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INTEGRABLE MAPPINGS DERIVED FROM SOLITON EQUATIONS

G.R.W. QUISPEL
Department of Mathematics, La Trobe Univeristy, Bundoora, Victoria 3083, Australia

H.W. CAPEL
Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

V.G. PAPAGEORGIOU and F.W. NIJHOFF
Department of Mathematics and Computer Science, and Institute for Nonlinear Studies, Clarkson University, Potsdam, NY 13767, USA

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We derive a hierarchy of integrable mappings (integrable ordinary difference equations) corresponding to solutions of the initial-value problem of an integrable partial difference equation with periodic initial data. For each $n \in \mathbb{N}$ this hierarchy contains at least one integrable mapping $\mathbb{R}^n \to \mathbb{R}^n$. The integrals of these mappings are constructed using the Lax pair of the underlying partial difference equation. Our approach is illustrated for the integrable partial difference analogues of the sine-Gordon and the (modified) Korteweg–de Vries equations.

1. Introduction

We study the infinite hierarchy of integrable difference equations, labelled by two integers z_1, $z_2 \in \mathbb{Z}$,

$$V_n V_{n+z_1} V_{n+z_2} V_{n+z_1+z_2} - pq(V_n V_{n+z_1+z_2} - V_{n+z_1} V_{n+z_2}) = 1 \,, \tag{1}$$

V, p, $q \in \mathbb{R}$. In eq. (1) the subscripts denote the values of the independent variable, $n \in \mathbb{Z}$, V is the dependent variable, and p and q are arbitrary constants. For given z_1 and z_2, eq. (1) is equivalent to a mapping $\mathbb{R}^{z_1+z_2} \to \mathbb{R}^{z_1+z_2}$. In this paper we derive an algorithm for calculating integrals of the mapping (1). Using the elegant procedure of ref. [1] this algorithm is constructed from the Lax representation of the integrable partial difference equation from which eq. (1) is derived. This procedure was first used in ref. [1]

to obtain the integrals of a family of mappings associated with a class of periodic solutions of the initial value problem of the two-dimensional lattice version of the Korteweg–de Vries equation.

Integrable dynamical systems with one or more continuous or *two or more* discrete independent variables have been extensively studied, cf. refs. [2–13]. In this paper we study integrable systems with *one* discrete variable. This is an area that only recently has received some attention, [1, 14–22]. We hope that this paper may be useful towards the construction of a general theory of integrable ordinary difference equations. The outline of this paper is as follows. In section 2 the hierarchy of integrable ordinary difference equations (1) is derived from an integrable *partial* difference equation called the ΔΔ-sine Gordon (ΔΔ-SG) equation by considering travelling wave solutions. It is shown that eq. (1) is reversible and (anti) measure-preserving. In section 3 it is shown that eq. (1) represents a solution of the initial-value problem of the ΔΔ-SG with periodic initial data. In section 4 the algorithm for constructing integrals of eq. (1) is derived. The integrals for the cases $z_1 = 1$, $z_2 = 1$; $z_1 = 1$, $z_2 = 2$; $z_1 = 1$, $z_2 = 3$ are explicitly given in section 5. In appendix A the algorithm for constructing integrals of the hierarchy of integrable mappings of the ΔΔ-modified-Korteweg–de Vries (ΔΔ-MKdV) equation is derived, and the integrals for the cases $z_1 = 1$, $z_2 = 2$; $z_1 = 1$, $z_2 = 3$ are explicitly given. In appendix B we give a generating expression for the invariants of the integrable mappings of the ΔΔ-SG hierarchy and of the ΔΔ-MKdV hierarchy for general $z_1, z_2 \in \mathbb{Z}$. Finally in appendix C we give the integrals for mappings of the ΔΔ-Korteweg–de Vries hierarchy for general z_1 and z_2. Similar results for a slightly different class of solutions of a ΔΔ-equation which is a mixed version of the lattice MKdV and the discrete-time Toda equation, can be found in ref. [1].

2. Hierarchy of integrable ΔΔ-sine Gordon mappings

Consider the ΔΔ-sine Gordon equation [11, 23]

$$V_{l,m}V_{l+1,m}V_{l,m+1}V_{l+1,m+1} - pq(V_{l,m}V_{l+1,m+1} - V_{l+1,m}V_{l,m+1}) = 1 \tag{2}$$

for fields $V_{l,m}$ defined at the sites (l, m) of a two-dimensional lattice $\mathbb{Z}^2$. Eq. (2) follows as the compatibility condition from the two relations

$$\begin{aligned}
(p-k)\begin{pmatrix} V_{l+1,m}(k) \\ U_{l+1,m}(k) \end{pmatrix} &= \begin{pmatrix} p & -V_{l+1,m} \\ -k^2/V_{l,m} & pV_{l+1,m}/V_{l,m} \end{pmatrix}\begin{pmatrix} V_{l,m}(k) \\ U_{l,m}(k) \end{pmatrix}, \\
(q-k^{-1})\begin{pmatrix} V_{l,m+1}(k) \\ U_{l,m+1}(k) \end{pmatrix} &= \begin{pmatrix} qV_{l,m+1}/V_{l,m} & -k^{-2}/V_{l,m} \\ -V_{l,m+1} & q \end{pmatrix}\begin{pmatrix} V_{l,m}(k) \\ U_{l,m}(k) \end{pmatrix},
\end{aligned} \tag{3}$$

for wave functions $V_{l,m}(k)$, $U_{l,m}(k)$ depending on a spectral parameter k at the sites of $\mathbb{Z}^2$.

Eq. (3) is called the Lax representation of eq. (2). Eq. (2) has been shown to be integrable by the direct linearization method [11], i.e. solutions can be obtained solving only linear integral equations.

Introducing (complex) fields $\Theta_{l,m}$ via $V_{l,m} = e^{\frac{1}{2}i\Theta_{l,m}}$, eq. (2) can be expressed as

$$\sin\tfrac{1}{4}(\Theta_{l,m} + \Theta_{l+1,m,+1} + \Theta_{l+1,m} + \Theta_{l,m+1}) \\ - pq\sin\tfrac{1}{4}(\Theta_{l,m} + \Theta_{l+1,m+1} - \Theta_{l+1,m} - \Theta_{l,m+1}) = 0 . \tag{4}$$

By considering two continuum limits, $q \to \infty$ and $p \to \infty$, eq. (4) reduces to $\partial_x \partial_t \Theta = \sin \Theta$, which is the well-known sine-Gordon (SG) equation. Eq. (2) is therefore an integrable two-dimensional lattice version of the SG equation which we call the ΔΔ-SG equation.

Travelling wave solutions of eq. (2) are obtained by the ansatz

$$V_{l,m} = V_n , \tag{5}$$

where

$$n = z_1 l + z_2 m , \tag{6}$$

z_1 and z_2 being relatively prime integers. From (2) we then obtain

$$V_n V_{n+z_2} V_{n+z_1} V_{n+z_1+z_2} - pq(V_n V_{n+z_1+z_2} - V_{n+z_1} V_{n+z_2}) = 1 . \tag{1}$$

Eq. (1) is invariant under $z_1 \to -z_1$, $p \to -p$ and under $z_1 \leftrightarrow z_2$. Furthermore, from (5) and (6) it is clear that $V_{l,m}$ satisfies the periodicity property

$$V_{l+z_2,m-z_1} = V_{l,m} . \tag{7}$$

This will be used in section 3 to solve an initial-value problem for the ΔΔ-SG (2).

Eq. (1) represents an infinite hierarchy of mappings labelled by z_1 and z_2. For fixed z_1 and z_2, eq. (1) is a mapping from $\mathbb{R}^{z_1+z_2} \to \mathbb{R}^{z_1+z_2}$.

