

Duality for discrete integrable systems

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Abstract

A new class of discrete dynamical systems is introduced via a duality relation for discrete dynamical systems with a number of explicitly known integrals. The dual equation can be defined via the difference of an arbitrary linear combination of integrals and its upshifted version. We give an example of an integrable mapping with 2 parameters and 4 integrals leading to a (four-dimensional) dual mapping with 4 parameters and 2 integrals. We also consider a more general class of higher-dimensional mappings arising via a travelling-wave reduction from the (integrable) MKdV partial-difference equation. By differencing the trace of the monodromy matrix we obtain a class of novel dual mappings which are shown to be integrable as level-set dependent versions of the original ones.

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1 Introduction

Discrete integrable systems have received a lot of attention in the last two decades. Areas of physics in which discrete integrable systems prominently feature, include statistical mechanics and discrete analogues of integrable systems in classical mechanics or solid state physics [5, 6, 7, 8, 22, 23, 24, 25].

Some of the early papers dealt with the problem of discretizing integrable partial differential equations, such as the (modified) Korteweg-de Vries (MKdV) equation and the Sine-Gordon equation, while retaining their integrability. This led to integrable partial difference equations (PΔEs) [12]–[15]. Later papers studied integrable ordinary difference equations (OΔEs), both autonomous (i.e. integrable maps) [5, 6, 7, 8, 10, 21] and non-autonomous (e.g. discrete Painleve equations, for a review see [11]).

In this paper we start from a (primary) OΔE with one or more first integrals, and construct a dual OΔE which also has one or more integrals. While neither the primary OΔE nor the dual need be integrable in order for this construction to work, here we will be mainly interested in the case where the primary OΔE is integrable. The question we subsequently seek to answer is whether or not the ensuing dual equation is also integrable.

Accordingly, we focus on a new class of discrete dynamical systems which can be obtained by means of a *duality* relation from a known discrete dynamical system which possesses a number of integrals. The general idea is as follows. We consider a discrete dynamical system given by the d th-order OΔE:

$$u_{n+d} = f(u_n, u_{n+1}, \dots, u_{n+d-1}, \{p_i\}), \quad (1)$$

where $\{p_i\}$ is a set of l parameters occurring in the system and $f : \mathbb{R}^{d+l} \rightarrow \mathbb{R}$. By the standard method, we can alternatively view (1) as defining a map

$$\mathbf{V}_{n+1} = \mathbf{F}[\mathbf{V}_n, \{p_i\}], \quad (2)$$

in which

$$\mathbf{V}_n := (u_n, u_{n+1}, \dots, u_{n+d-1}) \quad (3)$$

is a d -dimensional vector and $\mathbf{F} : \mathbb{R}^{d+l} \rightarrow \mathbb{R}^d$.

Suppose the dynamical system is known to have the integrals:

$$I_j := I_j(\mathbf{V}_n, \{p_i\}) = I_j(\mathbf{V}_{n+1}, \{p_i\}), \quad (4)$$

with $j = 1, \dots, m$. Then we can form a linear combination

$$\sum_{i=1}^m \alpha_j I_j =: I(\mathbf{V}_n, \{p_i\}, \{\alpha_j\}), \quad (5)$$

which, for arbitrary α_j , is an integral of the system (1).

Taking the difference between the integral \mathbf{I} and its upshifted version, we derive a relation of the type:

$$I(\mathbf{V}_{n+1}, \{p_i\}, \{\alpha_j\}) - I(\mathbf{V}_n, \{p_i\}, \{\alpha_j\}) = L(\mathbf{V}_n, \mathbf{V}_{n+1}, \{p_i\}) L^*(\mathbf{V}_n, \mathbf{V}_{n+1}, \{p_i\}, \{\alpha_j\}), \quad (6)$$

in which

$$L(\mathbf{V}_n, \mathbf{V}_{n+1}, \{p_i\}) = u_{n+d} - f(u_n, u_{n+1}, \dots, u_{n+d-1}, \{p_i\}) = 0 \quad (7)$$

is equivalent to the original dynamical equation (1).

In fact, for the system (1), the left-hand-side of (6) must vanish and one would expect the right-hand-side to contain a factor such that the vanishing of that factor is equivalent to the dynamical equation (1). Apart from this, the right-hand-side may contain another factor such that the vanishing of this second factor also ensures that the left hand side of (6) is zero.

