ik|jl} = A_{ij} B_{kl}$. The notation $\{\mathsf{A} \overset{\otimes}{,} \mathsf{B}\}$ denotes the matricial tensor product of A and B where we take the Poisson brackets between the corresponding entries in the tensor product, cf e.g. [26].

and

$$s^+ = s^+(\mu,\mu') = (\mathbf{1}\otimes\Lambda(\mu'))\, s\,(\mathbf{1}\otimes\Lambda(\mu')^{-1}) = \frac{1}{\mu'}\sum_{i=1}^{N-1} \mathsf{E}_{N,i}\otimes\mathsf{E}_{i,N} \tag{4.7a}$$

$$s^- = s^-(\mu,\mu') = (\Lambda(\mu)\otimes\mathbf{1})\, s\,(\Lambda(\mu)^{-1}\otimes\mathbf{1}) = \frac{1}{\mu}\sum_{i=1}^{N-1} \mathsf{E}_{i,N}\otimes\mathsf{E}_{N,i}. \tag{4.7b}$$

Using (3.3) and (3.4) we find the fundamental Poisson structure for the matrices L_n

$$\begin{aligned}\{\mathsf{L}_n(\mu) \overset{\otimes}{,} \mathsf{L}_m(\mu')\} = {} & \delta_{n,m+1}(\mathsf{L}_n(\mu)\otimes\mathbf{1})\, s^+\,(\mathbf{1}\otimes\mathsf{L}_m(\mu')) \\ & - \delta_{n+1,m}(\mathbf{1}\otimes\mathsf{L}_m(\mu'))\, s^-\,(\mathsf{L}_n(\mu)\otimes\mathbf{1}) \\ & - \delta_{n,m}\left[r^+\,(\mathsf{L}_n(\mu)\otimes\mathsf{L}_m(\mu')) - (\mathsf{L}_n(\mu)\otimes\mathsf{L}_m(\mu'))\, r^-\right].\end{aligned} \tag{4.8}$$

Equation (4.8) is easily deduced from (4.3), taking

$$\begin{aligned} r^- &= r^-(\mu,\mu') = \frac{P}{\mu-\mu'} \\ r^+ &= r^+(\mu,\mu') = r^-(\mu,\mu') + s^+(\mu,\mu') - s^-(\mu,\mu')\end{aligned} \tag{4.9}$$

and where use has been made of the fact that the the Poisson brackets between the matrices W_n at the same site are trivial.

Equation (4.8) forms a so-called non-ultralocal Poisson structure, cf [26]. In the discrete case fundamental Poisson brackets similar to (4.8) have been presented in e.g. [27, 28]. For integrable time-discrete systems, notably in the case of mappings of KdV type (i.e. the case $N = 2$), they were presented first in [8]. In the continuum case non-ultralocal Poisson brackets have been studied in a number of papers, cf e.g. [29–31]. In fact, equations (4.9) and (4.7) form a new solution of the non-ultralocal version of the classical Yang–Baxter equations (CYBEs) which read

$$[r^\pm_{12}, r^\pm_{13}] + [r^\pm_{12}, r^\pm_{23}] + [r^\pm_{13}, r^\pm_{23}] = 0 \tag{4.10a}$$

together with

$$[s^\pm_{12}, s^\pm_{13}] = [r^\pm_{23}, s^\pm_{12}] + [r^\pm_{23}, s^\pm_{13}]. \tag{4.10b}$$

Equations (4.10a), (4.10b) ensure that the Jacobi identities for the Poisson bracket (4.8) are satisfied. To ensure the skew-symmetry of the bracket (4.8), we have the relation

$$s^-(\mu,\mu') = P s^+(\mu',\mu) P \tag{4.11}$$

whereas both $r^\pm$ are anti-symmetric

$$r^\pm(\mu,\mu') = -P r^\pm(\mu',\mu) P. \tag{4.12}$$

The cancellation mechanism when calculating the Poisson brackets between the monodromy matrices (4.1) makes use of the relation (4.9) between the $r^\pm$ and the

$s^{\pm}$, from which it follows that the terms coming from the shifted Kronecker-deltas in (4.8) cancel against the s-terms coming from the difference between r^+ and r^-, leading to the result

$$\{\mathsf{T}(\mu) \overset{\otimes}{,} \mathsf{T}(\mu')\} = -r^+ \left(\mathsf{T}(\mu) \otimes \mathsf{T}(\mu')\right) + \left(\mathsf{T}(\mu) \otimes \mathsf{T}(\mu')\right) r^- \\ + \left(\mathsf{T}(\mu) \otimes 1\right) s^+ \left(1 \otimes \mathsf{T}(\mu')\right) - \left(1 \otimes \mathsf{T}(\mu')\right) s^- \left(\mathsf{T}(\mu) \otimes 1\right). \tag{4.13}$$

As a consequence of (4.13), taking the trace over both components of the tensor product and using (4.9), the right-hand side of (4.13) vanishes identically, and we find for all $n, m \in \mathbb{N}$

$$\{\mathrm{Tr}(\mathsf{T}(\mu)^n), \mathrm{Tr}(\mathsf{T}(\mu')^m)\} = 0 \tag{4.14}$$

implying that the integrals of the mapping are in involution. For involutivity of the invariants, it is clearly sufficient to prove that the Poisson bracket between the trace of the monodromy matrix (4.1) for different values of μ, μ and μ' say, vanishes. Thus, we establish integrability in the sense of Arnol'd–Darboux–Liouville, applying the general result of [10].

