INTEGRABLE MAPPINGS AND SOLITON EQUATIONS

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We report an 18-parameter family of integrable reversible mappings of the plane. These mappings are shown to occur in soliton theory and in statistical mechanics. We conjecture that all autonomous reductions of differential–difference soliton equations are integrable mappings.

In the theory of nonlinear dynamical systems infinite-dimensional integrable equations (solitons) and low-dimensional mappings ("chaos") have received a lot of attention. In this paper we attempt to make a link between these two areas. This link is provided by the observation that simple (e.g. stationary) solutions to many different soliton equations lead to integrable mappings. In this Letter we give some examples of integrable mappings and how they arise. We also report an 18-parameter family of integrable planar mappings. Finally we conjecture that generally simple solutions of differential–difference (DΔ) soliton equations lead to integrable mappings.

Consider the DΔ isotropic Heisenberg spin chain [1]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{S}_n=\frac{\boldsymbol{S}_n\times\boldsymbol{S}_{n+1}}{1+\boldsymbol{S}_n\cdot\boldsymbol{S}_{n+1}}-\frac{\boldsymbol{S}_{n-1}\times\boldsymbol{S}_n}{1+\boldsymbol{S}_{n-1}\cdot\boldsymbol{S}_n}\,. \tag{1}$$

This differential–difference soliton equation can be reduced to a difference equation assuming a simple time-dependence

$$\boldsymbol{S}_n(t)=(\cos\varphi_n\cos\omega t,\ \cos\varphi_n\sin\omega t,\ \sin\varphi_n)\,,$$
$$x_n\equiv\tan\tfrac{1}{2}\varphi_n\,. \tag{2}$$

We then obtain the 2D mapping

$$\begin{aligned}x_{n+1}=&[2x_n^3+\omega x_n^2+2x_n-\omega\\&-x_{n-1}(-x_n^4-\omega x_n^3+\omega x_n+1)]\\&\times[-x_n^4-\omega x_n^3+\omega x_n+1\\&-x_{n-1}(\omega x_n^4-2x_n^3-\omega x_n^2-2x_n)]^{-1}\,.\end{aligned} \tag{3}$$

This symmetric mapping turns out to be integrable. Its one-parameter family of invariant curves is given by the symmetric biquadratic relation

$$\begin{aligned}&(1+2K)x_n^2x_{n+1}^2+\omega(1+K)(x_n^2x_{n+1}+x_nx_{n+1}^2)\\&+(1+K)(x_n^2+x_{n+1}^2)+2Kx_nx_{n+1}\\&+\omega(1+K)(x_n+x_{n+1})+1+2K=0\,,\end{aligned} \tag{4}$$

where K is the invariant parametrizing the family of curves.

As a second example consider a discrete modified Korteweg–de Vries equation [2]:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x_n=&(1+x_n^2)[x_{n-1}-x_{n+1}\\&+\tfrac{1}{2}(x_{n+2}+x_n)(1+x_{n+1}^2)\\&-\tfrac{1}{2}(x_n+x_{n-2})(1+x_{n-1}^2)]\,.\end{aligned} \tag{5}$$

Stationary solutions of this equation are given by

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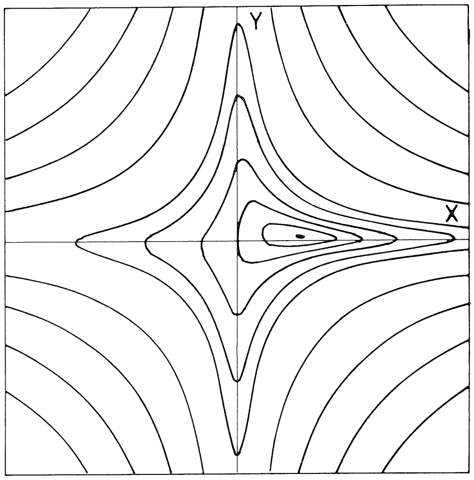

Fig. 1. Representative orbits of some points (x, y) of the mapping (7). Here $K_1=0.1$, $K_2=4$ and the area shown is $-15 \leqslant x \leqslant 15$, $-15 \leqslant y \leqslant 15$.

$$x_{n+1} - \tfrac{1}{2}(x_n + x_{n+2})(1 + x_{n+1}^2) = x_{n-1} - \tfrac{1}{2}(x_{n-2} + x_n)(1 + x_{n-1}^2) \; . \tag{6}$$

Note that this defines a 4D mapping. By inspection this mapping can be integrated to a 2D mapping

$$x_{2n} = \frac{2x_{2n-1} + 2K_2 - x_{2n-2}(1 + x_{2n-1}^2)}{1 + x_{2n-1}^2} \; ,$$

$$x_{2n+1} = \frac{2x_{2n} + 2K_1 - x_{2n-1}(1 + x_{2n}^2)}{1 + x_{2n}^2} \; . \tag{7}$$

where K_1, K_2 are integration constants. This mapping also turns out to be integrable. Its one-parameter family of invariant curves is given by the asymmetric biquadratic relation

$$x_{2n}^2 x_{2n+1}^2 + x_{2n}^2 + x_{2n+1}^2 - 2x_{2n}x_{2n+1} - 2K_1 x_{2n+1} - 2K_2 x_{2n} + K_3 = 0 \; , \tag{8}$$

where K_3 is the third integration constant. (See fig. 1.)