This feature is different in the case of integrable DΔ-equations for time dependent fields $V_n(t)$ defined at the sites n of the one-dimensional chain $\mathbb{Z}$. In that case a travelling-wave ansatz for solutions yields one mapping for each DΔ-equation [15, 16]. The fact that we have an infinite hierarchy of mappings associated with one ΔΔ-equation is presumably related to the fact that one integrable ΔΔ-equation yields an infinite hierarchy of DΔ-equations via appropriate continuum limits [12].

Eq. (1) is equivalent to the mapping

$$L:\begin{cases} V'_{z_1+z_2-1} = \dfrac{1-pqV_{z_1}V_{z_2}}{V_0(V_{z_1}V_{z_2}-pq)}\,, \\ V'_{z_1+z_2-2} = V_{z_1+z_2-1}\,, \\ \vdots \\ V'_1 = V_2\,, \\ V'_0 = V_1 \end{cases} \tag{8}$$

It is not difficult to show that the mapping L is reversible, cf. refs. [24–28],

$$LGL = G\,, \tag{9}$$

where the involution G is given by

$$G:\begin{cases} V'_{z_1+z_2-1} = V_0\,, \\ V'_{z_1+z_2-2} = V_1\,, \\ \vdots \\ V'_1 = V_{z_1+z_2-2}\,, \\ V'_0 = V_{z_1+z_2-1}\,. \end{cases} \tag{10}$$

The Jacobian determinant of L is given by

$$\begin{aligned} \operatorname{Det} \mathrm{d}L &\equiv \operatorname{Det}\begin{pmatrix} \dfrac{\partial V'_0}{\partial V_0} & \cdots & \dfrac{\partial V'_0}{\partial V_{z_1+z_2-1}} \\ \vdots & & \vdots \\ \dfrac{\partial V'_{z_1+z_2-1}}{\partial V_0} & \cdots & \dfrac{\partial V'_{z_1+z_2-1}}{\partial V_{z_1+z_2-1}} \end{pmatrix} \\ &= \varepsilon\,\frac{1-pqV_{z_1}V_{z_2}}{V_0^2(V_{z_1}V_{z_2}-pq)} = \varepsilon\,\frac{V'_{z_1+z_2-1}}{V_{z_0}}\,, \end{aligned} \tag{11}$$

where

$$\varepsilon = (-1)^{[(z_1+z_2+1)/2]} \tag{12}$$

with the square brackets denoting the integer part.

Eq. (11) can be rewritten in the form

$$\operatorname{Det} \mathrm{d}L = \varepsilon \frac{F(V'_0, V'_1, \ldots, V'_{z_1+z_2-1})}{F(V_0, V_1, \ldots, V_{z_1+z_2-1})} \tag{13}$$

with

$$F[V_0, V_1, \ldots, V_{z_1+z_2-1}] = V_0 V_1 \cdots V_{z_1+z_2-1} \,. \tag{14}$$

Eq. (13) shows that the mapping L is (anti) measure preserving [29, 30].

3. Standard staircase

In this section we will show that any travelling wave solution of the form (5), (6) corresponds to the solution of a certain initial value problem of the $\Delta\Delta$-sine Gordon equation (2). To this end we construct a "standard staircase" on the 2D square lattice. Here we shall assume that $z_1 > z_2$ without loss of generality.

We start in a point $A \in \mathbb{Z}^2$, which without loss of generality can be taken to be at the origin ($l = m = 0$). The point A corresponds to V_0. Go one step to the right ($l = 1$, $m = 0$). This point corresponds to V_{z_1}. Now we go down as many steps as possible without getting negative subscripts on the V's. The last point will be $V_{z_1 \bmod z_2}$. Then we go one step to the right ($V_{z_1+z_1 \pmod{z_2}}$) and repeat the procedure. Going down, the last point will be $V_{2z_1 \pmod{z_2}}$. The whole procedure of going one step to the right and going down is then repeated z_2 times, after which we arrive at a point B with coordinates $l = z_2$, $m = -z_1$, which again corresponds to V_0. The staircase leading from A to B is then repeated periodically by a subsequent shift consisting of z_2 steps in the horizontal direction, combined with z_1 steps in the vertical direction. In this way we obtain the standard staircase.

An example with $z_1 = 12$, $z_2 = 5$ is given in fig. 1. Note that on the part ranging from A to B every V_n with $1 \leqslant n \leqslant z_1 + z_2 - 1$ occurs once and only once.

Starting from the standard staircase with initial values of V given at the points of the staircase one can obtain a complete solution of the $\Delta\Delta$-SG equation (2) above and to the right (or below and to the left) of the staircase by repeated use of eq. (2). The periodicity property (7) holds not only on the staircase, but also for solutions obtained above (or below) the staircase. This implies that the initial-value problem for the *partial* difference equation (2) with initial data on a periodic standard staircase, is solved by the *ordinary* difference equation (1).

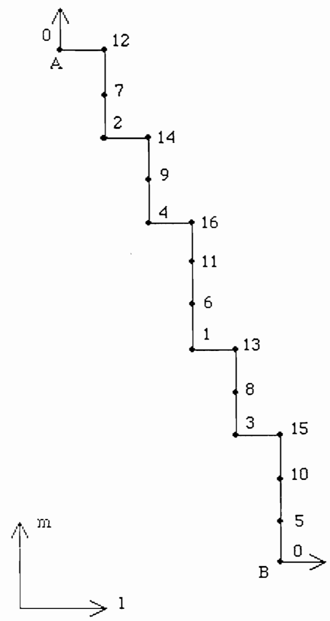

Fig. 1. Standard staircase for $z_1 = 12$, $z_2 = 5$. The integers denote the subscript n of the travelling wave solution V_n at the lattice sites.

For the mapping introduced in eq. (8) we have

$$V'_{l,m} = V_{l+p_1,m-p_2}\,, \tag{15}$$

with (l, m) corresponding to the points on the staircase ranging from A to B and where the translation on $\mathbb{R}^2$ over (p_1, p_2) corresponds to the shift $V_n \to V_{n+1}$, i.e. $z_1p_1 - z_2p_2 = 1$. (In fig. 1 we have the special case $p_1 = 3$, $p_2 = 7$.)

More generally one might consider a mapping satisfying (15) but now with $z_1p_1 - z_2p_2 = q$ $(q > 1)$. This mapping is simply L^q with L defined by (8). All integrals of the mapping L are integrals under L^q.

So far we have considered standard staircases with $z_1 > z_2$. The standard staircases with $z_2 > z_1$ follow from the ones with $z_1 > z_2$ by interchanging z_1 and z_2, or equivalently by a reflection around the diagonal $l - m =$ constant passing through B and by interchanging B and A.

Hence, we have shown that the hierarchy of difference equations (1) corresponds to the solution of the initial-value problem of the ΔΔ-SG equation (2) on a standard staircase satisfying the periodicity property (7).

Finally we note that the construction of standard staircases associated with travelling wave solutions of partial difference equations is by no means restricted to integrable ΔΔ-equations. In fact, one may consider a general

$\Delta\Delta$-equation of the type

$$V_{l+1,m+1} = h(V_{l,m}, V_{l+1,m}, V_{l,m+1}) . \tag{16}$$

With the travelling wave solutions of type (5), (6) one can associate the mapping

$$\begin{aligned} V'_{z_1+z_2-1} &= h(V_0, V_{z_1}, V_{z_2}) , \\ V'_{z_1+z_2-2} &= V_{z_1+z_2-1} , \\ &\vdots \\ V'_1 &= V_2 , \\ V'_0 &= V_1 , \end{aligned} \tag{17}$$

which in general is not integrable. The travelling wave solutions are directly related to the initial value problem of the $\Delta\Delta$-equation associated with a standard staircase going from $P(0, 0)$ to $Q(z_2, -z_1)$ and periodically repeated.

