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REVERSIBLE MAPPINGS OF THE PLANE

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The critical exponents for period-doubling and for the breaking of KAM-tori in areapreserving reversible systems are known. We show that symmetric orbits in non-areapreserving reversible systems have the same exponents.

1. Reversible dynamical systems have been shown to be *qualitatively* very similar to hamiltonian systems [1–8]. In particular, KAM theorems hold for reversible flows [1] and mappings [4,5,7]. A major qualitative difference is that reversible systems can possess attractors whereas conservative systems cannot.

In this Letter we explore the *quantitative* similarities between reversible mappings of the plane and areapreserving mappings of the plane numerically. That is to say we study period-doubling and the breaking of KAM-tori in reversible mappings that are not areapreserving and (in some cases) possess attractors. Consider the Venn-diagram in fig. 1. To our knowledge [#1] numerical studies of period-doubling and the breaking of KAM-tori have only been done on a rather restricted class of systems, namely systems that are *both* areapreserving and reversible, i.e. region II of fig. 1[#2]. The systems we study here are in region I of fig. 1.

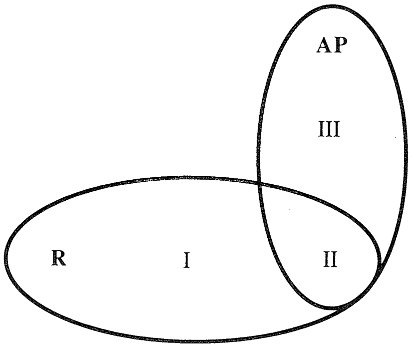

Fig. 1. R denotes reversible systems, AP denotes areapreserving systems.

2. The method we use for determining the breaking of KAM-tori with "noble" winding number $\omega_\infty = (3-\sqrt{5})/2$ is the method Greene developed for areapreserving maps [12–16]. We approximate ω_∞ successively by $\omega_i = F_i/F_{i+2}$, where F_i are the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13.... For each ω_i we find a symmetric periodic orbit (this will be defined below) with that winding number and calculate the 2 eigenvalues λ_i and μ_i of the linearization about that orbit. Since $\lambda_i \mu_i = 1$ for a symmetric periodic orbit the orbit can be characterised solely by its residue

$$R_i := \tfrac{1}{4}(2-\lambda_i-\mu_i)\ . \qquad (1)$$

[#1] With the exception of an unpublished calculation [9] of the Feigenbaum exponent δ for Rannou's map [10].

[#2] The reason that reversible systems are nice to study is that they possess a symmetry which makes it easier to find their symmetric periodic orbits numerically [11].

Table 1
Three reversible mappings of the plane we have studied numerically: $x' = [f_1(y)+x]/[1-xf_3(y)]$, $y' = [g_1(x')+y]/[1-yg_3(x')]$.

	Example 1	Example 2	Example 3
$f_1(x)$	$-x+1.5\epsilon^2x^3-0.5\epsilon^4x^5$	$-x/(1+\epsilon^2x^2)$	$-x/(1+\epsilon^2x^2)$
$f_3(x)$	0	$\epsilon^2x^3/(\epsilon^2+x^2)$	$\epsilon^2x^3/(\epsilon^2+x^2)$
$g_1(x)$	$2x-2Cx-2x^2$	$2x-2Cx-2x^2$	$2x-2Cx-2x^2$
$g_3(x)$	$2C\epsilon^2x$	0	$-\epsilon$
ϵ	0.05	0.02	0.02

We then plot R_i as a function of a parameter C of the reversible mapping and calculate the intersection points of R_i and R_{i+1}. These intersection points, and their associated values of C, converge to R_{cr} and C_{cr} with convergence rates δ_R and δ_C respectively. R_{cr}, δ_R and δ_C are universal numbers whereas C_{cr} is not [17]. Greene's criterion for areapreserving maps then says that C_{cr} is the parameter value where the KAM torus of winding number ω_∞ breaks. We conjecture that this is also true for reversible systems.

