LOCAL REVERSIBILITY IN DYNAMICAL SYSTEMS

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A reversible mapping is the product of two involutions. In this Letter we introduce a local version of reversibility in which this product property is satisfied up to a certain order in an expansion around a fixed point. We derive some necessary conditions for local reversibility, and given an example how these conditions may be applied as negative criteria to show that a given mapping is not globally reversible.

1. A dynamical system (not necessarily Hamiltonian or symplectic) is called reversible if there is an involution in phase space which reverses the direction of time [1]. Historically, reversibility of dynamical systems has played, and still plays, an important role in physics. Many of the equations of physics are time-reversible. This was first noticed by Loschmidt for particles moving in a velocity-independent forcefield. Boltzmann soon realised the importance of time-reversibility and showed that the Maxwell equations are also reversible (if one also takes $B \to -B$) [2]. Also the Einstein equations are reversible [3]. A very readable account of the role of time reversal in physics is given in ref. [4].

Only more recently has reversibility been studied in dynamical systems that are not Hamiltonian or symplectic. Moser proved that Kolmogorov–Arnol'd–Moser tori also exist in reversible non-Hamiltonian flows [5,6]. Further important work was done by Devaney who gave a coordinate invariant definition of reversibility [1]. Very recently reversible systems that are not symplectic or Hamiltonian have come to the fore in a variety of contexts [7–20].

Whereas it is easy to check if a certain system is reversible under a given involution, the more important question: "Is there an involution under which a given system is reversible?" is more difficult to answer. Apart from the classification of dynamical systems this question is of interest because reversibility is heuristically found to be a necessary condition for a dynamical system to be integrable [21]. In this Letter we address the above question for the case of mappings of the plane. (These mappings can be thought of e.g. as arising from a Poincaré section of the flow of an ordinary differential equation.)

In section 2 we introduce the new concept of local reversibility. A mapping is defined to be locally reversible if, to a certain order, its Taylor expansion about a fixed point is equal to the composition of the expansions of two involutions about a common fixed point. In section 3 we give an example showing how the necessary conditions of a local reversibility may be used as negative criteria to show that a given system is *not* reversible. Some aspects of the relationships between local reversibility and a local form of measure preservation are discussed in section 4.

2. *Definition.* A mapping L: $\mathbb{R}^2 \to \mathbb{R}^2$ is reversible if it can be written as the composition of two involutions

$$L = W \circ V , \tag{1}$$

where

$$V \circ V = \mathrm{Id} , \quad W \circ W = \mathrm{Id} . \tag{2}$$

Some general properties of reversible mappings are:

(i) if L is reversible, so is L^{z_1}, $\forall z_1 \in \mathbb{Z}$;

(ii) if V is a reversing involution of L, so is $L^{z_2} V$, $\forall z_2 \in \mathbb{Z}$;

(iii) reversibility is invariant under coordinate transformations.

Very generally we now take the (orientation reversing) involution W to be

$$W = T^{-1} \circ U \circ T , \tag{3}$$

where U is given by

$$U: \quad x' = x , \quad y' = -y \tag{4}$$

and T is an arbitrary coordinate transformation (locally such a T exists if $\mathrm{d}W = U$, it is given by $T = U + W$). Eq. (1) can then be rewritten

$$T \circ L = U \circ T \circ V . \tag{5}$$

We assume the mapping L has a symmetric fixed point $\boldsymbol{x}_0$, i.e. a point for which $L\boldsymbol{x}_0 = V\boldsymbol{x}_0 = \boldsymbol{x}_0$. (It is sufficient for a fixed point to be symmetric if it has Jacobian determinant equal to 1 (see below) and if there is no other fixed point with the same eigenvalues.) Without loss of generality we take $\boldsymbol{x}_0 = T\boldsymbol{x}_0 = \boldsymbol{0}$ and use the following formal expansion for the mapping L about the fixed point $\boldsymbol{0}$,

$$\begin{aligned} L: \quad x' &= Ax + By + Ex^2 + Fxy + Gy^2 + Lx^3 + Mx^2y \\ &\quad + Nxy^2 + Oy^3 + \dots , \\ y' &= Cx + Dy + Hx^2 + Jxy + Ky^2 + Px^3 + Qx^2y \\ &\quad + Rxy^2 + Sy^2 + \dots . \end{aligned} \tag{6}$$