4. Construction of integrals

In ref. [1] the mappings associated with the solutions of the $\Delta\Delta$-Korteweg–de Vries equation ($\Delta\Delta$-KdV) satisfying $V_{l,m} = V_{l+P,m-p}$, $P = 2, 3$, were investigated with the use of the Lax representation of the $\Delta\Delta$-KdV. It was possible to construct a number of integrals equal to half the dimension of the mapping. In this section we shall apply the method of ref. [1] to the hierarchy of mappings associated with the $\Delta\Delta$-SG equation. (The $\Delta\Delta$-MKdV and the $\Delta\Delta$-KdV equations are treated in appendix A and C, respectively.)

For this purpose the Lax representation (3) is written as

$$\begin{aligned} (p-k)\begin{pmatrix} V_{l+1,m}(k) \\ U_{l+1,m}(k) \end{pmatrix} &= M^{\mathrm{hor}}_{l,m}(k)\begin{pmatrix} V_{l,m}(k) \\ U_{l,m}(k) \end{pmatrix} , \\ (q-k^{-1})\begin{pmatrix} V_{l,m+1}(k) \\ U_{l,m+1}(k) \end{pmatrix} &= M^{\mathrm{vert}}_{l,m}(k)\begin{pmatrix} V_{l,m}(k) \\ U_{l,m}(k) \end{pmatrix} , \end{aligned} \tag{18}$$

with

$$\begin{aligned} M^{\mathrm{hor}}_{l,m}(k) &= \begin{pmatrix} p & -V_{n+z_1} \\ -k^2/V_n & pV_{n+z_1}/V_n \end{pmatrix} , \\ M^{\mathrm{vert}}_{l,m}(k) &= \begin{pmatrix} qV_{n+z_2}/V_n & -k^{-2}/V_n \\ -V_{n+z_2} & q \end{pmatrix} . \end{aligned} \tag{19}$$

Consider now a standard staircase going from a point A with coordinates (l, m) via intermediate points $A_1, A_2, \ldots, A_{z_1+z_2-1}$ with coordinates $(l+i_2, m-i_1)$ to the point B with coordinates $(l+z_2, m-z_1)$. An example is given in fig. 2.

With every step to the right from $(l+i_1, m-i_2)$ to $(l+i_1+1, m-i_2)$ we associate the matrix

$$S_{l+i_1,m-i_2}(k) \equiv M^{\mathrm{hor}}_{l+i_1,m-i_2}(k) \tag{20a}$$

and with every step down from $(l+i_1, m-i_2)$ to $(l+i_1, m-i_2-1)$ the matrix

$$S_{l+i_1,m-i_2}(k) = M^{\mathrm{vert}^{-1}}_{l+i_1,m-i_2-1}(k)\,. \tag{20b}$$

For the staircase going from A to B we form the product

$$L^{z_1,z_2}_{l,m}(k) = \overset{\frown}{\prod_{i_1,i_2}} S_{l+i_1,m-i_2}(k) \tag{21}$$

over the points $A, A_1, \ldots, A_{z_1+z_2-1}$ (but not over B), with coordinates $(l+i_1, m-i_2)$. The symbol $\frown$ denotes that the matrices on the right-hand side of (21) are ordered from the right to the left. So the matrix S associated with A occurs on the right. Left of this matrix we have the matrix associated with A_1 and so on. (The matrix A on the extreme left corresponds to $A_{z_1+z_2-1}$.)

From eqs. (18) and (21) it follows that

$$(p-k)^{z_2}(q-k^{-1})^{-z_1}\begin{pmatrix} V_{l+z_2,m-z_1}(k) \\ U_{l+z_2,m-z_1}(k)\end{pmatrix} = L^{z_1,z_2}_{l,m}(k)\begin{pmatrix} V_{l,m}(k) \\ U_{l,m}(k)\end{pmatrix}. \tag{22}$$

Because of the *periodicity condition* for the potentials $V_{l,m}$ (but not necessarily for the wave functions $V_{l,m}(k)$, $U_{l,m}(k)$) we have

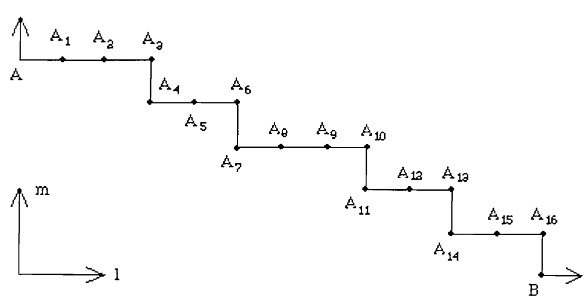

Fig. 2. Standard staircase from A to B with intermediate points $A_1, A_2, \ldots, A_{z_1+z_2-1}$, with $z_1 = 5$, $z_2 = 12$.

$$M^{\text{hor}}_{l+z_2,m-z_1}(k) = M^{\text{hor}}_{l,m} \tag{23}$$

and therefore

$$L^{z_1,z_2}_{l+1,m}(k) = M^{\text{hor}}_{l,m}(k) \cdot L^{z_1,z_2}_{l,m}(k) \cdot M^{\text{hor}^{-1}}_{l,m}(k) \tag{24a}$$

and similarly

$$L^{z_1,z_2}_{l,m+1}(k) = M^{\text{vert}}_{l,m}(k) \cdot L^{z_1,z_2}_{l,m}(k) \cdot M^{\text{vert}^{-1}}_{l,m} , \tag{24b}$$

Combining (24a), (24b), it follows that

$$L^{z_1,z_2}_{l+p_1,m-p_2}(k) = R \cdot L^{z_1,z_2}_{l,m}(k) \cdot R^{-1} \tag{25}$$

for any p_1 and p_2, in which the matrix R is composed of matrices M^{hor} and M^{vert}. As a consequence Tr $L^{z_1,z_2}_{l,m}(k)$ is *invariant* under all translations on $\mathbb{Z}^2$, and therefore also under mapping (8). Hence, we will omit the subscrips l, m denoting the lattice sites.

The invariants of the mapping derived by eq. (8) can thus be found evaluating Tr $L^{z_1,z_2}(k)$, with the matrices M^{hor} and M^{vert} given by (19), over the standard staircase ranging from A to B. As the spectral parameter k occurring in the Lax representation (3) is an arbitrary complex constant, the integrals of the mapping L are given as the coefficients in the expansion of Tr $L^{z_1,z_2}(k)$ in powers of k.

5. Integrals for the sine-Gordon mapping

In this section we discuss the invariants of the mapping L given in eq. (8) associated with the ΔΔ-SG equation for the cases $z_1 = 1$, $z_2 = 1, 2, 3$. For more general z_1 and z_2, see appendix B.

1) $z_1 = 1$, $z_2 = 1$

The mapping in this case is

$$\begin{aligned} V_1' &= \frac{1 - pqV_1^2}{V_0(V_1^2 - pq)} , \\ V_0' &= V_1 , \end{aligned} \tag{26}$$

and the invariant is given by

$$
\begin{aligned}
(q^2-k^{-2})\,\mathrm{Tr}\,L^{1,1}(k) &= \mathrm{Tr}\begin{pmatrix} qV_0/V_1 & k^{-2}/V_1 \\ V_0 & q \end{pmatrix}\begin{pmatrix} p & -V_1 \\ -k^2V_q^0 & pV_1/V_0 \end{pmatrix} \\
&= pq\left[\frac{V_0}{V_1}+\left(\frac{V_0}{V_1}\right)^{-1}\right] - V_0V_1-(V_0V_1)^{-1}\,. \qquad (27)
\end{aligned}
$$

The mapping (26) with invariant (27) is a special case of the 18-parameter family of integrable mappings given in refs. [15, 16].