We mention that the classical Poisson structure given above can be deformed into a non-ultralocal quantum Yang–Baxter structure, leading to an appropriate quantization scheme of the mappings defined in the previous section, cf [32], cf also [33] for the case of mappings of KdV type. Similar quantum structures were investigated in [28] in the case of the Toda lattice.

5. Big Lax and max

In order to obtain the final reduction of the mapping it is convenient to work with a representation in terms of a 'big' Lax pair, i.e. involving matrices of the size of the number of steps over a one-period staircase. Thus, we introduce $2P \times 2P$ matrices L and M, for the various members ($N = 2, 3, \ldots$) of the GD hierarchy

$$\mathsf{L} = -\Sigma_\eta^N + \sum_{\alpha=1}^{N-1} \Sigma_\eta^\alpha \mathsf{X}^{(\alpha)} + \Delta \qquad \mathsf{M} = \Sigma_\eta + \mathsf{D}. \tag{5.1}$$

Equation (5.1) is expressed in terms of powers α, ($\alpha = 1, \ldots, N$), of the $2P \times 2P$ matrix Σ_η, which is the analogue of the ($N \times N$) matrix $\Lambda(\lambda)$, i.e.

$$\Sigma_\eta = \begin{pmatrix} 0 & 1 & & & \\ \vdots & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & 0 & 1 \\ \eta & \cdots & \cdots & \cdots & 0 \end{pmatrix} \tag{5.2}$$

and the matrices $\mathsf{X}^{(\alpha)}$, D and Δ are diagonal matrices with

$$\begin{aligned} &\mathsf{X}^{(\alpha)} = \mathrm{diag}(X_1^{(\alpha)}, \ldots, X_{2P}^{(\alpha)}) && (\alpha = 1, \ldots, N-1) \\ &\Delta = \mathrm{diag}(\lambda_1 - \lambda_2, 0, \lambda_3 - \lambda_4, 0, \ldots, \lambda_{2P-1} - \lambda_{2P}, 0) && \lambda_{2n} = \lambda_{2n+2} \\ &\mathsf{D} = \mathrm{diag}(d_1, 0, d_3, 0, \ldots, d_{2P-1}, 0). \end{aligned} \tag{5.3}$$

The usefulness of the big Lax representation in connection with the mappings corresponding to the various members of the lattice GD hierarchy, that were derived in section 3, stems from the following result.

Proposition 5.1. The ZS system (3.2) for L_n and M_n given by equation (3.3) and (3.7) is equivalent to the Lax equation

$$\mathsf{L}' = \mathsf{M} \cdot \mathsf{L} \cdot \mathsf{M}^{-1} \tag{5.4}$$

in which L and M are given by (5.1) together with (5.3). Equation (5.4) is the consistency condition for the following linear problem

$$\mathsf{L} \cdot \Phi + \mu\Phi = 0 \qquad \Phi' = \mathsf{M} \cdot \Phi. \tag{5.5}$$

Furthermore, (5.4) gives rise to the following $(N-1) \times 2P$-dimensional rational mapping

$$\begin{aligned} &X_{2n-1}^{(\alpha)\prime} = X_{2n}^{(\alpha)} + d_{2n-1-\alpha} X_{2n}^{(\alpha+1)} \\ &X_{2n}^{(\alpha)\prime} + X_{2n+1}^{(\alpha+1)\prime} d_{2n+1} = X_{2n+1}^{(\alpha)} + d_{2n-\alpha} X_{2n+1}^{(\alpha+1)} \qquad (\alpha = 1, \ldots, N-2) \\ &X_{2n-1}^{(N-1)\prime} = X_{2n}^{(N-1)} - d_{2n-N} \\ &X_{2n}^{(N-1)\prime} = X_{2n+1}^{(N-1)} + d_{2n+1} - d_{2n+1-N} \end{aligned} \tag{5.6}$$

in which

$$d_{2n-1} = \frac{\lambda_{2n-1} - \lambda_{2n}}{X_{2n}^{(1)}} \qquad d_{2n} = 0 \qquad n = 1, \ldots, P. \tag{5.7}$$

In the proof of this proposition use will be made of the following lemma.

Lemma. There exists a gauge transformation between the Zakharov–Shabat system (3.2) and a similar system containing only the reduced variables $X_n^{(\alpha)}$ of equation (5.1).

Proof. The gauge transformation simplifies the matrices W_n of equation (3.5) with variables $v_{i,j}(n)$ in the lower triangular part to matrices with variables $X_n^{(\alpha)}$ ocurring only in the first column. This is achieved by the following gauge transformation

$$\mathsf{V}_n = \Gamma_n \cdot \bar{\mathsf{V}}_n \cdot \Gamma_{n-1}^{-1} \tag{5.8}$$

in which

$$\bar{\mathsf{V}}_n = \Lambda_n \left(1 + \sum_{\alpha=1}^{N-1} X_n^{(\alpha)} \mathsf{E}_{N-\alpha+1,1} \right) \tag{5.9a}$$

$$\Gamma_n = \prod_{\alpha=1}^{\overset{\curvearrowleft}{N-2}} \Gamma_{n+1-\alpha}^{(\alpha)} \tag{5.9b}$$

$$\Gamma_n^{(\alpha)} = 1 + \sum_{i>j=1}^{N-\alpha} v_{i+\alpha,j+\alpha}(n) \mathsf{E}_{i,j}. \tag{5.9c}$$

Furthermore, we have

$$\mathbf{M}_n = \mathbf{\Gamma}'_{2n-2}\,\bar{\mathbf{M}}_n\,\mathbf{\Gamma}^{-1}_{2n-2} = \Lambda_{2n}\left(\mathbf{1} + \sum_{i>j=1}^{N} \bar{u}_{i,j}\,\mathbf{E}_{i,j}\right) \tag{5.10}$$

and the gauge-transformed ZS equations, i.e.