For T and V we use analogous general expansions with coefficients a, ..., s and $\hat{a}$, ..., $\hat{s}$ respectively, instead of A, ..., S. Equating like terms on the left hand side and the right hand side of eq. (5) it is then a matter of straightforward albeit rather tedious algebra to eliminate a,..., s and $\hat{a}$, ... $\hat{s}$ from the ensuing equations, and thus obtain necessary conditions on the coefficients of the mapping L ensuring the existence of solutions for V and T satisfying eq. (5) up to n'th order terms in x and y. If these conditions are satisfied we call the mapping L *locally* reversible up to order n at the fixed point $\boldsymbol{0}$. In this Letter we present the results up to and including the order $n = 3$. (We restrict ourselves to the case that the involution V is orientation reversing.) Our aim will be to show a system is not globally reversible, by showing it is not locally reversible.

At first order we get the necessary condition on the mapping L that the determinant of its linear part is 1:

$$AD - BC = 1 . \tag{7}$$

Taking this into account the linear part of L can be brought into one of the three following Jordan normal forms,

$$(1) \quad \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} , \qquad (2) \quad \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} , \qquad (3) \quad \begin{pmatrix} -1 & B \\ 0 & -1 \end{pmatrix} . \tag{8}$$

(We will assume in this Letter that $B \neq 0$.) The necessary conditions that we then derive on the quadratic and cubic coefficients of the mapping L (after this linear transformation) are given in table 1. It turns out that in the case that L has two eigenvalues $+1$ one has to distinguish two subcases depending on whether the linear part of V is

$$(2a) \quad \begin{pmatrix} 1 & \hat{b} \\ 0 & -1 \end{pmatrix} , \qquad (2b) \quad \begin{pmatrix} -1 & \hat{b} \\ 0 & 1 \end{pmatrix} .$$

So far, we have given the necessary conditions for local reversibility up to third order. In the next section we present an example showing how such conditions may be used to prove a mapping is not globally reversible. A systematic approach to obtain conditions at higher order might be to use the theory of normal forms. For example, in each of the different cases as determined by the linear part of the

Table 1
Necessary conditions for local reversibility of a mapping L given by the expansion (6), after a linear transformation to Jordan normal form. The first column gives the Jordan normal form of the linear part of L in the various cases. In the third column we specify the necessary conditions at second order. The third order conditions are given in the fourth column.

	Case	Second order	Third order
$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}, A\neq\pm1$	(1)	no condition	$(A-1)(AR+A^{-1}M-FJ-2GH)$ $+EF(A^{-2}-2A^{-1})+JK(2A^{2}-A^{3})=0$
$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$	(2a)	$J+2E-2BH=0$	if $H=0$ then $3B^{2}P-BQ-3BL+2EK+EF-3BEJ=0$
	(2b)	$H=0$	$6B^{2}P-2BQ-6BL+2EK+BEJ$ $+6BE^{2}+2BJ^{2}-FJ-JK=0$
$\begin{pmatrix} -1 & B \\ 0 & -1 \end{pmatrix}$	(3)	no condition	$12BP+4Q+12L+2J^{2}+2EJ+22BEH+10B^{2}H^{2}$ $+10FH+11BHJ+4HK+12E^{2}=0$

mapping one could make a transformation to the corresponding nonlinear normal form [22–24], and then work out the necessary conditions for local reversibility on the coefficients in the normal form. Alternatively, one can start from a reversible system in normal form (see refs. [25–28] and ref. [9], ch.2), apply an arbitrary coordinate transformation, and determine what conditions one gets on the ensuing mapping. At higher order we expect to obtain extra conditions when the eigenvalue A is a root of unity (corresponding to extra resonant terms in the normal form expansion).