We also study the period-doubling of symmetric periodic orbits in reversible maps. (For areapreserving systems cf. refs. [18,19].) We calculate the parameter values C_i at which period-doublings occur. These parameter values converge at a rate δ_{PD}. Finally we calculate the orbit-scaling factors α_{PD} and β_{PD}.

3. A mapping L is called reversible [3,11,20] if there is a symmetric G such that [21]

$$\mathrm{L}\circ\mathrm{G}\circ\mathrm{L}=\mathrm{G} \tag{2}$$

and G is an involution

$$\mathrm{G}\circ\mathrm{G}=\mathrm{Id}\,. \tag{3}$$

We call an orbit symmetric if it contains a point x such that $\mathrm{G}x=x$. Note that it follows from (2) and (3) that L is the product of two involutions

$$\mathrm{L}=\mathrm{H}\circ\mathrm{G},\quad \mathrm{H}\circ\mathrm{H}=\mathrm{G}\circ\mathrm{G}=\mathrm{Id}\,, \tag{4}$$

where $\mathrm{H}=\mathrm{L}\circ\mathrm{G}$.

Consider the class of reversible mappings given by

$$\mathrm{L}:\quad x'=\frac{f_1(y)+x}{1-xf_3(y)},\quad y'=\frac{g_1(x')+y}{1-yg_3(x')}\,, \tag{5}$$

with symmetries

$$\mathrm{H}:\quad x'=x,\quad y'=\frac{g_1(x)-y}{1+yg_3(x)}\,,$$

$$\mathrm{G}:\quad x'=\frac{f_1(y)+x}{1-xf_3(y)},\quad y'=-y\,, \tag{6}$$

where g_1 and g_3 are completely arbitrary functions and f_1 and f_3 are arbitrary odd functions.

Tables 1 and 2 summarise our numerical results for 3 reversible perturbations of the areapreserving Hénon map:

Table 2
Numerical results for period-doubling and the breaking of KAM-tori in reversible mappings of the plane.

	Example 1	Example 2	Example 3	Areapreserving reversible maps [17,18]
R_{cr}	0.25008...	0.25008...		0.2500888...
C_{cr}	−0.8395555030...	2.841404026...		
δ_R	−1.637...	−1.63...		−1.6371161...
δ_C	−2.66...	−2.665...		−2.6651429...
δ_{PD}		8.7210972...	8.72109...	8.721097200...
α_{PD}		−4.0180767...	−4.01807...	−4.018076704...
β_{PD}		16.3639...	16.363...	16.363896879...

$$x' = -y + x\,,$$
$$y' = 2x' - 2Cx' - 2x'^2 + y \qquad (7)$$

(this map is equivalent to the one given by Hénon [23]). These 3 examples have been constructed such that they possess (asymmetric) attracting periodic orbits. Hence these mappings cannot be conjugate to areapreserving ones: example 1 has an attracting fixed point at $(1, +20)$; example 2 has an attracting fixed point which e.g. for $C = -1.266$ is at $(2.266, -1102.1...)$; in example 3 we have numerically found an attracting 2 cycle which e.g. for $C = -1.266$ is given by $(-21.9418664..., -452.17714...)$, $(5.551303..., 60.750913...)$. Although these mappings do have singularities elsewhere in the phase plane, this does not seem to affect our numerics (cf. ref. [24]).

4. We infer from table 2 that symmetric orbits in non-areapreserving reversible maps and areapreserving reversible maps (regions I and II in fig. 1 respectively) exhibit identical universal behaviour as far as period-doubling and the breaking of KAM-tori are concerned. The evidence in ref. [9] would indicate that non-reversible areapreserving maps (region III in fig. 1) might also belong to the same universality class.

In principle the definition (2), (3) of reversible mappings can be generalised to the case where the symmetry G is not an involution [5,7,25]. We hope to report on this in the future.

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