2) $z_1=1,\ z_2=2$

In this case the mapping

$$
\begin{aligned}
V_2' &= \frac{1-pqV_1V_2}{V_0(V_1V_2-pq)}\,, \\
V_1' &= V_2\,, \qquad (28)\\
V_0' &= V_1
\end{aligned}
$$

can be reduced to a 2-dimensional mapping. Introducing $W_i=V_iV_{i+1}$, $i=0,\ 1$, eq. (28) can be rewritten as

$$
\begin{aligned}
W_1' &= \frac{1-pqW_1}{W_1-pq}\,\frac{W_1}{W_0}\,, \\
W_0' &= W_1\,,
\end{aligned}
\qquad (29)
$$

and the invariant is

$$
\begin{aligned}
&(q^2-k^{-2})\,\mathrm{Tr}\,L^{1,2}(k) \\
&\quad= \mathrm{Tr}\begin{pmatrix} qV_0/V_2 & k^{-2}/V_2 \\ V_0 & q \end{pmatrix}\begin{pmatrix} p & -V_2 \\ -k_2/V_1 & pV_2/V_1 \end{pmatrix}\begin{pmatrix} p & -V_1 \\ k^2/V_0 & pV_1/V_0 \end{pmatrix} \\
&\quad= p\left[pq\left(\frac{V_2}{V_0}+\frac{V_0}{V_2}\right) - V_0V_1 - V_0^{-1}V_1^{-1} - V_1V_2 - V_1^{-1}V_2^{-1}\right] + 2k^2q \\
&\quad= p\left[pq\left(\frac{W_1}{W_0}+\frac{W_0}{W_1}\right) - W_0 - W_0^{-1} - W_1 - W_1^{-1}\right] + 2k^2q\,. \qquad (30)
\end{aligned}
$$

Eq. (29) with the invariant (30) is again a special case of the 18-parameter family of integrable mappings studied in refs. [15, 16].

3) $z_1 = 1,\ z_2 = 3$

In this case the mapping is

$$\begin{aligned} V_3' &= \frac{1 - pqV_1V_3}{V_0(V_1V_3 - pq)}\ , \\ V_2' &= V_3\ , \\ V_1' &= V_2\ , \\ V_0' &= V_1\ , \end{aligned} \tag{31}$$

and the invariants are obtained from

$$\begin{aligned} &(q^2 - k^{-2})\,\mathrm{Tr}\,L^{1,3}(k) \\ &\quad = \mathrm{Tr}\begin{pmatrix} qV_0/V_3 & k^{-2}/V_3 \\ V_0 & q \end{pmatrix}\begin{pmatrix} p & -V_3 \\ -k^2/V_2 & pV_3/V_2 \end{pmatrix} \\ &\qquad \times \begin{pmatrix} p & -V_2 \\ -k^2/V_1 & pV_2/V_1 \end{pmatrix}\begin{pmatrix} p & -V_1 \\ -k^2/V_0 & pV_1/V_0 \end{pmatrix} \\ &\quad = p^2\Big\{ pq\Big[\frac{V_3}{V_0} + \Big(\frac{V_3}{V_0}\Big)^{-1}\Big] - V_3V_2 - (V_3V_2)^{-1} - V_2V_1 \\ &\qquad - (V_2V_1)^{-1} - V_1V_0 - (V_1V_0)^{-1}\Big\} \\ &\qquad + k^2\Big\{ pq\Big[\frac{V_3}{V_2} + \Big(\frac{V_3}{V_2}\Big)^{-1}\Big] + \frac{V_2}{V_1} + \Big(\frac{V_2}{V_1}\Big)^{-1} \\ &\qquad + \frac{V_1}{V_0} + \Big(\frac{V_1}{V_0}\Big)^{-1} - V_3V_0 - (V_3V_0)^{-1}\Big\}\ . \end{aligned} \tag{32}$$

We see that there are two independent invariants for the four-dimensonal mapping, i.e. the constant term and the coefficient of k^2 in the right-hand side of (32).

Proceeding in this way the invariants for larger values of z_2 can be evaluated. For odd values $z_2 = 2p + 1$ and $z_1 = 1$ there are $p + 1$ independent invariants for the $(2p + 2)$-dimensional mapping. For even values $z_2 = 2p$ the $(2p + 1)$-dimensional mapping in terms of the V's can be expressed as a $2p$-dimensional mapping in terms of the variables $W_i = V_iV_{i+1}$, $i = 0, \ldots, 2p$. The evaluation of

$(q^2 - k^2)\,\mathrm{Tr}\, L^{1,2p}(k)$ leads to p independent invariants, which can be expressed in terms of the variables W_i.

6. Concluding remarks

1) Since the mappings we consider in this paper are reductions of integrable partial difference equations, we conjecture that the mappings themselves are also integrable. Supporting evidence is the fact that in those cases where we explicitly calculate the integrals of the mappings, their number is equal to half the dimension of the system of difference equations. This is analogous to the case of integrable Hamiltonian ordinary differential equations. Recently interesting results have been obtained for Lagrangian and Hamiltonian structures of certain types of discrete-time systems [17–20]. In particular, an analog of the Liouville–Arnold theorem has been proved for Lagrangian mappings by Veselov. In ref. [21] the complete integrability for the class of mappings considered in ref. [1] has been proved by establishing the Lagrangian and Hamiltonian structures of these mappings, and in ref. [21] similar results were obtained for a class of discrete-time Toda lattices on the basis of an r-matrix formalism. It would be gratifying to obtain similar results for the general class of mappings studied in this paper.

2) We have restricted the discussion to standard staircases from A to B. It is also possible to consider non-standard staircases and define a mapping by an arbitrary translation in $\mathbb{Z}^2$ of this non-standard staircase. The integrals for such a non-standard mapping can be obtained from those of the standard mapping using eq. (1) and the fact that eq. (2) represents the compatibility condition of the Lax representation (3).

3) A more general reduction of $\Delta\Delta$-equations is given by

$$V_{l,m} = V_{n,k}\,, \tag{33}$$

where

$$\begin{aligned} n &= z_1 l + z_2 m\,, \qquad z_1,\, z_2 \text{ relatively prime}\,, \\ k &= z_3 l + z_4\, m \end{aligned} \tag{34}$$

with

$$V_{n,k} = V_{n,k+P}\,, \qquad P = 1, 2, 3, 4\ldots \tag{35}$$

and

$$z_1 z_4 - z_2 z_3 = 1 . \tag{36}$$

The examples treated in this paper all have $P = 1$. In ref. [1] mappings derived from the ΔΔ-KdV are treated for general P and $z_1 = z_2 = 1$.

4) We have treated the simplest type of integrable partial difference equation,

$$V_{l+1,m+1} = h(V_{l,m}, V_{l+1,m}, V_{l,m+1}) . \tag{16}$$

More complicated integrable partial difference equations also exist [31, 32]. An example is the ΔΔ-Boussinesq equation [31], for which

$$V_{l+2,m+2} = h(V_{l,m}, V_{l+1,m}, V_{l+2,m}, V_{l,m+1}, V_{l+1,m+1}, V_{l+1,m+2}, V_{l+2,m+1}, V_{l,m+2}) .$$

Such examples may be formulated in terms of a generalized staircase construction.

5) Very recently, the travelling wave solutions of integrable ΔΔΔ-equations for (scalar) fields $V_{l,m,n}$ at the sites (l, m, n) of $\mathbb{Z}^3$ have been studied [32]. However, the relation with the initial value problem on suitably defined staircases, and the construction of invariants on the basis of Lax representations is as yet an open problem. It would also be of interest to investigate the *matrix-valued* ΔΔΔ-equations of ref. [13].

6) Eq. (1) is also well defined if z_1 and z_2 are arbitrary (irrational) real numbers. The integrability of eq. (1) in this more general case has not yet been established.

7) Presumably, the methods used in this paper can be extended to find integrals of mappings derived from integrable DΔ-equations [15, 16], cf. also ref. [20] for some interesting results on completely integrable mappings of this type.

8) All examples of integrable mappings in this paper are reversible and (anti) measure preserving. We have heuristically found that both these properties are very common in integrable dynamical systems in general. In particular the property of reversibility can sometimes be used to find integrals of integrable difference equations (cf. e.g. refs. [15, 16]) and of integrable differential equations.