$$\bar{\mathbf{L}}'_n(k)\cdot\bar{\mathbf{M}}_n(k) = \bar{\mathbf{M}}_{n+1}(k)\cdot\bar{\mathbf{L}}_n(k) \tag{5.11}$$

yields a system of equations only in terms of the entries $X_n^{(\alpha)}$ in the matrix $\bar{\mathbf{V}}_n$, identifying

$$\bar{\mathbf{L}}_n = \bar{\mathbf{V}}_{2n}\cdot\bar{\mathbf{V}}_{2n-1} \tag{5.12}$$

analogous to equation (3.3). The proof of the identity (5.8) in the lemma is based on a succession of ordered n-dependent Gaussian eliminations, and we refer to appendix B for the details.

The proof of proposition 5.1 proceeds via a number of steps, which we will briefly summarize here.

1. In order to prove the first part of proposition 5.1 we start from the relation (3.9) after the gauge-transformation (5.8), i.e.

$$\bar{\phi}_{n+1} = \bar{\mathbf{V}}_n\bar{\phi}_n \qquad \bar{\phi}'_{2n-1} = \bar{\mathbf{M}}_n\bar{\phi}_{2n-1} \tag{5.13}$$

with

$$\phi_n = \mathbf{\Gamma}_{n-1}\bar{\phi}_n\,. \tag{5.14}$$

Note that $\mathbf{\Gamma}_n$ in equation (5.9*b*),(5.9*c*) is such that

$$\bar{\phi}_n^{(1)} = \phi_n^{(1)} \tag{5.15}$$

From equation (5.13) with the special form of $\bar{\mathbf{V}}_n$ as given in equation (5.9*a*) we have

$$\begin{aligned}\bar{\phi}_{n+1}^{(\alpha)} &= X_n^{(N-\alpha)}\bar{\phi}_n^{(1)} + \bar{\phi}_n^{(\alpha+1)} \qquad (\alpha = 1,\ldots,N-1)\\ \bar{\phi}_{n+1}^{(N)} &= \lambda_n\bar{\phi}_n^{(1)}\end{aligned} \tag{5.16}$$

From equation (5.16) we immediately obtain that

$$\lambda_n\phi_n^{(1)} + \sum_{\alpha=1}^{N-1} X_{n+\alpha}^{(\alpha)}\phi_{n+\alpha}^{(1)} - \phi_{n+N}^{(1)} = 0 \tag{5.17}$$

Equation (5.17) is equivalent to the $\mathbf{L}$-part of the big Lax representation. In fact, introducing the $2P$-dimensional vector

$$\Phi = (\phi_1^{(1)},\ldots,\phi_{2P}^{(1)}) \tag{5.18}$$

together with the ansatz

$$\phi^{(1)}_{n+2P} = \eta\phi^{(1)}_{n} \tag{5.19}$$

we have $\mathbf{L}\Phi = -\mu\Phi$ with $\mathbf{L}$ given by (5.1).

2. To derive the $\mathbf{M}$-part we first note with the explicit form of $\bar{\mathbf{M}}_n$ in equation (5.10) that

$$\begin{aligned}\phi^{(1)\prime}_{2n-1} &= \bar{u}_{2,1}\phi^{(1)}_{2n-1} + \ddot{\phi}^{(2)}_{2n-1} \\ &= (\bar{u}_{2,1}(n) - X^{(N-1)}_{2n-1})\phi^{(1)}_{2n-1} + \phi^{(1)}_{2n}\end{aligned} \tag{5.20}$$

In the same way, we find with the use of equation (5.16) for $\alpha = 1,2$ and equation (5.13)

$$\begin{aligned}\phi^{(1)\prime}_{2n} &= X^{(N-1)\prime}_{2n-1}\phi^{(1)\prime}_{2n-1} + \bar{\phi}^{(2)\prime}_{2n-1} \\ &= \phi^{(1)}_{2n+1} + (X^{(N-1)\prime}_{2n-1} + \bar{u}_{3,2}(n) - X^{(N-1)}_{2n})\phi^{(1)}_{2n} + (\bar{u}_{3,1}(n) - \bar{u}_{3,2}(n)X^{(N-1)}_{2n-1} \\ &\qquad - X^{(N-2)}_{2n-1} + X^{(N-1)\prime}_{2n-1}(\bar{u}_{2,1}(n) - X^{(N-1)}_{2n-1}))\phi^{(1)}_{2n-1}\end{aligned} \tag{5.21}$$

with the form

$$\begin{aligned}\bar{\mathbf{L}}_n = \sum_{i=1}^{N-2} \mathbf{E}_{i,i+2} + \lambda_{2n+1}\mathbf{E}_{N-1,1} + \lambda_{2n}\mathbf{E}_{N,2} + \lambda_{2n}X^{(N-1)}_{2n-1}\mathbf{E}_{N,1} \\ + \sum_{i=1}^{N-2} X^{(N-i-1)}_{2n-1}\mathbf{E}_{i,1} + \sum_{i=1}^{N-1} X^{(N-i)}_{2n}(\mathbf{E}_{i,2} + X^{(N-1)}_{2n-1}\mathbf{E}_{i,1})\end{aligned} \tag{5.22}$$

it is easy to work out the $(N,3)$-, $(N,2)$- and $(N,1)$-elements of the compatibility relation (5.11) yielding

$$\begin{aligned}&\lambda_{2n+2} = \lambda_{2n} \equiv \mu \\ &X^{(N-1)}_{2n} = X^{(N-1)\prime}_{2n-1} + \bar{u}_{3,2}(n) \\ &X^{(N-2)}_{2n-1} + X^{(N-1)}_{2n}X^{(N-1)}_{2n-1} = X^{(N-1)\prime}_{2n-1}\bar{u}_{2,1}(n) + \bar{u}_{3,1}(n)\end{aligned} \tag{5.23}$$

Inserting (5.23) into (5.21) yields

$$\phi^{(1)\prime}_{2n} = \phi^{(1)}_{2n+1} \tag{5.24}$$

which in combination with (5.20) gives the $\mathbf{M}$-part of the big Lax representation.