Both mappings (3) and (7) are reversible [3], i.e. they can be written as the product of two involutions. They are both special cases of an 18-parameter family of integrable reversible mappings of the plane that we have found [#1]

$$x' = \frac{f_1(y) - x f_2(y)}{f_2(y) - x f_3(y)} \; ,$$

$$y' = \frac{g_1(x') - y g_2(x')}{g_2(x') - y g_3(x')} \; , \tag{9}$$

where

$$\boldsymbol{f}(x) = (\mathbf{A}_0 \boldsymbol{X}) \times (\mathbf{A}_1 \boldsymbol{X}) \; ,$$

$$\boldsymbol{g}(x) = (\mathbf{A}_0^{\mathrm{T}} \boldsymbol{X}) \times (\mathbf{A}_0^{\mathrm{T}} \boldsymbol{X}) \; , \tag{10}$$

with

$$\boldsymbol{X} \equiv \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} ,$$

$$\mathbf{A}_0 \equiv \begin{pmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \delta_0 & \epsilon_0 & \zeta_0 \\ \kappa_0 & \lambda_0 & \mu_0 \end{pmatrix} ,$$

$$\mathbf{A}_1 \equiv \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \delta_1 & \epsilon_1 & \zeta_1 \\ \kappa_1 & \lambda_1 & \mu_1 \end{pmatrix} . \tag{11}$$

Each member of this family possesses a one-parameter family of invariant curves given by the biquadratic relation

$$(\alpha_0 + K\alpha_1)x^2y^2 + (\beta_0 + K\beta_1)x^2y + (\gamma_0 + K\gamma_1)x^2 + (\delta_0 + K\delta_1)xy^2 + (\epsilon_0 + K\epsilon_1)xy + (\zeta_0 + K\zeta_1)x + (\kappa_0 + K\kappa_1)y^2 + (\lambda_0 + K\lambda_1)y + (\mu_0 + K\mu_1) = 0 \; . \tag{12}$$

This equation can be parametrized in terms of elliptic functions [#2] yielding the final integration constant. It describes periodic, quasiperiodic and solitonic behaviour.

Other examples of differential–difference equations that possess solutions described by (9)–(12) include the DΔKdV, DΔMKdV, DΔNLS, DΔAHSC [6]. This leads us to make the following conjecture: *Consider a differential–difference equation. Then every autonomous difference equation obtained by an exact reduction of the* DΔE *is an integrable mapping.*

Some remarks:

[#1] The case $f_1 = g_1$; $f_2 = g_2$; $f_3 = g_3 = 0$ was discovered by McMillan [4].

[#2] See e.g. ref. [5], where it is also shown that in general some of the arbitrary parameters may be removed by applying bilinear transformations.

(1) This conjecture is a discrete analogue of the Painlevé conjecture of Ablowitz et al. [7]. Our conjecture is weaker however since we only consider autonomous reductions. The connection with the even stronger Painlevé property for DΔEs [8] is presently being investigated.

(2) What the above conjecture really says is that starting from a DΔE integrable in the (continuous) time variable after a reduction the resulting difference equation is integrable in the (discrete) space variable.

(3) One should note that the reverse of the above conjecture does not necessarily hold.

Finally we give an application of integrable mappings to the Yang–Baxter equation in statistical mechanics. In the solution of the hard hexagon model occurs the following equation for the scaled Boltzmann weights

Equation [#3]:

$$\theta_n^2 = \delta\theta_{n-1}\theta_{n+1} - \delta^{-1}\theta_{n-2}\theta_{n+2} \, ;$$

Transformation:

$$x_n \equiv \theta_{n+1}\theta_{n-1}/\theta_n^2 \, ;$$

Mapping:

$$x_{n+1}x_{n-1}x_n^2 + \delta - \delta^2 x_n = 0 \, ;$$

Invariant:

$$x_n^2 x_{n+1}^2 + K x_n x_{n+1} + \delta^2 (x_n + x_{n+1}) - \delta = 0 \, .$$

We conclude this Letter by noting that the mapping (9) is reversible for arbitrary ***f*** and ***g***, though in general it is not area-preserving. Nevertheless, a KAM theorem holds near typical elliptic fixed points [10]. A basic difference with area-preserving systems however is the fact that reversible systems can also have attractors. We hope to report on this in the near future [6,11].

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[#3] Eq. (13.102) of ref. [9]. Actually for the hard hexagon model one also imposes $\theta_n = \theta_{n+5}$.

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