Starting from a specific example of a dynamical mapping (1) possessing integrals, it is not clear to which extent the second factor would contain an interesting dependence on the field \mathbf{V}_n . However, if this dependence is interesting, the vanishing of the second factor could be equivalent to another dynamical equation which by construction of (6) may be called the *dual equation* of the original (1). The dual equation automatically has 1 integral which is given by $I(\mathbf{V}_n, \{p_i\}, \{\alpha_j\})$ but, depending on the presence or absence of the original parameters $\{p_i\}$ in L^* , there may be more integrals.

At this stage the description of how to obtain the dual equation and the nature of the resulting dual equation is rather general ¹. However, in the present paper, we will show on the basis of some more sophisticated specific examples that indeed new dynamical systems with interesting properties can be derived from the differencing of integrals described by (6).

In Section 2, we will first consider what we call a motivating example. Our starting point is a low-dimensional mapping arising from the integrable partial difference MKdV equation of [6] by a travelling-wave reduction. It has four integrals and the dual mapping has 4 parameters and 2 integrals but may not be integrable.

The following Sections 3–6 are devoted to a more general class of higher-dimensional mappings arising from the MKdV partial-difference equations treated in [6]. We consider the integrals arising from different powers of the spectral parameter occurring in the trace of the monodromy matrix T . Defining the dual equation by (6) with $I = \text{Trace } T$, we find that these provide some new dynamical systems which can be considered as generalizations of known integrable mappings and which have a number of interesting integrals. In fact, we can establish the integrability of the ensuing dual equations. We do this by deriving a Lax representation for it which can be obtained from the Lax representation of the original system by some simple substitutions. Furthermore, it is interesting to note that in this case the dual equation is a so-called *level-set-dependent* (LSD) version of the original equation. This terminology refers to a higher-dimensional generalization of the work of [3, 4] on QRT [7, 8] and other mappings, where it was shown that a large class of such mappings amounts to a LSD version of the McMillan mapping. The work of [3, 4] and of [9] represents another way of associating two dynamical systems with integrals to one another.

The treatment given in this paper can also be applied to a large variety of other dynamical systems and various possible extensions and specific comments are given in

¹e.g. the dual equation may be almost trivial

a final discussion.

2 A motivating example: creating the dual of a 4D map

Consider the following 4th order difference equation:

$$V_4 = V_0 \frac{qV_1 - pV_3}{qV_3 - pV_1}. \quad (8)$$

In (8) and throughout this section, V_j , $j = 0, 1, 2, 3, 4$ is shorthand for V_{n+j} , $n \in \mathbf{Z}$, and p and q are parameters. This equation can be obtained as a reduction of the so-called MKdV P Δ E [6]. It is equivalent to the following map $(V_0, V_1, V_2, V_3) \mapsto (V'_0, V'_1, V'_2, V'_3)$ in 4 dimensions:

$$\begin{aligned} L : \quad V'_0 &= V_1 \\ V'_1 &= V_2 \\ V'_2 &= V_3 \\ V'_3 &= V_0 \frac{qV_1 - pV_3}{qV_3 - pV_1}. \end{aligned} \quad (9)$$

One checks that the map L has the following 4 integrals of motion (i.e. $I_\alpha(V_1, V_2, V_3, V_4) = I_\alpha(V_0, V_1, V_2, V_3)$ etc.). They each depend linearly on the parameters p and q , which we highlight in writing them:

$$I_\alpha = I_{\alpha,q}q - I_{\alpha,p}p \quad (10)$$

$$I_\beta = I_{\beta,q}q - I_{\beta,p}p \quad (11)$$

$$I_\gamma = I_{\gamma,q}q - I_{\gamma,p}p \quad (12)$$

$$I_\delta = I_{\delta,q}q - I_{\delta,p}p, \quad (13)$$

with

$$I_{\alpha,p} = \frac{V_1}{V_0} + \frac{V_2}{V_1} + \frac{V_3}{V_2} + \frac{V_0}{V_1} + \frac{V_1}{V_2} + \frac{V_2}{V_3} \quad (14)$$

$$I_{\beta,p} = \frac{V_2}{V_0} + \frac{V_3}{V_1} + \frac{V_0}{V_2} + \frac{V_1}{V_3} + \frac{V_3V_0}{V_2V_1} + \frac{V_2V_1}{V_3V_0} \quad (15)$$