3. The mapping (5.6) in terms of the $X^{(\alpha)}_n$, $(\alpha = 1, 2, \ldots, N-1,\ n = 1, \ldots, 2P)$ together with (5.7) can be found from equation (5.4) together with (5.1). The same mapping together with the entries of the matrix $\bar{\mathbf{M}}_n$ can be found from the

ZS condition (5.11) in combination with (5.10) and (5.22). The entries $\bar{u}_{i,j}(n)$ of (5.10) are given by

$$\bar{u}_{i+1,1}(n) = X_{2n-2}^{(N-i)\prime} \qquad (i = 1,\ldots,N-1) \tag{5.25a}$$

$$\bar{u}_{N-p+1,N-p}(n) = \frac{\lambda_{2n-p-2}-\mu}{X_{2n-p-1}^{(1)}} \qquad \begin{cases} p = 1,3,\ldots N-3 & \text{if } N \text{ is even} \\ p = 1,3,\ldots,N-2 & \text{if } N \text{ is odd} \end{cases} \tag{5.25b}$$

$$\bar{u}_{2,1}(n) = X_{2n-1}^{(N-1)} + \frac{\lambda_{2n-1}-\mu}{X_{2n}^{(1)}} \tag{5.25c}$$

$$\bar{u}_{i,j}(n) = 0 \qquad \text{otherwise.} \tag{5.25d}$$

With this we have proven proposition 5.1. A next step is to establish the canonical nature of the mapping, which is given by the following:

Proposition 5.2. The mapping (5.6) is symplectic with respect to the following Poisson bracket structure

$$\begin{aligned} \{X_n^{(\alpha)}, X_m^{(\beta)}\} &= 0 & \alpha+\beta > N \\ \{X_n^{(\alpha)}, X_m^{(\beta)}\} &= \delta_{n,m+\alpha} - \delta_{m,n+\beta} & \alpha+\beta = N \\ \{X_n^{(\alpha)}, X_m^{(\beta)}\} &= \delta_{m,n+\beta} X_m^{(\alpha+\beta)} - \delta_{n,m+\alpha} X_n^{(\alpha+\beta)} & \alpha+\beta < N \end{aligned} \tag{5.26}$$

in which

$$\delta_{n,m} = \begin{cases} 1 & n = m \pmod{2P} \\ 0 & \text{otherwise} \end{cases} .$$

The proof of proposition 4.2 consists of two steps. Firstly, one should verify the Poisson brackets (5.26) on the basis of the explicit relation between the variables $X_n^{(\alpha)}$ and the variables $v_{i,j}(n)$ of the previous section. Secondly, one needs to establish that the mapping (5.6) leaves the Poisson brackets (5.26) invariant. The details of this can be found in appendix C.

As a consequence of the gauge transformation (5.9) the traces of powers of the monodromy matrix are expressed in terms of the variables $X_n^{(\alpha)}$, leading to invariants by expansion in the spectral parameter μ. Alternatively, using the identity†

$$\det_{2P\times 2P}(\mathbf{L} + \mu\mathbf{1}) = \det_{N\times N}(\mathbf{T} - \eta\mathbf{1}) \tag{5.27}$$

we can find invariants from the characteristic polynomial of the big Lax matrix. We shall defer the problem of the actual counting of independent invariants of the mapping to a future study. The investigation of the lower-dimensional examples indicate that this is actually the case. Since these invariants are involution according to equation (4.14) the mapping (5.6) is then completely integrable in the sense of

† In fact, both sides of (5.27) are polynomials of order N in the parameter η, and due to the gauge transformation (taking into account equations (5.15) and (5.19)), we have that the condition for the left-hand side to vanish is the same as the one under which the right-hand side vanishes.

Arnol'd–Liouville, cf [10]. In principle they can be used to integrate the discrete-time dynamics. The explicit construction of the finite-gap solutions, cf e.g. [34], is under current investigation.

Let us finish this section by working out an explicit example of the above construction, namely the mappings of 'least period' in the GD hierarchy. For mappings coming from the lattice KdV equation the case of period $P = 1$ leads to trivial results†. For the other members of the GD hierarchy this is no longer the case. The 'big' Lax matrix for $P = 1$ is actually smaller than the corresponding 'small' ZS matrix, and is for $N = 3$ given by

$$\mathsf{L} = \begin{pmatrix} \eta X_1 + \lambda_1 & -\eta + Y_2 \\ -\eta^2 + \eta Y_1 & \eta X_2 + \lambda_2 \end{pmatrix} \qquad \mathsf{M} = \begin{pmatrix} d_1 & 1 \\ \eta & 0 \end{pmatrix} \tag{5.28}$$

whose determinant leads to the following Casimirs

$$C_1 = X_1 + X_2 \qquad C_2 = Y_1 + Y_2 + X_2 X_1 \tag{5.29}$$

and to the following invariant

$$I = (\lambda_1 - \lambda_2) X_1 + Y_2 Y_1 . \tag{5.30}$$

Eliminating X_2 and Y_2 from the resulting mapping, using the Casimirs C_1 and C_2 we arrive at the following 2-dimensional rational mapping of the plane