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Appendix A

In this appendix we discuss integrable mappings related to the ΔΔ-MKdV equation and give their integrals for $z_1 = 1$, $z_2 = 2$ and for $z_1 = 1$, $z_2 = 3$ in explicit form. A generating expression for the integrals of general $z_1, z_2 \in \mathbb{Z}$ is given in appendix B.

The Lax representation for the ΔΔ-MKdV equation is

$$
\begin{aligned}
(p-k)\begin{pmatrix} V_{l+1,m}(k) \\ U_{l+1,m}(k) \end{pmatrix} &= \begin{pmatrix} p & -V_{l+1,m} \\ -k^2/V_{l,m} & pV_{l+1,m}/V_{l,m} \end{pmatrix} \begin{pmatrix} V_{l,m}(k) \\ U_{l,m}(k) \end{pmatrix} , \\
(q-k)\begin{pmatrix} V_{l,m+1}(k) \\ U_{l,m+1}(k) \end{pmatrix} &= \begin{pmatrix} q & -V_{l,m+1} \\ -k^2/V_{l,m} & qV_{l,m+1}/V_{l,m} \end{pmatrix} \begin{pmatrix} V_{l,m}(k) \\ U_{l,m}(k) \end{pmatrix} .
\end{aligned}
\tag{A.1}
$$

The compatibility of both relations (A.1) yields

$$
p(V_{l,m}V_{l,m+1} - V_{l+1,m}V_{l+1,m+1}) = q(V_{l,m}V_{l+1,m} - V_{l,m+1}V_{l+1,m+1}) . \tag{A.2}
$$

Eq. (A.2) is the ΔΔ-MKdV equation (performing two suitable continuum limits it can be transformed into the well-known MKdV equation). The solutions of the ΔΔ-MKdV can be obtained from a linear integral equation [6].

For travelling wave solutions of the type (5), (6), we obtain the mapping

$$
L: \begin{cases} V'_{z_1+z_2-1} = \dfrac{pV_{z_2} - qV_{z_1}}{pV_{z_1} - qV_{z_2}} V_0 , \\ V'_{z_1+z_2-2} = V_{z_1+z_2-1} , \\ \vdots \\ V'_1 \qquad = V_2 , \\ V'_0 \qquad = V_1 . \end{cases} \tag{A.3}
$$

This mapping is again reversible, i.e. we have $LGL = G$, where the involution G is given by (10). Measure preservation follows in a similar way as in the case of the ΔΔ-SG, cf. eq. (13).

Introducing the variables

$$
a_i = \frac{V_{i+1}}{V_i} , \qquad i = 0, 1, \ldots, z_2 - 1 , \tag{A.4}
$$

eq. (A.3) reduces to the following $(z_1 + z_2 - 1)$-dimensional mapping (taking $z_2 \geqslant z_1$):

$$L:\begin{cases} a'_{z_1+z_2-2} = \dfrac{pa_{z_2-1}\cdots a_{z_1} - q}{p - qa_{z_2-1}\cdots a_{z_1}}\,(a_0a_1\cdots a_{z_1+z_2-2})^{-1}\,, \\ a'_{z_1+z_2-3} = a_{z_1+z_2-2}\,, \\ \vdots \\ a'_1 \quad = a_2\,, \\ a'_0 \quad = a_1\,. \end{cases} \tag{A.5}$$

The invariants of the mapping can be found evaluating $\mathrm{Tr}\,L^{z_1,z_2}(k)$, as defined in eq. (21), but now with the matrices $M^{\mathrm{hor}}(k)$ and $M^{\mathrm{vert}}(k)$ given by

$$\begin{aligned} M^{\mathrm{hor}}_{l,m}(k) &= \begin{pmatrix} p & -V_{n+z_1} \\ -k^2/V_n & pV_{n+z_1}/V_n \end{pmatrix}, \\ M^{\mathrm{vert}}_{l,m}(k) &= \begin{pmatrix} q & -V_{n+z_2} \\ -k^2/V_n & qV_{n+z_2}/V_n \end{pmatrix}. \end{aligned} \tag{A.6}$$

As examples we treat the cases $z_1 = 1$ and $z_2 = 2, 3$. (For $z_1 = z_2$, the mapping (A.3) is trivial.)

For $z_1 = 1$, $z_2 = 2$ we have the mapping

$$\begin{aligned} a'_1 &= \frac{pa_1 - q}{p - qa_1}\,(a_0a_1)^{-1}\,, \\ a'_0 &= a_1\,, \end{aligned} \tag{A.7}$$

and the invariant is given by

$$\begin{aligned} &(q^2 - k^2)\,\mathrm{Tr}\,L^{1,2}(k) \\ &= \mathrm{Tr}\begin{pmatrix} q & V_0 \\ k^2/V_2 & qV_0/V_2 \end{pmatrix}\begin{pmatrix} p & -V_2 \\ -k^2/V_1 & pV_2/V_1 \end{pmatrix}\begin{pmatrix} p & -V_1 \\ -k^2/V_0 & pV_1/V_0 \end{pmatrix} \\ &= 2p^2q + k^2[qa_1a_0 + q(a_1a_0)^{-1} - p(a_0 + a_0^{-1} + a_1 + a_1^{-1})]\,. \end{aligned} \tag{A.8}$$

For $z_1 = 1$, $z_2 = 3$, the mapping (A.5) can be reduced to a 2-dimensional mapping in terms of the variables

$$b_i := a_i a_{i+1}, \qquad i = 0, 1;$$
$$b_1' = \frac{pb_1 - q}{p - qb_1} b_0^{-1}, \tag{A.9}$$
$$b_0' = b_1 .$$

The invariant follows from

$$\begin{aligned}(q^2 - k^2) \operatorname{Tr} L^{1,3}(k) &= \operatorname{Tr}\begin{pmatrix} q & V_0 \\ k^2/V_3 & qV_0/V_3 \end{pmatrix}\begin{pmatrix} p & -V_3 \\ -k^2/V_2 & pV_3/V_2 \end{pmatrix} \\ &\quad \times \begin{pmatrix} p & -V_2 \\ -k^2/V_1 & pV_2/V_1 \end{pmatrix}\begin{pmatrix} p & -V_1 \\ -k^2/V_0 & pV_1/V_0 \end{pmatrix} \\ &= 2p^3 q - 2k^4 + k^2\{pq[b_0 + b_1 + b_0^{-1} + b_1^{-1} + b_0 b_1 + (b_0 b_1)^{-1}] \\ &\quad - p^2(b_0 + b_1 + b_0^{-1} + b_1^{-1} + b_1 b_0^{-1} + b_1^{-1} b_0)\} .\end{aligned} \tag{A.10}$$

More generally, for $z_1 = 1$ and $z_2 = 2p$, the $2p$-dimensional mapping (A.5) has p invariants. For $z_1 = 1$ and $z_2 = 2p + 1$ the mapping can be reduced to a $2p$-dimensional mapping in terms of the fields $b_i := a_i a_{i+1}$, $i = 0, \ldots, z_2 - 2$, and there are p invariants that can be expressed in terms of the b_i. The invariants for general z_1 and z_2 are treated in appendix B.

Finally it should be noted that the ΔΔ-MKdV is equivalent to the ΔΔ-SG. In fact with the substitution

$$V_{l,m} \to \begin{cases} V_{l,m}, & l \text{ even}, \\ V_{l,m}^{-1}, & l \text{ odd}, \end{cases}$$
$$p \to p^{-1}, \tag{A.11}$$

eq. (A.2) changes into the ΔΔ-SG given in eq. (2). Furthermore for even z_2, the periodicity condition (7) is invariant under the substitution (A.11). Therefore, the mapping (A.3) for the ΔΔ-MKdV is equivalent to the mapping (8) for the ΔΔ-SG, in the case of even z_2, cf. also eqs. (A.7), (A.8) and (29) and (30) in the special case $z_2 = 2$. For odd z_2, eq. (7) is not invariant and both mappings are different.