$$I_{\gamma,p} = V_3V_0 \quad (16)$$

$$I_{\delta,p} = V_3^{-1}V_0^{-1} \quad (17)$$

$$I_{\alpha,q} = \frac{V_3}{V_0} + \frac{V_0}{V_3} \quad (18)$$

$$I_{\beta,q} = \frac{V_2}{V_0} + \frac{V_3}{V_1} + \frac{V_0}{V_2} + \frac{V_1}{V_3} + \frac{V_2V_3}{V_0V_1} + \frac{V_0V_1}{V_2V_3} \quad (19)$$

$$I_{\gamma,q} = V_0V_1 + V_1V_2 + V_2V_3 \quad (20)$$

$$I_{\delta,q} = V_0^{-1}V_1^{-1} + V_1^{-1}V_2^{-1} + V_2^{-1}V_3^{-1}. \quad (21)$$

These integrals may be inferred from the work of Hydon [2], but they can also be checked directly using two obvious symmetries of (8): $S_1 : V_i \mapsto \lambda V_i$, $\lambda \in \mathbb{R}$, and $S_2 : V_i \mapsto V_i^{-1}$. This is equivalent to saying that L of (9) commutes with S_1 and S_2 , whence if L has an integral I it also has an integral $I \circ S_i$, $i = 1, 2$. Using this, the integrals follow by constructing homogeneous expressions in the V_i 's of degree 0 (i.e. I_α, I_β), 2 (i.e. I_γ) and -2 (i.e. I_δ). We now take the linear combination of these integrals²:

$$I(V_0, V_1, V_2, V_3; p, q; \alpha, \beta, \gamma, \delta) = \alpha I_\alpha + \beta I_\beta + \gamma I_\gamma + \delta I_\delta, \quad (22)$$

and *difference* it, meaning we consider the difference between I and its upshifted version I' with $V_j \mapsto V_{j+1}$. Since the separate integrals satisfy:

$$I_\alpha - I'_\alpha = \left[\left(q \frac{V_3}{V_1} - p \right) - \frac{V_0}{V_4} \left(q - p \frac{V_3}{V_1} \right) \right] \left(\frac{V_1}{V_0} - \frac{V_4}{V_3} \right), \quad (23)$$

$$I_\beta - I'_\beta = \left[\left(q \frac{V_3}{V_1} - p \right) - \frac{V_0}{V_4} \left(q - p \frac{V_3}{V_1} \right) \right] \left(\frac{V_2}{V_0} \left(1 + \frac{V_1}{V_3} \right) - \frac{V_4}{V_2} \right) \left(1 + \frac{V_1}{V_3} \right), \quad (24)$$

$$I_\gamma - I'_\gamma = \left[\left(q \frac{V_3}{V_1} - p \right) - \frac{V_0}{V_4} \left(q - p \frac{V_3}{V_1} \right) \right] (-V_1 V_4), \quad (25)$$

$$I_\delta - I'_\delta = \left[\left(q \frac{V_3}{V_1} - p \right) - \frac{V_0}{V_4} \left(q - p \frac{V_3}{V_1} \right) \right] (V_0^{-1} V_3^{-1}), \quad (26)$$

and so vanish if (8) is satisfied, we find

$$\begin{aligned} I(V_1, V_2, V_3, V_4; p, q; \alpha, \beta, \gamma, \delta) - I(V_0, V_1, V_2, V_3; p, q; \alpha, \beta, \gamma, \delta) = \\ L(V_0, V_1, V_2, V_3, V_4; p, q) L^*(V_0, V_1, V_2, V_3, V_4; \alpha, \beta, \gamma, \delta), \end{aligned} \quad (27)$$

where

$$L(V_0, V_1, V_2, V_3, V_4; p, q) = \left[\left(q \frac{V_3}{V_1} - p \right) - \frac{V_0}{V_4} \left(q - p \frac{V_3}{V_1} \right) \right] \quad (28)$$

$$\begin{aligned} L^*(V_0, V_1, V_2, V_3, V_4; \alpha, \beta, \gamma, \delta) = & \left[\frac{1}{V_0} (\alpha V_1 V_3 + \beta V_2 V_3 + \beta V_1 V_2 + \delta) \right. \\ & \left. - \frac{V_4}{V_2} (\alpha V_2 + \beta (V_3 + V_1) + \gamma V_1 V_2 V_3) \right] \frac{1}{V_3}. \end{aligned} \quad (29)$$