$$X' = C_1 - X + \frac{\lambda_2 - \lambda_1}{Y'} \qquad Y' = C_2 - Y - X(C_1 - X) \tag{5.31}$$

with $X = X_1$ and $Y = Y_1$, carrying the following invariant

$$I = Y(C_2 - Y - C_1 X + X^2) + (\lambda_1 - \lambda_2) X . \tag{5.32}$$

A continuous-time interpolating flow to this mapping is generated by (5.32) together with the Poisson brackets $\{X, Y\} = 1$. For any member of the GD hierarchy with $N > 2$, we can thus give a 'lowest-dimensional' mapping by considering period $P = 1$, and surprisingly enough they give actually the *same* mapping (5.31) for all N. This mapping is a special case of the 18-parameter family of mappings of the plane presented in [9].

6. Conclusions

The big Lax representation is convenient to note the analogy between the GD mappings and the conventional continuous GD hierarchy. In fact, the shift matrix $\mathbf{\Sigma}_\eta$ can be viewed as the direct analogue of the differential operator ∂ that occurs in the Nth order linear differential system that is at the basis of the GD hierarchy, [15–17]. The big Lax representation (5.1) is the discrete analogue of that system. Of course we could have started right away by posing (5.1) as the starting point of our investigation.

† The first non-trivial mapping in the KdV-case $N = 2$ occurs for period $P = 2$, leading to two-dimensional rational mapping which is actually the McMillan map, [35]. The KdV situation was extensively covered in [6, 7], and we will not go into that case explicitly here.

We have not chosen to do so, because we wanted to make clear the relations between this discrete hierarchy of mappings and the original lattice equations that are derived from the DLT method. As we have pointed out in this paper these relations are far from trivial, and the lattice discretization opens up some new ways of thought on this type of systems. First of all, it is at the intermediate level of the variables $v_{i,j}(n)$ that the r-matrix structure is most easily expressed, and in terms of which involutivity of invariants is established. It is at this level also that the quantization of the lattice GD hierarchy can be accomplished, leading to a hierarchy of exactly integrable quantum mappings [32], cf also [33].

It is at the level of the big Lax representation that the connection with the work of Moser and Veselov [13] can be most easily established. Considering the Lax representation (5.1) from the point of view of factorization problems and QR-type of mechanisms, cf also [12], we have in the discrete GD hierarchy a situation completely analogous to the continuous case, where Darboux type of factorizations of the Nth-order differential system leads to the introduction of Miura transformations, [36–38]. In the present case this would lead to the problem of finding a factorization of the form

$$\mathbf{L} = -\prod_{\alpha=1}^{\stackrel{\curvearrowleft}{N}} (\Sigma_\eta + \mathbf{D}^{(\alpha)}) \tag{6.1}$$

in which the $\mathbf{D}^{(\alpha)}$ are diagonal matrices, for the big Lax matrix (5.1). Equation (6.1) leads to expressions of the variables $X_n^{(\alpha)}$ in terms of the entries $d_n^{(\alpha)}$ of the matrices $\mathbf{D}^{(\alpha)}$, which can be interpreted as a system of Miura transformations between the GD mappings (5.6) and a system of mappings in terms of the variables $d_n^{(\alpha)}$. In fact, identifying the matrix $\mathbf{D}^{(1)}$ with the matrix $\mathbf{D}$ in the M-part of the Lax pair, we obtain the mapping by a cyclic permutation of factors†, leading to

$$\mathbf{L}' = -\prod_{\alpha=2}^{\stackrel{\curvearrowleft}{N+1}} (\Sigma_\eta + \mathbf{D}^{(\alpha)}) = -\prod_{\alpha=1}^{\stackrel{\curvearrowleft}{N}} (\Sigma_\eta + \mathbf{D}^{(\alpha)\prime}) \tag{6.2}$$

posing the periodicity condition $\mathbf{D}^{(N+\alpha)} = \mathbf{D}^{(\alpha)}$. In general the inversion of the relations between the variables $X_n^{(\alpha)}$ and the variables $d_n^{(\alpha)}$ is non-unique, so one can expect that the mapping resulting from (6.2) is a *correspondence* rather than a single-valued discrete-time flow, cf also [13]. It is of interest to investigate these Miura relations between the $X_n^{(\alpha)}$ and $d_n^{(\alpha)}$ variables more closely, in particular in connection with the possible occurrence of a bi-hamiltonian structures and the emergence of discrete W-algebras, cf also [39], in these systems. Furthermore, it would be of interest to extend the results on mappings to mappings coming from lattice equations in $2+1$ dimensions, notably the lattice KP equation, cf [40]. These questions are under current investigation.

Appendix A

In this appendix we present some results on a modified GD hierarchy which is gauge-related to the hierarchy given in section 2. From equations (2.4a), (2.4b) together

† Note that in the QR-algorithm we have the special case of the interchangement of factors.