Appendix B

In this appendix we evaluate $\operatorname{Tr} L^{z_1,z_2}(k)$ for the SG hierarchy of mappings, and for the MKdV hierarchy of mappings, for general z_1 and z_2. For this purpose we consider the staircase as indicated in fig. 2 from (0,0) to $(z_2, -z_1)$ with the initial data

$$W_j = V_{n(A_j)}\,, \qquad j = 0, 1, \ldots, z_1 + z_2 - 1\,, \tag{B.1}$$

where $n(A_j)$ denotes the value of $n = z_1 l + z_2 m$ at the point with coordinates (l, m) corresponding to A_j. A horizontal step from A_j to A_{j+1} corresponds to the matrix, cf. (19),

$$\begin{aligned} S_j(k) = M_j^{\mathrm{hor}}(k) &= \begin{pmatrix} p & -W_{j+1} \\ -k^2/W_j & pW_{j+1}/W_j \end{pmatrix} \\ &= -W_{j+1}\left[\frac{k^2 - p^2}{W_j W_{j+1}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -p/W_{j+1} & 1 \\ p^2/(W_j W_{j+1}) & -p/W_j \end{pmatrix}\right]. \end{aligned} \tag{B.2}$$

A vertical step going down from A_j to A_{j+1} corresponds to the matrix, cf. (19),

$$\begin{aligned} S_j(k) = M_j^{\mathrm{vert}^{-1}}(k) &= \begin{pmatrix} qW_j/W_{j+1} & -k^{-2}/W_{j+1} \\ -W_j & q \end{pmatrix}^{-1} \\ &= -\frac{(1 - q^2k^2)^{-1}}{W_j}\left[k^2(1 - q^2k^2)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} qk^2W_{j+1} & 1 \\ q^2k^4W_jW_{j+1} & qk^2W_j \end{pmatrix}\right]. \end{aligned} \tag{B.3}$$

Both expressions can be replaced by a single one, introduce variables ρ_j and p_j such that

$$\begin{aligned} &\rho_j = 1\,, \quad p_j = p\,, \qquad \text{for a horizontal step from } A_j \text{ to } A_{j+1}\,, \\ &\rho_j = -1\,, \quad p_j = -qk^2\,, \quad \text{for a vertical step from } A_j \text{ to } A_{j+1}\,. \end{aligned} \tag{B.4}$$

Then

$$(1 - q^2k^2)^{\frac{1}{2}(1-\rho_j)} S_j(k) = -W_j^{\frac{1}{2}(1-\rho_j)} W_{j+1}^{\frac{1}{2}(1+\rho_{j+1})} \mathbf{H}_j \tag{B.5}$$

with

$$\begin{aligned} &\mathbf{H}_j = (k^2 - p_j^2)(W_jW_{j+1})^{-\rho_j}\mathbf{S}_- + \mathbf{Z}_j\,, \\ &\mathbf{S}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{Z}_j = \begin{pmatrix} -p_jW_{j+1}^{-\rho_j} & 1 \\ p_j^2(W_jW_{j+1})^{-\rho_j} & -p_jW_j^{-\rho_j} \end{pmatrix} \end{aligned} \tag{B.6}$$

and $\mathrm{Tr}\, L^{z_1,z_2}(k)$, cf. (21), is given by

$$(1-q^2k^2)^{z_1}\,\mathrm{Tr}\,L^{z_1,z_2}(k)=(1-q^2k^2)^{z_1}\,\mathrm{Tr}\overset{\curvearrowleft}{\prod_{j=0}^{z_1+z_2-1}}\mathbf{S}_j(k)$$

$$=(-1)^{z_1+z_2}\prod_{j=0}^{z_1+z_2-1}W_j^{\frac{1}{2}(\rho_j+\rho_{j-1})}\,\mathrm{Tr}\overset{\curvearrowleft}{\prod_{j=0}^{z_+z_2-1}}\mathbf{H}_j\,. \tag{B.7}$$

To evaluate $\mathrm{Tr}\prod\mathbf{H}_j$ we use an expansion in terms of the matrices $\mathbf{S}_-$ in (B.6). Because $\mathbf{S}_-^2=\mathbf{0}$ we only can have matrices $\mathbf{S}_-$ at values j belonging to a subset V with integers $i_1, i_2, \ldots, i_{n(\mathrm{V})}$ satisfying

$$\begin{aligned}&i_1\geqslant 0\,,\qquad i_{n(\mathrm{V})}\leqslant z_1+z_2-1\,,\\&i_{\nu+1}-i_\nu\geqslant 2\,,\qquad \nu=1,2,\ldots,n(\mathrm{V})\quad (i_{n(\mathrm{V})+1}\equiv i_1+z_1+z_2)\,,\end{aligned} \tag{B.8}$$

then

$$\mathrm{Tr}\overset{\curvearrowleft}{\prod_{j=0}^{z_1+z_2-1}}\mathbf{H}_j=\mathrm{Tr}\overset{\curvearrowleft}{\prod_{j=0}^{z_1+z_2-1}}\mathbf{Z}_j$$

$$+\sum_{\mathrm{V}}\prod_{\nu=1}^{n(\mathrm{V})}(k^2-p_{i_\nu}^2)(W_{i_\nu}W_{i_\nu+1})^{-\rho_{i_\nu}}\,\mathrm{Tr}\overset{\curvearrowleft}{\prod_{\nu=1}^{n(\mathrm{V})}}\left[\left(\overset{\curvearrowleft}{\prod_{j=i_\nu+1}^{i_{\nu+1}-1}}\mathbf{Z}_j\right)\cdot\mathbf{S}_-\right], \tag{B.9}$$

in which it is understood that $\mathbf{Z}_{j+z_1+z_2}=\mathbf{Z}_j$.

With the use of the property

$$\mathrm{Tr}\prod_{m=1}^{s}(\mathbf{A}_m\cdot\mathbf{S}_-)=\prod_{m=1}^{s}(A_m)_{12}\,, \tag{B.10}$$

for general $s\geqslant 1$ and 2×2 matrices $\mathbf{A}_m$, eq. (B.9) can be simplified to

$$\mathrm{Tr}\overset{\curvearrowleft}{\prod_{j=0}^{z_1+z_2-1}}\mathbf{H}_j=\mathrm{Tr}\overset{\curvearrowleft}{\prod_{j=0}^{z_1+z_2-1}}\mathbf{Z}_j$$

$$+\sum_{\mathrm{V}}\prod_{\nu=1}^{n(\mathrm{V})}(k^2-p_{i_\nu}^2)(W_{i_\nu}W_{i_{\nu+1}})^{-\rho_{i_\nu}}\prod_{\nu=1}^{n(\mathrm{V})}\left(\overset{\curvearrowleft}{\prod_{j=i_\nu+1}^{i_{\nu+1}-1}}\mathbf{Z}_j\right)_{12}. \tag{B.11}$$

The matrix elements in (B.11) can be obtained with the relation

$$\overset{\curvearrowleft}{\prod_{m=1}^{s}}\begin{pmatrix}a_m & 1\\ a_mb_m & b_m\end{pmatrix}=\left(\prod_{m=1}^{s-1}(b_m+a_{m+1})\right)\begin{pmatrix}a_1 & 1\\ b_sa_1 & b_s\end{pmatrix}. \tag{B.12}$$