The equation

$$L(V_0, V_1, V_2, V_3, V_4; p, q) = 0, \quad (30)$$

solved for V_4 gives precisely (8). With (27), this reminds us that I of (22) is an integral of this map. On the other hand, the equation

$$L^*(V_0, V_1, V_2, V_3, V_4; \alpha, \beta, \gamma, \delta) = 0, \quad (31)$$

defines a *different* 4th order difference equation:

$$V_4 = \frac{V_2}{V_0} \frac{\alpha V_1 V_3 + \beta (V_1 V_2 + V_2 V_3) + \delta}{\alpha V_2 + \beta (V_1 + V_3) + \gamma V_1 V_2 V_3}. \quad (32)$$

²Note that the four integrals are not independent, i.e. $I_\gamma I_\delta = 3q^2 + p^2 + q I_\beta$. However, since I_β is linear in p and q we still use this integral in the construction of a dual mapping

We call (32) the *dual map* corresponding to (8). It follows from (27) that I is also an integral of the dual. But significantly in eq. (27), the parameter sets $\{p, q\}$ and $\{\alpha, \beta, \gamma, \delta\}$ *dissociate* from one another in the right-hand side. Since p and q do not appear in the dual map (32), we can conclude that their coefficients in I are separately integrals of (32). More precisely, we can use (10–13) to rewrite the expression for I of (22) as:

$$I(V_0, V_1, V_2, V_3; p, q; \alpha, \beta, \gamma, \delta) = qI_q - pI_p, \quad (33)$$

where I_q and I_p given by

$$\begin{aligned} I_q &= (\alpha I_{\alpha,q} + \beta I_{\beta,q} + \gamma I_{\gamma,q} + \delta I_{\delta,q}) \\ &= \alpha \left(\frac{V_3}{V_0} + \frac{V_0}{V_3} \right) + \beta \left(\frac{V_2}{V_0} + \frac{V_3}{V_1} + \frac{V_0}{V_2} + \frac{V_1}{V_3} + \frac{V_2V_3}{V_0V_1} + \frac{V_0V_1}{V_2V_3} \right) \\ &\quad + \gamma(V_0V_1 + V_1V_2 + V_2V_3) + \delta(V_0^{-1}V_1^{-1} + V_1^{-1}V_2^{-1} + V_2^{-1}V_3^{-1}) \end{aligned} \quad (34)$$

$$\begin{aligned} I_p &= (\alpha I_{\alpha,p} + \beta I_{\beta,p} + \gamma I_{\gamma,p} + \delta I_{\delta,p}) \\ &= \alpha \left(\frac{V_1}{V_0} + \frac{V_2}{V_1} + \frac{V_3}{V_2} + \frac{V_0}{V_1} + \frac{V_1}{V_2} + \frac{V_2}{V_3} \right) \\ &\quad + \beta \left(\frac{V_2}{V_0} + \frac{V_3}{V_1} + \frac{V_0}{V_2} + \frac{V_1}{V_3} + \frac{V_0V_3}{V_1V_2} + \frac{V_1V_2}{V_0V_3} \right) + \gamma V_0V_3 + \delta V_0^{-1}V_3^{-1} \end{aligned} \quad (35)$$

are integrals of the dual.

Let us make some remarks about this process:

Remarks:

1. It has been convenient that the parameters p and q entered the integrals of (10–13) in a linear way. One sees that the number of integrals of the dual is equal to the number of parameters appearing in the original map and vice-versa.

2. The original map is actually degenerate, and can be reduced to a second order difference equation. This is achieved by introducing the reduced variables: $W_0 = \frac{V_2}{V_0}$, $W_1 = \frac{V_3}{V_1}$. Then (8) reduces to

$$W_2 = \frac{1}{W_0} \frac{q - pW_1}{qW_1 - p}. \quad (36)$$

Of the four integrals $I_\alpha, I_\beta, I_\gamma, I_\delta$ of the original map, only I_β can be expressed in terms of the reduced variables. However, differencing just I_β in (22)-(27) would lead to a completely trivial dual equation determined by $L^*(V_0, V_1, V_2, V_3, V_4; 0, \beta, 0, 0) = 0$.