with (2.6*a*), (2.6*b*), one derives the following Lax system

$$(p+\omega k)\tilde{\psi}_k = \mathsf{L}_k \cdot \psi_k \qquad (q+\omega k)\hat{\psi}_k = \mathsf{M}_k \cdot \psi_k \tag{A.1}$$

in which

$$\psi_k = \begin{pmatrix} u_k^{[r]} \\ u_k^{(0)} \\ u_k^{(1)} \\ \vdots \\ u_k^{(N-2)} \end{pmatrix}$$

$$\mathsf{L}_k = \begin{pmatrix} p-r & \tilde{v} & & & & \\ 0 & p-\tilde{u}^{0,0} & 1 & & & \\ 0 & -\tilde{u}^{1,0} & p & 1 & & \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & -\tilde{u}^{N-3,0} & 0 & \cdots & p & 1 \\ (k^N-(-r)^N)/v & * & -s^{(N-2)} & \cdots & -s^{(2)} & p-s^{(1)} \end{pmatrix} \tag{A.2}$$

and similarly for M_k replacing $\tilde{}$ by $\hat{}$ and p by q. In (A.2) $*$ is determined by the condition that $\det(L_k) = (p^N-(-k)^N)\tilde{v}/v$. The fields $s^{(j)}$ in (A.2) are given by

$$s^{(j)} = \frac{v^{(j)}}{v} \qquad v^{(j)} \equiv (-r)^j - \omega^j\left((r+\Lambda)^{-1}\cdot \mathsf{U}\cdot {}^t\Lambda^j\right)^{(0,0)}$$

and $v \equiv v^{(0)}$. There is a gauge transformation connecting the two Lax systems, namely

$$\psi_k = \mathsf{U}_k \cdot \phi_k \qquad \mathsf{U}_k = \mathsf{K}^{-1}\cdot \begin{pmatrix} v^{(N-1)} & v^{(N-2)} & \cdots & v^{(1)} & v^{(0)} \\ 1 & 0 & \cdots & 0 & 0 \\ & 1 & \ddots & \vdots & \vdots \\ & & \ddots & 0 & 0 \\ & & & 1 & 0 \end{pmatrix} \tag{A.3}$$

in which K is a diagonal matrix

$$\mathsf{K} = \mathrm{diag}\left(k^N-(-r)^N, 1, 1, \cdots, 1\right).$$

For general N the compatibility equations for the system containing (A.2) and a similar linear problem with p replaced by q and $\tilde{}$ by $\hat{}$ leads to a coupled system of equations referred to as the 'modified lattice GD hiertarchy'. We will not give any specifics here on this system of equations for general N. However, again in the special case of $N=3$ one finds a closed scalar equation which one could consider to be a mixed Toda–MBSQ equation. In fact, the compatibility relations of the Lax pair (A.2) are the following in that case

$$p-q+\hat{u}-\tilde{u} = (p-r)\frac{\hat{v}}{\hat{\tilde{v}}} - (q-r)\frac{\tilde{v}}{\hat{\tilde{v}}} \tag{A.4a}$$

$$p-q-\hat{s}+\tilde{s} = (p-r)\frac{v}{\tilde{v}} - (q-r)\frac{v}{\hat{v}} \tag{A.4b}$$

$$(p+q-s-\hat{\tilde{u}})(p-q+\hat{u}-\tilde{u}) = (p^2+pr+r^2)\frac{\tilde{v}}{v} - (q^2+qr+r^2)\frac{\hat{v}}{v} \tag{A.4c}$$

in which $s \equiv s^{(1)}$, leading to the equation

$$\frac{(p^2+pr+r^2)\,\hat{\tilde{v}} - (q^2+qr+r^2)\,\hat{v}}{(p-r)\,\hat{v} - (q-r)\,\hat{\tilde{v}}}\,\frac{\hat{\bar{\tilde{v}}}}{\hat{v}} - \frac{(p^2+pr+r^2)\,\tilde{\bar{v}} - (q^2+qr+r^2)\,\hat{\bar{v}}}{(p-r)\,\hat{\bar{v}} - (q-r)\,\tilde{\bar{v}}}\,\frac{\hat{\bar{\tilde{v}}}}{\bar{v}}$$
$$= (p-r)\left(\frac{v}{\bar{\tilde{v}}} - \frac{\hat{\bar{\tilde{v}}}}{\hat{\bar{\tilde{\bar{v}}}}}\right) - (q-r)\left(\frac{v}{\hat{\bar{v}}} - \frac{\hat{\bar{\tilde{v}}}}{\hat{\bar{\tilde{\bar{v}}}}}\right). \tag{A.5}$$

Equation (A.5) is a new integrable lattice equation that leads in appropriate continuum limits to the modified BSQ equation, cf also [23]. The relations (A.4) can be considered to constitute the Miura relations between the lattice BSQ equation (1.3) and the equation (A.5).

Appendix B

To derive equations (5.8) and (5.9) we perform the following iterated Gaussian elimination procedure

$$\mathbf{V}_n^{(\alpha)} = \mathbf{\Lambda}_n + \sum_{i>j=1}^{N} v_{i,j}^{(\alpha)}(n)\mathbf{E}_{i-1,j} = \mathbf{\Gamma}_n^{(\alpha+1)}\left(\mathbf{\Lambda}_n + \sum_{j=1}^{N-1} y_j^{(\alpha)}(n)\mathbf{E}_{j,1}\right) \tag{B.1}$$

where

$$\mathbf{\Gamma}_n^{(\alpha+1)} \equiv \left(\mathbf{1} + \sum_{i>j=1}^{N-1} v_{i+1,j+1}^{(\alpha)}(n)\mathbf{E}_{i,j}\right) \tag{B.2}$$

leading to the relation

$$y_i^{(\alpha)}(n) + \sum_{j=1}^{i-1} v_{i+1,j+1}^{(\alpha)}(n)y_j^{(\alpha)}(n) = v_{i+1,1}^{(\alpha)}(n) \qquad (i = 1,\ldots,N-1). \tag{B.3}$$