This gives

$$\mathrm{Tr}\ \overset{\curvearrowleft}{\prod_{j=0}^{z_1+z_2-1}} \mathbf{Z}_j = (-1)^{z_1+z_2} \prod_{j=0}^{z_1+z_2-1} (p_{j+1} W_{j+2}^{-\rho_{j+1}} W_j^{\rho_j} + p_j) W_j^{-\rho_j} ,$$

$$\Big(\overset{\curvearrowleft}{\prod_{j=i_{\nu+1}}^{i_{\nu+1}-1}} \mathbf{Z}_j \Big)_{12} = \begin{cases} 1 & \text{if } i_{\nu+1} - i_\nu = 2 , \\ -\displaystyle\prod_{j=i_\nu+1}^{i_{\nu+1}-1} (p_{j+1} W_{j+2}^{-\rho_{j+1}} W_j^{\rho_j} + p_j) W_j^{-\rho_j} & \text{if } i_{\nu+1} - i_\nu \geqslant 3 . \end{cases} \tag{B.13}$$

Inserting (B.13), (B.11), (B.9) into (B.7) we find the general expression

$$\begin{aligned} (1-q^2k^2)^{z_1}\ \mathrm{Tr}\ L^{z_1,z_2}(k) &= \prod_{i=0}^{z_1+z_2-1} V_{n(A_i)}^{\frac{1}{2}(\rho_{i-1}-\rho_i)} \\ &\times \Big(\prod_{i=0}^{z_1+z_2-1} (p_{i+1} V_{n(A_{i+2})}^{-\rho_{i+1}} V_{n(A_i)}^{\rho_i} + p_i) \\ &+ \sum_{\mathrm{V}} \prod_{j\in \mathrm{V}} (k^2 - p_j^2) V_{n(A_{j+1})}^{-\rho_j} V_{n(A_{j-1})}^{\rho_j - 1} \prod_{i,i+1\notin \mathrm{V}} (p_{i+1} V_{n(A_{i+2})}^{-\rho_{i+1}} V_{n(A_i)}^{\rho_i} + p_i) \Big) \end{aligned} \tag{B.14}$$

with V and p_j given by (B.8) and (B.4).

For $z_1 + z_2$ odd, the right-hand side of (B.14) can be expressed in terms of the variables $W_i \equiv V_i V_{i+1}$, $i = 0, 1, \ldots, z_1 + z_2 - 2$. In fact, if $\rho_j = \rho_{j+1}$ we have $\{V_{n(A_{j+2})}/V_{n(A_j)}\}^{-\rho_{j+1}}$ with $n(A_{j+2}) - n(A_j)$ even, and if $\rho_j = -\rho_{j+1}$ we have $(V_{n(A_{j+2})} V_{n(A_j)})^{-\rho_{j+1}}$ with $n(A_{j+2}) - n(A_j)$ odd.

For odd z_1 and z_2 ($\geqslant z_1$) one finds nonvanishing contributions from the powers $k^{z_1+1}, k^{z_1+3}, \ldots, k^{2z_1+z_2-1}$, thus providing the $\frac{1}{2}(z_1+z_2)$ integrals of the (z_1+z_2)-dimensional mapping. For z_1 odd and z_2 even, nonvanishing contributions arise from the powers $k^{z_1+1}, k^{z_1+3}, \ldots, k^{2z_1+z_2}$, but the coefficient of $k^{2z_1+z_2}$ is a trivial constant. For z_1 even and z_2 odd there are nonvanishing contributions from the powers $k^{z_1}, k^{z_1+2}, \ldots, k^{2z_1+z_2-1}$, with the coefficient of k^{z_1} being trivial. In both cases with z_1+z_2 odd there are $\frac{1}{2}(z_1+z_2-1)$ nontrivial integrals for the (z_1+z_2-1)-dimensional mapping in terms of the variables $W_i = V_i V_{i+1}$.

For the MKdV the evaluation of $\mathrm{Tr}\ L^{z_1,z_2}(k)$ is also straightforward. We consider again the staircase of fig. 2 with the notation (B.1). From the Lax representation (A.6) it follows that the $S_j(k)$ for the horizontal step from A_j to A_{j+1} is still given by (B.2), but for the vertical step from A_j to A_{j+1} we have now

$$S_j(k) = \begin{pmatrix} q & -W_j \\ -k^2/W_{j+1} & qW_j/W_{j+1} \end{pmatrix}^{-1}$$
$$= -(k^2-q^2)^{-1} W_{j+1} \left[\frac{k^2-q^2}{W_j W_{j+1}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} q/W_{j+1} & 1 \\ q^2/W_j W_{j+1} & q/W_j \end{pmatrix} \right]. \tag{B.15}$$

Eqs. (B.2) and (B.15) can be combined to

$$(k^2-q^2)^{\frac{1}{2}(1-\rho_j)} S_j(k) = -W_{j+1} \left[(k^2-p_j^2)(W_j W_{j+1})^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -p_j/W_{j+1} & 1 \\ p_j^2/(W_j W_{j+1}) & -p_j/W_j \end{pmatrix} \right] \tag{B.16}$$

with $p_j = p$, if $\rho_j = 1$ and $p_j = -q$, if $\rho_j = -1$. The right-hand side of (B.16) is the special case of (B.5) and (B.6) with $\rho_j = 1$ and keeping p_j. Then from (B.14) we have the invariant

$$(k^2-q^2)^{z_1} \operatorname{Tr} L^{z_1, z_2}(k) = \prod_{i=0}^{z_1+z_2-1} (p_{i+1} V^{-1}_{n(A_{i+2})} V_{n(A_i)} + p_i)$$
$$+ \sum_{\mathrm{V}} \prod_{j \in \mathrm{V}} (k^2 - p_j^2) V^{-1}_{n(A_{j+1})} V_{n(A_{j-1})} \prod_{i, i+1 \notin \mathrm{V}} (p_{i+1} V^{-1}_{n(A_{i+2})} V_{n(A_i)} + p_i) \tag{B.17}$$

with V defined by (B.8) and $p_j = p$ or $p_j = -q$ depending on whether the step between A_j and A_{j+1} is a horizontal or vertical step, respectively. From eq. (B.17) it is clear that the right-hand side can be expressed in terms of the variables $a_i = V_{i+1}/V_i$, $i = 0, 1, \ldots, z_1 + z_2 - 2$. Furthermore when $z_1 + z_2$ is even, $n(A_{i+2}) - n(A_i)$ is even, and (B.17) can be expressed in terms of the variables $b_i = a_i a_{i+1}$, $i = 0, 1, \ldots, z_1 + z_2 - 3$. This is also the case with the mapping (A.3). For $z_1 + z_2 =$ odd it can be expressed as a $(z_2 + z_1 - 1)$-dimensional mapping in the variables a_i and for $z_1 + z_2 =$ even it reduces to a $(z_1 + z_2 - 2)$-dimensional mapping for the variables b_i. Furthermore the right-hand side of (B.17) yields nonvanishing coefficients of the powers k^0, $k^2, \ldots, k^{(z_1+z_2-1)}$ when $z_1 + z_2$ is odd, and of the powers k^0, $k^2, \ldots, k^{z_1+z_2}$ when $z_1 + z_2$ is even. The coefficient of k^0 is always a trivial constant $2p^{z_2}q^{z_1}$ and when $z_1 + z_2$ is even, the coefficient of $k^{z_1+z_2}$ is 2. Therefore we find $\frac{1}{2}(z_1 + z_2 - 1)$ nontrivial integrals if $z_1 + z_2 =$ odd, and $\frac{1}{2}(z_1 + z_2 - 2)$ nontrivial integrals if $z_1 + z_2 =$ even.