3. Consider the mapping

$$L:\quad w'=w+\omega(z)\quad (\text{modulo } 1),$$

$$z'=\frac{z+(k/2\pi)\sin(2\pi w')}{1-zh(w')}\tag{9}$$

(cf. ref. [29]). For $h(w')=0$ and $\omega(z)=z$ eq. (9) reduces to the well-known (area-preserving) Chirikov–Taylor or standard map. The mapping (9) is reversible when $h(w')$ is an odd function or when $\omega(z)$ is an odd function [14]. We now consider the case when h and ω are *not* odd but both contain an even part, i.e.

$$\omega(z)=\omega_{\text{odd}}(z)+\omega_2 z^2\,,$$

$$h(w)=h_0+h_{\text{odd}}(w)\,,\tag{10}$$

where ω_{odd} and h_{odd} are arbitrary odd functions and ω_2 and h_0 are constants. We want to know whether the mapping (9) with ω and h given by (10) is reversible.The mapping (9) has fixed points under the conditions $\omega(z)\in\mathbb{Z}$, $(k/2\pi)\sin(2\pi w)=-z^2h(w)$. For $z=0$ we have fixed points at $w=0$ and $w=\frac{1}{2}$. Expanding ω and h about 0,

$$\omega(z)=\omega_1 z+\omega_2 z^2+\omega_3 z^3+\mathrm{O}(z^5)\,,$$

$$h(w)=h_0(1+\eta w)+\mathrm{O}(w^3)\,,\tag{11}$$

the eigenvalues of $\mathrm{d}L$ at $\omega=z=0$ are determined by $(1-A)^2=kA\omega_1$ (hence $\det \mathrm{d}L=1$). These eigenvalues are different from those at $w=\frac{1}{2}$, $z=0$ if $k\omega_1\neq0$, and are in general also different from the eigenvalues at possible fixed points with $z\neq0$. Therefore, if the mapping (9) is to be reversible, $w=z=0$ should be a symmetric fixed point, and the conditions for local reversibility should all be satisfied.

Applying a diagonalising transformation

$$w=\frac{A^{-1}x+y}{1+A^{-1}}\,,$$

$$z=\frac{(1-A^{-1})\omega_1^{-1}(x-y)}{1+A^{-1}}\,,\quad (1-A)^2=kA\omega_1\,,\tag{12}$$

we obtain eq. (6) with

$$B=C=0\,,\quad D=A^{-1}\,,$$

$$E=\frac{A(\Omega-\epsilon)}{1+A^{-1}}\,,\quad F=\frac{\epsilon(A+A^{-1})-2A\Omega}{1+A^{-1}}\,,$$

$$G=\frac{A\Omega-A^{-1}\epsilon}{1+A^{-1}}\,,\quad H=\frac{A^{-1}\Omega+\epsilon}{1+A^{-1}}\,,$$

$$J=\frac{-A^{-1}[2\Omega+\epsilon(A+A^{-1})]}{1+A^{-1}},$$

$$K=\frac{A^{-1}(\Omega+A^{-1}\epsilon)}{1+A^{-1}},$$

$$A^{-1}M+AR$$
$$=\frac{3\Omega\epsilon(2-A-A^{-1})-\epsilon^2(A+2+2A^{-1}+A^{-2})}{1+A^{-1}}, \tag{13}$$

where

$$\Omega:=\frac{\omega_2(2-A^{-1})^2\omega_1^{-2}}{1+A^{-1}},$$

$$\epsilon:=\frac{-h_0(1-A^{-1})\omega_1^{-1}}{1+A^{-1}}.$$

With eq. (13) the third-order local reversibility condition of case (1) in table 1 (with $A\neq\pm1$) reduces to $\Omega\epsilon=0$. Therefore, the mapping L is not reversible if $\omega_2 h_0\neq0$, i.e. when both $\omega(z)$ and $h(w')$ contain an even part.

Similar considerations can be applied to other mappings.