In order to obtain a gauge-type of transformation we need to multiply (B.1) from the right by a shifted matrix of the same form. Thus, taking

$$\mathbf{V}_n^{(\alpha+1)} = \left(\mathbf{\Lambda}_n + \sum_{j=1}^{N-1} y_j^{(\alpha)}(n)\mathbf{E}_{j,1}\right)\mathbf{\Gamma}_{n-1}^{(\alpha+1)}$$
$$= \mathbf{\Lambda}_n + \sum_{j=1}^{N-1} y_j^{(\alpha)}(n)\mathbf{E}_{j,1} + \sum_{i>j=1}^{N-1} v_{i+1,j+1}^{(\alpha)}(n-1)\mathbf{E}_{i-1,j}. \tag{B.4}$$

Comparing this with (B.1) for $\alpha + 1$ we are led to the identifications

$$\begin{cases} v_{i,j}^{(\alpha+1)}(n) = v_{i+1,j+1}^{(\alpha)}(n-1) + \delta_{j,1}y_{i-1}^{(\alpha)}(n) & 1 \leqslant j < i \leqslant N-1 \\ v_{N,j}^{(\alpha+1)}(n) = \delta_{j,1}y_{N-1}^{(\alpha)}(n) & j = 1,\ldots,N \\ v_{i,j}^{(\alpha+1)}(n) = 0 & \text{otherwise} \end{cases} \tag{B.5}$$

where the variables $y_i^{(\alpha)}$, $i = 1, \ldots N-1$ are solved from (B.3). Notice from (B.5) that the entries v of the matrix $\mathbf{V}_n^{(\alpha+1)}$ appear in one row less (counted from the last row) than the entries of the preceding matrix $\mathbf{V}_n^{(\alpha)}$, except for the corresponding entry in the first column. Thus, by successive applications of the procedure we can 'empty' the matrix $\mathbf{V}_n$ in $N-2$ steps, except for the first column. In fact, starting from $\mathbf{V}_n^{(0)} = \mathbf{V}_n$, and applying successive gauge transformations with matrices $\mathbf{\Gamma}_n^{(\alpha)}$, $\alpha = 1, \ldots, N-2$, we have in this case

$$v_{i,j}^{(\alpha)}(n) = v_{i+\alpha,j+\alpha}(n-\alpha) \qquad \text{for } 2 \leqslant j < i \leqslant N-\alpha$$

$$v_{i,j}^{(\alpha)} = 0 \qquad \text{for } j \neq 1, i = N-\alpha+1, \ldots, N$$

in which case (B.2) reduces to (5.9*c*). Thus we are led to equation (5.8) together with (5.9). The resulting entries $y_i^{(N-2)}(n) \equiv X_n^{(i)}$ are obtained by solving (B.3) in the successive steps.

Appendix C

To derive the commutation relations (5.26) from equations (3.10) for the v-variables, we start from the Lax representation (3.9), and we use the following convenient operator notation. Introducing a shift-operator σ, acting by $(\sigma\phi)_n = \phi_{n+1}$, and omitting the subscript n, we have

$$\phi^{(\alpha+1)} = (\sigma - v_{\alpha+1,\alpha})\phi^{(\alpha)} - \sum_{j=1}^{\alpha-1} v_{\alpha+1,j}\,\phi^{(j)} \qquad \text{(C.1)}$$

for $\alpha = 1, \ldots, N-1$. Equation (C.1) gives

$$\phi^{(\alpha+1)} = \overset{\frown}{\det}\left(\sigma\mathbf{1} - \sum_{i \geqslant j=1}^{\alpha} (-1)^{i+j} v_{i+1,j} \mathbf{E}_{i,j}\right)\phi^{(1)} \qquad \text{(C.2)}$$

in which $\overset{\frown}{\det}$ denotes an 'ordered' determinant which must be evaluated using a row- (or column-) expansion starting from the last row (column) and continuing with the other rows (columns) in decreasing order. The result is a higher-order translation operator involving terms with powers of σ, which, when retained in the proper order, acts on everything on its right by producing a shift $n \to n+1$ in the argument of the corresponding factor $v_{i,j}$. Using the relation $\sigma\phi^{(N)} = \lambda\phi^{(1)}$ we are led to the identity

$$\left[\lambda - \sigma\,\overset{\frown}{\det}\left(\sigma\mathbf{1} - \sum_{i \geqslant j=1}^{N-1} (-1)^{i+j} v_{i+1,j} \mathbf{E}_{i,j}\right)\right]\phi^{(1)} = 0. \qquad \text{(C.3)}$$

Comparing this with equation (5.17) we find that $X^{(\alpha+1)}$ is determined by the α-fold principal minors of the diagonal elements of the matrix $\mathbf{V}$, i.e.

$$X^{(\alpha+1)} = (-1)^{N-\alpha} \sum_{i_1 < i_2 < \ldots < i_\alpha} \frac{\partial}{\partial v_{i_1+1,i_1}} \frac{\partial}{\partial v_{i_2+1,i_2}} \cdots \frac{\partial}{\partial v_{i_\alpha+1,i_\alpha}} \det(\mathbf{V}). \qquad \text{(C.4)}$$