Appendix C

In this appendix we evaluate integrals for mappings of the ΔΔ-KdV hierarchy for general z_1 and z_2. For the ΔΔ-KdV we have the Lax representation

$$(p-k)\begin{pmatrix} U_{l+1,m}(k) \\ V_{l+1,m}(k) \end{pmatrix} = \mathbf{M}^{\text{hor}}_{l,m}(k)\cdot\begin{pmatrix} U_{l,m}(k) \\ V_{l,m}(k) \end{pmatrix},$$
$$(q-k)\begin{pmatrix} U_{l,m+1}(k) \\ V_{l,m+1}(k) \end{pmatrix} = \mathbf{M}^{\text{vert}}_{l,m}(k)\cdot\begin{pmatrix} U_{l,m}(k) \\ V_{l,m}(k) \end{pmatrix}, \tag{C.1}$$

with

$$\mathbf{M}^{\text{hor}}_{l,m}(k) = \begin{pmatrix} p-U_{l+1,m} & 1 \\ k^2-p^2+(p+U_{l,m})(p-U_{l+1,m}) & p+U_{l,m} \end{pmatrix},$$
$$\mathbf{M}^{\text{vert}}_{l,m}(k) = \begin{pmatrix} q-U_{l,m+1} & 1 \\ k^2-q^2+(q+U_{l,m})(q-U_{l,m+1}) & q+U_{l,m} \end{pmatrix}. \tag{C.2}$$

The compatibility of (C.1) yields

$$(p+q+U_{l,m}-U_{l+1,m+1})(q-p+U_{l+1,m}-U_{l,m+1}) = q^2-p^2\,. \tag{C.3}$$

For the travelling wave solutions of the type (5), (6) we have the mapping

$$L:\begin{cases} U'_{z_1+z_2-1} = p+q+U_0-\dfrac{q^2-p^2}{q-p+U_{z_1}-U_{z_2}}\,, \\ U'_{z_1+z_2-2} = U_{z_1+z_2-1}\,, \\ \vdots \\ U'_1 = U_2\,, \\ U'_0 = U_1\,. \end{cases} \tag{C.4}$$

This mapping is reversible since $LGL = G$ with the involution G given by

$$G:\begin{cases} U'_0 = -U_{z_1+z_2-1}\,, \\ U'_1 = -U_{z_1+z_2-2}\,, \\ \vdots \\ U'_{x_1+z_2-2} = -U_1\,, \\ U'_{z_1+z_2-1} = -U_0\,. \end{cases} \tag{C.5}$$

Furthermore the mapping is volume preserving, the Jacobian determinant having absolute value 1.

Introducing

$$U_j - U_{j+1} =: b_j\,, \qquad j = 0, 1, \ldots, z_2 + z_1 - 2\,, \tag{C.6}$$

and taking $z_2 > z_1$, we obtain the $(z_1 + z_2 - 1)$-dimensional mapping

$$L: \begin{cases} b'_{z_1+z_2-2} = -(b_0 + b_1 + \ldots b_{z_1+z_2-2}) - (p+q) \\ \qquad\qquad + \dfrac{q^2 - p^2}{q - p + b_{z_1} + \ldots b_{z_2-1}}\,, \\ b'_{z_1+z_2-3} = b_{z_1+z_2-2}\,, \\ \vdots \\ b'_1 \qquad = b_2\,, \\ b'_0 \qquad = b_1\,. \end{cases} \tag{C.7}$$

To evaluate the invariants we consider again Tr $L^{z_1,z_2}(k)$ as given by eq. (21) along the staircase of fig. 2 with the fields $W_j = U_{n(A_j)}$, $j = 0, 1, , \ldots, z_1 + z_2 - 1$. A horizontal step from A_j to A_{j+1} corresponds to the matrix

$$\mathbf{S}_j(k) = \mathbf{M}_j^{\text{hor}}(k) = \begin{pmatrix} p - W_{j+1} & 1 \\ k^2 - p^2 + (p + W_j)(p - W_{j+1}) & p + W_j \end{pmatrix} \tag{C.8}$$

and a vertical step down from A_j to A_{j+1} corresponds to

$$\begin{aligned} \mathbf{S}_j(k) &= \mathbf{M}_j^{\text{vert}\ -1}(k) \\ &= \frac{1}{k^2 - q^2}\begin{pmatrix} -q - W_{j+1} & 1 \\ k^2 - q^2 + (-q + W_j)(-q - W_{j+1}) & -q + W_j \end{pmatrix}, \end{aligned} \tag{C.9}$$

cf. eqs (B.15) and (B.16) with the Lax representation (A.6) replaced by (C.2). Eqs. (C.8) and (C.9) can be combined to

$$(k^2 - q^2)^{\frac{1}{2}(1-\rho_j)}\mathbf{S}_j(k) = (k^2 - p_j^2)\cdot\mathbf{S}_- + \mathbf{Z}_j \tag{C.10}$$

with

$\rho_j = 1, \quad p_j = p$ for a horizontal step from A_j to A_{j+1},

$\rho_j = -1\,, \quad p_j = -q$ for a vertical step from A_j to A_{j+1},

and

$$\mathbf{Z}_j = \begin{pmatrix} p_j - W_{j+1} & 1 \\ (p_j - W_{j+1})(p_j + W_j) & p_j + W_j \end{pmatrix} . \tag{C.11}$$

Because of (C.10) and (C.11), Tr $L^{z_1,z_2}(k)$ can be evaluated following the treatment of appendix B, cf. eqs. (B.8)–(B.13). We find

$$(k^2 - q^2)^{z_1} \operatorname{Tr} L^{z_1,z_2}(k) = \prod_{j=0}^{z_1+z_2-1} (p_j + p_{j+1} + U_{n(A_j)} - U_{n(A_{j+2})})$$
$$+ \sum_{V} \prod_{j\in V} (k^2 - p_j^2) \prod_{i,i+1\notin V} (p_i + p_{i+1} + U_{n(A_i)} - U_{n(A_{i+2})}) . \tag{C.12}$$

The right-hand side of (C.12) can be expressed in terms of the fields $b_j = U_j - U_{j+1}$ for arbitrary z_1 and z_2, and in terms of the fields $c_j = b_j + b_{j+1}$, $j = 0, 1, \ldots, z_1 + z_2 - 3$, if $z_1 + z_2$ is even. In fact, we always have $n(A_i) - n(A_{i+2}) =$ even if $z_1 + z_2$ is even.

The integrals given in eqs. (C.12) and eqs. (B.13) are directly related to each other via the Miura transformation connecting the solutions of the ΔΔ-KdV and the ΔΔ-MKdV. This relation has been mentioned in ref. [1] in connection with the mappings that can be obtained in the case $z_1 = z_2 = 1$ for general P, cf. eqs. (33)–(36). From the Lax representation (C.1), (C.2) for the ΔΔ-KdV one finds that $V_{l,m} = U_{l,m}(0)$, i.e. the wave function for the special value $k = 0$, satisfies the ΔΔ-MKdV as given in eq. (A.2), in combination with the Miura transformation

$$p - q + U_{l,m+1} - U_{l+1,m} = \frac{pV_{l+1,m} - qV_{l,m+1}}{V_{l,m}} = \frac{pV_{l,m+1} - qV_{l+1,m}}{V_{l+1,m}} , \tag{C.13}$$

cf. eq. (A.2). From eqs. (C.13), (C.3) and (A.2) one also finds that

$$2p + U_{l,m} - U_{l+2,m} = \frac{p(V_{l+2,m} + V_{l,m})}{V_{l+1,m}} . \tag{C.14}$$

Both equations (C.13) and (C.14) can be combined to

$$p_j + p_{j+1} + U_{n(A_j)} - U_{n(A_{j+2})} = (p_{j+1} V_{n(A_{j+2})} V^{-1}_{n(A_j)} + p_j) V^{-1}_{n(A_{j+1})} V_{n(A_j)} . \tag{C.15}$$

Inserting eq. (C.15) into eq. (C.12) one obtains eq. (B.17) with $V_{n(A_j)}$ replaced by $V^{-1}_{n(A_j)}$, but the mapping (A.3) is invariant under the replacement $V_n \to V_n^{-1}$, so is Tr $L^{z_1,z_2}(k)$. This means that the integrals of the MKdV and the KdV mappings given in (B.17) and (C.12) are related via the Miura transformation.

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