3. For general parameters α, β, γ and δ , the dual map (32) has two integrals. It can also be checked to be measure-preserving. But this is not enough to ensure integrability. In fact, no symplectic structure and no additional integrals have been found so far for the general case. Numerically, some special cases of (32) have been shown to have zero algebraic entropy [20]. It may also be interesting to investigate

the possible integrability of the dual by extending to four dimensions the arithmetic integrability tests of [16, 17].

We mention some special cases of the dual map that we have proved to be integrable.

- For $\beta = 0$, (32) reduces to the four-dimensional Sine-Gordon (SG) mapping [1]. It can be derived from the SG difference equation using a periodicity constraint [6, 1].
- For $\gamma = \delta = 0$, (32) reduces to a three-dimensional mapping in terms of the variables $W_0 = \frac{V_1}{V_0}$, $W_1 = \frac{V_2}{V_1}$, $W_2 = \frac{V_3}{V_2}$:

$$W_3 = \frac{W_1 W_0}{W_2} \frac{\alpha W_2 + \beta(1 + W_1 W_2)}{\alpha W_1 + \beta(1 + W_1 W_2)}. \quad (37)$$

For this reduced mapping, there are two integrals I_q and I_p with $\gamma = \delta = 0$, both of which can be expressed in terms of W_0 , W_1 and W_2 . In combination with measure preservation this implies integrability.

- For $\alpha = 0$, (32) reduces to a three-dimensional mapping in terms of the variables $W_0 = V_1 V_0$, $W_1 = V_2 V_1$, $W_2 = V_3 V_2$:

$$W_3 = \frac{W_1 W_2}{W_0} \frac{\beta(W_1 + W_2) + \delta}{\beta(W_1 + W_2) + \gamma W_1 W_2}. \quad (38)$$

Again, for $\alpha = 0$, both of the integrals I_q and I_p of (32) can be expressed in terms of W_0 , W_1 and W_2 .

4. Finally, it is worthwhile to note that we can achieve four normal forms for (32) with $\gamma = \delta = 0$; $\gamma = \delta = 1$; $\gamma = 1, \delta = 0$; resp. $\gamma = 1, \delta = -1$. This can be done using the rescaling $V_i \mapsto \lambda V_i$ and the symmetry $V_i \mapsto V_i^{-1}, \gamma \leftrightarrow \delta$.

The exercise of constructing the dual of (8) is not without some mystery! From (36) it follows that (8) is in a sense a trivial four-dimensional mapping since it can be reduced to a two-dimensional one. And yet, ignoring this fact and taking its four non-independent integrals I_α , I_β , I_γ and I_δ produces, in special cases, the duals (37) and (38) which appear to be genuinely new and nontrivial integrable three-dimensional mappings (see [18] for some other examples of three-dimensional integrable mappings).

3 Integrable O Δ Es derived from soliton equations have a dual

In this section and in section 4, we present a method to obtain the dual equations of a general class of higher-dimensional mappings arising from the MKdV partial difference equation [6]. However, we first use a general formulation on the basis of a given Lax representation for general integrable partial difference equations. This is done to make

the treatment directly applicable to other integrable dynamical systems as well. The remaining Sections 5 and 6 will specialize to the MKdV case.

A (scalar) PΔE on a two-dimensional lattice $f_{\ell,m} = 0, \ell, m \in \mathbb{Z}$ has a *Lax representation* if there are matrices $L_{\ell,m}(k), M_{\ell,m}^{-1}(k), N_{\ell,m}(k)$ depending on a spectral parameter k such that

$$L_{\ell,m}(k)M_{\ell,m}^{-1}(k) - M_{\ell+1,m}^{-1}(k)L_{\ell,m+1}(k) = f_{\ell,m}N_{\ell,m}(k) \quad (39)$$

in which $f_{\ell,m}$ does not depend on k , and $N_{\ell,m}$ is nonsingular on $f_{\ell,m} = 0$.

We restrict ourselves to PΔE's

$$f_{\ell,m} = 0$$

with $f_{\ell,m}$ of the form

$$f_{\ell,m} = f[u_{\ell,m}, u_{\ell+1,m}, u_{\ell,m+1}, u_{\ell+1,m+1}, \underline{p}_{\ell,m}] \quad (40)$$

where the $\underline{p}_{\ell,m}$ denote any additional parameters arising from the matrices L and M , and the fields $u_{\ell,m}$, for simplicity, are taken to be scalars. The subscripts ℓ, m allow for the possibility that the parameters \underline{p} depend on the lattice sites (ℓ, m) . This general setting is investigated in order to obtain dual equations and their integrals with a sufficient amount of generality.