4. A mapping L: $\mathbb{R}^2\to\mathbb{R}^2$ is measure preserving [30] when there is a function $F(x, y)$ such that the integral $\int \mathrm{d}x\,\mathrm{d}y\, 1/F(x, y)$ over an arbitrary domain D is invariant under the mapping, or equivalently, when there is a function $F(x, y)$ such that $F(x', y')=J(x, y)F(x, y)$ with $J(x, y)$ being the Jacobian determinant of the mapping. If a mapping L is measure preserving so is L^z, $\forall z\in\mathbb{Z}$. Furthermore measure preservation is invariant under coordinate transformations: for the mapping $\bar{L}=T^{-1}\circ L\circ T$, the new function $\bar{F}$ is given by $\bar{F}=F/\det \mathrm{d}T$, in which $\det \mathrm{d}T$ is the Jacobian determinant of T.

We can now expand an arbitrary mapping L around a fixed point $\boldsymbol{x}_0$, which we take to be at $\boldsymbol{x}_0=0$, using eq. (6). Assuming $F(x, y)$ to be analytic, we can expand it, as well as the Jacobian $J(x, y)$, in a similar Taylor series starting with nonzero constant term (the $(n-1)$st order coefficients $J(x, y)$ being determined by the n'th order coefficients of the mapping L). We can then eliminate the coefficients of the measure $F(x, y)$, yielding necessary conditions on the coefficients of the mapping L. If these conditions are satisfied we call the mapping L locally measure preserving up to the order n at the fixed point.

At first order the determinant of the linear part of L must be unity, i.e. $AD-BC=1$, the necessary conditions for the quadratic and cubic coefficients are the ones given in table 1, in the cases (1), (2a), and (3). Thus local measure preservation up to 3rd order implies local reversibility. (In non-resonant cases with $A^n\neq1$, $\forall n\in\mathbb{Z}_+$ this property is known to hold to arbitrary order as can be shown e.g. by performing a transformation to a locally area preserving system and a subsequent transformation to a Birkhoff normal form [25,26,30] i.e. $x'=y/f(x, y)$, $y'=x/f(x, y)$. In resonant cases the situation is less clear and a systematic investigation of higher orders would be worthwhile. This problem, however, is outside the scope of the present Letter.) The converse is not true, cf. cases (2b) of table 1, in which the condition $H=0$ for local reversibility is not related to a local form of measure preservation.

5. *Some remarks.* (i) The necessary conditions for local reversibility play a role in the bifurcation analysis of dynamical systems [20]. In second order, e.g., a mapping L with $\mathrm{d}L$ as in case (2) is locally reversible under the condition $H=0$, also when there is no measure preservation, i.e. $J+2E-2BH\neq0$. This case corresponds to a general type of Rimmer bifurcation [20] in which there are two bifurcating solutions, one with Jacobian $\geqslant1$, the other with Jacobian $\leqslant1$ (see also refs. [31–33]). Such bifurcations are of interest, because of the interplay of dissipative and conservative features. Furthermore, the condition $H=0$ for maps with linear part $\left(\begin{smallmatrix}1 & B\\ 0 & 1\end{smallmatrix}\right)$ is a necessary condition for nonlinear stability of the fixed point [23,24].

(ii) The case in which both $H=0$ and $J+2E-2BH=0$ are satisfied for a mapping with linear part $\left(\begin{smallmatrix}1 & B\\ 0 & 1\end{smallmatrix}\right)$ is (in general) related to a period doubling bifurcation. In fact, with a mapping L, with $\mathrm{d}L$ as in case (3), both conditions are automatically satisfied for the mapping L^2. The two bifurcating fixed points of L^2 both have Jacobian $\geqslant1$ or both <1.

(iii) In case 1 with $A\neq\pm1$, the Taylor coefficients after the linear transformations as they appear in table 1 are not real. Nevertheless, despite appearances, the condition given is equivalent to the real condition.

(iv) For mappings in $\mathbb{R}^m \to \mathbb{R}^m$ with $m>2$ and $A^n \neq 1$, being locally symplectic is in general not equivalent to being locally reversible, in contrast to the case $m=2$.

(v) It would seem that an approach similar to the one presented in this Letter should be applicable to derive necessary conditions for reversibility in ordinary differential equations. We hope to report on this in the future.

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