Physics Letters A 161 (1992) 419–422
North-Holland

PHYSICS LETTERS A

Integrable two-dimensional quantum mappings

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Received 30 September 1991; accepted for publication 12 November 1991
Communicated by A.P. Fordy

An 8-parameter family of integrable quantum mappings, in two quantum variables obeying Heisenberg type of commutation relations, is derived. These mappings are automorphisms of the quantum algebra possessing an exact quantum invariant. The classical limit of these mappings yields an area-preserving integrable family of mappings, discovered before by one of the authors.

1. Introduction

Systems that are either "integrable" or "solvable" are important in many areas of physics and mathematics, e.g. soliton theory, field theory, statistical mechanics, *S*-matrix theory, *n*-particle systems, and knot theory. Between these areas there are many connections.

Traditionally, the integrable systems studied were ordinary differential equations, followed by partial differential, partial differential–difference and partial difference equations.

Recently there has been a growing interest in exactly integrable mappings [1–17]. In ref. [1] *classical* integrable mappings, i.e. integrable finite-dimensional dynamical systems with discrete time, were proposed as a separate area of study. Many of the well-known features of continuous-time integrable systems carry over to the case of discrete-time integrable systems [9,10,12,13], whereas – due to their discreteness – integrable mappings are in a sense at once more simple *and* more fundamental than their continuous-time counterparts. Furthermore, their connections with soliton equations have been exploited to construct their invariants and find associated classical Yang–Baxter structures. Finally, they are increasingly more important for various applications, notably in the theory of dynamical systems (perturbations around integrable maps), cellular automata, the investigation of numerically induced chaos [18] and – as remarked in ref. [19] – as flows of integrable ODEs.

In a recent note [20] (cf. also ref. [21]), a theory of exactly integrable *quantum* mappings was proposed, and a class of multi-dimensional quantum mappings was presented. In the case of two-dimensional mappings (i.e. when the system involves two quantum variables) one recovers in the classical limit special cases of the 18-parameter family of integrable mappings of ref. [1]. This 18-parameter family is a measure-preserving set of mappings of the plane into itself whose invariants are biquadratic curves. A natural question, therefore, is to ask whether the whole 18-parameter family of *classical* mappings can be deformed to an 18-parameter family of *quantum* mappings, retaining integrability. Although a complete answer to this question remains as yet open, we give a partial answer in this paper, in the form of an 8-

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parameter family of quantum mappings together with their exact biquadratic quantum invariants.

Integrable quantum mappings could also be useful in a more systematic approach to the theory of quantum chaos [22]. In fact, they could be the appropriate starting point for the investigation of perturbations, in the study of nearly-integrable quantum systems.

2. Integrable classical map

Our starting point is the following classical integrable area-preserving mapping [16,1,5],

$$L_C: \quad x' = -x - g(y), \quad y' = -y - h(x') , \tag{1a}$$

with

$$g(x) := (\delta x^2 + \epsilon x + \xi)(\alpha x^2 + \beta x + \gamma)^{-1} ,$$

$$h(x) := (\beta x^2 + \epsilon x + \lambda)(\alpha x^2 + \delta x + \kappa)^{-1} . \tag{1b}$$

Here x and y are real classical (commuting) variables and α, β, γ, δ, ϵ, κ, λ and ξ are 8 real parameters (that is to say, eq. (1) is an 8-parameter family of integrable maps; the parameters α, β, γ, δ and κ should be chosen such that the denominators in (1b) are not identically zero). In eq. (1) and below, a subscript C denotes a classical quantity, and a subscript Q a quantum one.

The classical invariant of the map L_C is

$$I_C(x, y) = \alpha x^2 y^2 + \beta x^2 y + \gamma x^2 + \delta x y^2 + \epsilon x y + \xi x + \kappa y^2 + \lambda y . \tag{2}$$

The map L_C is not only integrable and area-preserving, but also reversible [23], i.e.

$$L_C = H_C \circ G_C , \tag{3}$$

where H_C and G_C are involutions (i.e. $H_C \circ H_C = G_C \circ G_C = \mathrm{Id}$) given by

$$G_C: \quad x' = -x - g(y), \quad y' = y , \tag{4a}$$

$$H_C: \quad x' = x, \quad y' = -y - h(x) . \tag{4b}$$

Each of these involutions also preserves the invariant I_C, i.e.

$$I_C(\boldsymbol{x}) = I_C(G_C(\boldsymbol{x})), \quad I_C(\boldsymbol{x}) = I_C(H_C(\boldsymbol{x})) , \tag{5}$$

where $\boldsymbol{x} := (x, y)$.

Although one can simply think of x and y as coordinates in $\mathbb{R}^2$, for the sequel it may be more appropriate to think of x and y as canonically conjugate variables, satisfying the Poisson bracket relation

$$\{x, y\} = 1 . \tag{6}$$

It follows that the mapping L_C is a canonical transformation, whereas the involutions G_C and H_C are anti-canonical transformations.

3. Integrable quantum map

We assume that our integrable quantum map has the same form as the classical map,

$$L_Q: \quad X' = -X - g(Y), \quad Y' = -Y - h(X') , \tag{7a}$$

with

$$g(X) := (\delta X^2 + \epsilon X + \xi)(\alpha X^2 + \beta X + \gamma)^{-1} ,$$

$$h(X) := (\beta X^2 + \epsilon X + \lambda)(\alpha X^2 + \delta X + \kappa)^{-1} , \tag{7b}$$

where α, β, γ, δ, ϵ, κ, λ, ξ again denote 8 real parameters. The only difference between (7) and (1) is that in (7) X and Y no longer commute, but are now Hermitian operators satisfying the canonical commutation relation

$$[X, Y] = \mathrm{i}\hbar , \tag{8}$$

and, e.g., $(\alpha X^2 + \beta X + \gamma)^{-1}$ denotes the operator inverse of $\alpha X^2 + \beta X + \gamma$. It follows that the mapping L_Q is a unitary transformation.