Having obtained the expressions for the $X_n^{(\alpha)}$ in terms of the v_{ij} we first note that the Poisson brackets (5.26) can be generated from the following ones

$$\begin{aligned} &\{X_n^{(1)}, X_{n+p}^{(1)}\} = 0 \qquad \text{for } p \neq \pm 1 \pmod{2P} \\ &\{X_n^{(1)}, X_n^{(2)}\} = \{X_{n-1}^{(1)}, X_n^{(2)}\} = \{X_{n-1}^{(2)}, X_n^{(2)}\} = 0 \end{aligned} \tag{C.5}$$

together with

$$X_n^{(\alpha+1)} = \{X_{n-1}^{(\alpha)}, X_n^{(1)}\} \tag{C.6}$$

and $X_n^{(N)} = -1$. Equations (C.5) are trivially satisfied using the expressions

$$X^{(1)} = v_{N,1} + \sum_{j=1}^{N-2} v_{N,j+1} \det(\mathbf{V}^{(1,j)}) \tag{C.7}$$

$$\begin{aligned} X^{(2)} = (-1)^{N-1}\Big[&\det(\mathbf{V}^{(1,N-2)}) + \sigma^{-1}\det(\mathbf{V}^{(2,N-1)}) \\ &+ \sum_{i=2}^{N-2} \det(\mathbf{V}^{(1,i-1)})\sigma^{-1}\det(\mathbf{V}^{(i+1,N-1)})\Big] \end{aligned} \tag{C.7}$$

with the notation

$$\mathbf{V}^{(i,j)} = \begin{pmatrix} v_{i+1,i} & 1 & & & \\ v_{i+2,i} & v_{i+2,i+1} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & 1 \\ v_{j+1,i} & v_{j+1,i+1} & \cdots & \cdots & v_{j+1,j} \end{pmatrix}. \tag{C.9}$$

Equation (C.6) can be checked for arbitrary N. To see how the mechanism works let us consider as an example the case $N = 5$, where we find $\{\sigma^{-1}X^{(1)}, X^{(1)}\} = X^{(2)}$. Evaluating the bracket $\{\sigma^{-1}X^{(2)}, X^{(1)}\}$ with

$$\begin{aligned} \sigma^{-1}X^{(2)} = (\sigma^{-1}v_{2,1})\Bigg(&\begin{vmatrix} \sigma^{-1}v_{3,2} & 0 \\ 0 & \sigma^{-1}v_{5,4} \end{vmatrix} + \begin{vmatrix} \sigma^{-1}v_{3,2} & 1 \\ \sigma^{-1}v_{4,2} & \sigma^{-1}v_{4,3} \end{vmatrix} \\ &+ \begin{vmatrix} \sigma^{-2}v_{4,3} & 1 \\ \sigma^{-2}v_{5,3} & \sigma^{-2}v_{5,4} \end{vmatrix} + (\sigma^{-1}v_{3,1})(\sigma^{-1}v_{4,3}) - (\sigma^{-1}v_{3,1})(\sigma^{-2}v_{5,4}) \\ &+ (\sigma^{-1}v_{4,1})(\sigma^{-2}v_{5,4})\Bigg) \end{aligned} \tag{C.10}$$

and $X^{(1)}$ given by (C.7) for $N = 5$, it is found that

$$\begin{aligned} \{\sigma^{-1}X^{(2)}, X^{(1)}\} = &-\begin{vmatrix} \sigma^{-1}v_{3,2} & 0 \\ 1 & \sigma^{-1}v_{4,3} \end{vmatrix} - \begin{vmatrix} \sigma^{-1}v_{3,2} & 0 \\ 0 & \sigma^{-2}v_{5,4} \end{vmatrix} \\ &- \begin{vmatrix} \sigma^{-2}v_{4,3} & 1 \\ \sigma^{-2}v_{5,3} & \sigma^{-2}v_{5,4} \end{vmatrix} - \begin{vmatrix} v_{2,1} & 0 \\ 0 & \sigma^{-2}v_{5,4} \end{vmatrix} - \begin{vmatrix} v_{2,1} & 0 \\ 0 & \sigma^{-1}v_{4,3} \end{vmatrix} \\ &- \begin{vmatrix} v_{2,1} & 1 \\ v_{3,1} & v_{3,2} \end{vmatrix} = X^{(3)}. \end{aligned} \tag{C.11}$$

Finally from (C.11) we have

$$\{\sigma^{-1}X^{(3)}, X^{(1)}\} = v_{2,1} + \sigma^{-1}v_{32} + \sigma^{-2}v_{4,3} + \sigma^{-3}v_{5,4} = X^{(4)} \tag{C.12}$$

To prove that the mapping is symplectic one uses the explicit expressions of $X^{(\alpha)}_{2n-1}$ and $X^{(\alpha)}_{2n}$ in terms of the $X^{(\alpha)}_m$, which can be derived from (5.6) inserting the relation for $X^{(\alpha+1)\prime}_{2n+1}$ in the expression for $X^{(\alpha)\prime}_{2n}$. For even α e.g. one then finds

$$\begin{aligned}\{X^{(\alpha-1)\prime}_{2n-1}, X^{(1)\prime}_{2n}\} &= \Big\{X^{(\alpha-1)}_{2n}, X^{(1)}_{2n+1} + \frac{\lambda_{2n-1}-\mu}{X^{(1)}_{2n}}X^{(2)}_{2n+1} \\ &\quad - \frac{\lambda_{2n+1}-\mu}{X^{(1)}_{2n+2}}X^{(2)}_{2n+2} - \frac{(\lambda_{2n+1}-\mu)(\lambda_{2n-1}-\mu)}{X^{(1)}_{2n}X^{(1)}_{2n+2}}X^{(3)}_{2n+2}\Big\} \\ &= X^{(\alpha)}_{2n+1} - \frac{\lambda_{2n+1}-\mu}{X^{(1)}_{2n+2}}X^{(\alpha+1)}_{2n+2} = X^{(\alpha)\prime}_{2n}\end{aligned} \tag{C.13}$$

In a similar way one can justify equation (C.6) with $X^{(\alpha)}_n \to X^{(\alpha)\prime}_n$ for odd values of α. Equations (C.5) in terms of the updated variables follow after a tedious but straightforward calculation.

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