To find the quantum invariant of L_Q we write it in the following form:

$$I_Q = I_C + \mathrm{i}\hbar f(X, Y) , \tag{9}$$

where $f(X, Y)$ denotes the quantum correction to the invariant, and is to be determined. The map L_Q can again be factorized in terms of involutions G_Q and H_Q, which have the same form as their classical counterparts,

$$G_Q: \quad X' = -X - g(Y), \quad Y' = Y , \tag{10a}$$

$$H_Q: \quad X' = X, \quad Y' = -Y - h(X) , \tag{10b}$$

as G_Q and H_Q are anti-unitary transformations [#1], it

[#1] For a discussion of anti-unitary and anti-canonical transformations see ref. [24] and references therein.

is natural to look for a quantum invariant of the mapping L_Q satisfying

$$I_Q(X)=I_Q^*(G_Q(X)) , \tag{11a}$$

$$I_Q(X)=I_Q^*(H_Q(X)) , \tag{11b}$$

where the superscript asterisk denotes complex conjugation. Using the form (9) for the invariant, eq. (11a) yields

$$I_C(X, Y)+\mathrm{i}\hbar f(X, Y)=I_C(X', Y')-\mathrm{i}\hbar f^*(X', Y') , \tag{12}$$

where X' and Y' are given by eq. (10a). Using the commutation relation (8), we obtain the following functional equation for the quantum correction $f(X, Y)$:

$$-2\alpha Y g(Y)-\beta g(Y)+2\delta Y+\epsilon$$
$$+f(X, Y)+f^*(-X-g(Y), Y)=0 . \tag{13a}$$

Similarly, eq. (11b) leads to

$$-2\alpha X h(X)-\delta h(X)+2\beta X+\epsilon$$
$$+f(X, Y)+f^*(X, -Y-h(X))=0 . \tag{13b}$$

These two equations for the quantum correction $f(X, Y)$ are solved by

$$f(X, Y)=-2\alpha XY-\beta X-\delta Y-\tfrac{1}{2}\epsilon . \tag{14}$$

Inserting this in eq. (9) we get the quantum invariant of the map (7),(8),

$$I_Q=\alpha X^2Y^2+\beta X^2Y+\gamma X^2+\delta XY^2+\epsilon XY+\xi X+\kappa Y^2$$
$$+\lambda Y+\mathrm{i}\hbar(-2\alpha XY-\beta X-\delta Y-\tfrac{1}{2}\epsilon) . \tag{15}$$

Using the fact that the operators X and Y are Hermitian [#2] it is easy to verify that the quantum invariant I_Q is also Hermitian. This means that we can equivalently write the quantum invariant in an explicitly Hermitian form:

$$I_Q=\tfrac{1}{2}\alpha(X^2Y^2+Y^2X^2)+\tfrac{1}{2}\beta(X^2Y+YX^2)+\gamma X^2$$
$$+\tfrac{1}{2}\delta(XY^2+Y^2X)+\tfrac{1}{2}\epsilon(XY+YX)$$
$$+\xi X+\kappa Y^2+\lambda Y .$$

This way of writing the quantum invariant has the advantage that $I_Q(X, Y)=I_Q^*(X, Y)$.

[#2] Note that the Hermiticity of X and Y has not been used to derive eq. (15).

Some previously discovered integrable quantum maps obtain as special cases of eq. (7). The (second iterate of) the quantum McMillan map [20] obtains for $\alpha=1$, $\beta=0$, $\gamma=-\tilde{\epsilon}^2$, $\delta=0$, $\epsilon=-2\tilde{\epsilon}\tilde{\delta}$, $\xi=0$, $\kappa=-\tilde{\epsilon}^2$, $\lambda=0$. The (inverse of) the quantum Boussinesq map [21] obtains for $\alpha=0$, $\beta=1$, $\gamma=0$, $\delta=0$, $\epsilon=-\tilde{\beta}$, $\xi=-\tilde{\gamma}$, $\kappa=-1$, $\lambda=\tilde{\alpha}$.

4. Conclusions and outlook

In this Letter we have presented an 8-parameter family of exactly integrable two-dimensional quantum mappings, which is the quantum analogue of an interesting subfamily of the classical 18-parameter family of Quispel et al. [15,17]. At the quantum level these mappings can be interpreted to give rise to an iterative set of automorphisms of the Heisenberg algebra, possessing an exact invariant.

Of course one should make it clear what one means by quantum integrability of mappings. As has been pointed out by Kruskal and co-workers [25], it is not even trivial to define integrability for a classical system. One should be even more careful at the quantum level. In refs. [21,22] quantum integrability is a consequence of the existence of a well-defined (non-ultralocal) quantum Yang–Baxter structure, which provides the system with a construction of commuting families of operators which are invariant under the mapping, and hence the mapping – being a unitary transformation, preserving the quantum algebra – also allows for an invariant diagonalization procedure of this family. In this Letter we have used the term integrable for a two-dimensional quantum map in analogy with the classical case. Nevertheless, we expect the 8-parameter family of quantum maps given above, to possess enough structure to be exactly integrable, as is true for the special cases cited before. These questions are under investigation.

Time-reversal symmetry (reversibility) plays a significant role in many classical *chaotic* dynamical systems [23]. In classical *integrable* systems reversibility is even more ubiquitous. We expect reversibility to play an equally important role in integrable quantum systems.

Acknowledgement

We thank J.A.G. Roberts for useful discussions in the early stages of this work. We are grateful to the Department of Theoretical Physics at the University of Amsterdam for its hospitality during our visit, when this work was started.

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