PΔE's can be reduced to ordinary difference equations (OΔE's) through *travelling-wave reductions* [6]. This can be done considering 2 integers z_1 and $z_2 > z_1$ which are relatively prime. In the (z_1, z_2) travelling-wave reduction the parameters $\underline{p}_{\ell,m}$ in the matrices $L_{\ell,m}(k)$ and $M_{\ell,m}^{-1}(k)$ depend on the sites (ℓ, m) via the similarity variable $n = z_1\ell + z_2m$ and we consider periodic solutions of $f_{\ell,m} = 0$ satisfying $u_{\ell,m} = u_{\ell-z_2, m+z_1}$, i.e.

$$\underline{p}_{\ell,m} = \underline{p}_n, u_{\ell,m} = u_n, n = z_1\ell + z_2m \quad (41)$$

and these solutions can be obtained from the OΔE

$$f_n = f[u_n, u_{n+z_1}, u_{n+z_2}, u_{n+z_1+z_2}, \underline{p}_n] = 0. \quad (42)$$

The OΔE can be solved specifying *initial values* on a *standard staircase* [6] consisting of points $(\ell_i, m_i), i = 0, 1, \dots, z_1 + z_2 - 1$ with $n(\ell_i, m_i) = n + n_i$ such that every value $n_i = 0, 1, \dots, z_1 + z_2 - 1$ occurs exactly once among the n_i values on the staircase. In fact, the u_n with $n \geq z_1 + z_2$ can be obtained from the OΔE $f_{n+\nu} = 0$ ($\nu \geq 0$), expressing $u_{n+\nu+z_1+z_2}$ in terms of $u_{n+\nu}, u_{n+\nu+z_1}, u_{n+\nu+z_2}$.

The *monodromy matrix* \mathcal{L}_n is defined to be the ordered product of Lax matrices along a standard staircase (more detail will follow in the next section). From (39) it can be shown that

$$\text{Trace } \mathcal{L}_n^\nu = \text{Trace } \mathcal{L}_{n+1}^\nu, \nu = 1, 2, \dots \quad (43)$$

In the special case of 2×2 matrices L_n, M_n^{-1} , such that $\det \mathcal{L}_n$ is trivial, we can restrict ourselves to $\nu = 1$.

Equation (43) is satisfied independently of the value of the spectral parameter k , and the coefficients of the various powers of k appearing in $\text{Trace } \mathcal{L}_n$ give integrals of

the O Δ E $f_n = 0$. (Note that it is not generally true that all integrals of the O Δ E can be obtained that way. In fact of the four integrals $I_\alpha, I_\beta, I_\gamma, I_\delta$ of the mapping (8) in the example of section 2, only the integral I_β follows from the Lax representation of the MKdV reduction associated with (8)). On the other hand, taking the difference between Trace \mathcal{L}_n and its shifted version we obtain a relation

$$\text{Trace } \mathcal{L}_n - \text{Trace } \mathcal{L}_{n+1} = f_n f_n^* \quad (44)$$

containing a factor f_n^* , ensuring that $f_n^* = 0$ implies the vanishing of the l.h.s as well. By analogy with (6) the equation $f_n^* = 0$ will be called the dual O Δ E.

4 Standard staircase and dual O Δ E

We now give a prescription for the dual O Δ E in the case that $z_1 = 1, z_2 = z, n = l + zm$. In this case, the standard staircase as introduced in [6] can be constructed in the following steps, see e.g. Figure 1.

To construct a standard staircase for $z_1 = 1, z_2 = z$:

- a) We start with the point (ℓ_0, m_0) with
 $n(\ell_0, m_0) = n + n_0, n_0 = 1$;
- b) We do a step to the left, to the point $(\ell_1, m_1) = (\ell_0 - 1, m_0)$
with $n(\ell_1, m_1) = n + n_1, n_1 = 0$;
- c) Then we do a step upward to the point $(\ell_2, m_2) = (\ell_1, m_1 + 1)$
with $n(\ell_2, m_2) = n + n_2, n_2 = z$;
- d) Next we do $z - 2$ steps to the left via points $(\ell_i, m_i) = (\ell_0 - i + 1, m_0 + 1)$
with $n(\ell_i, m_i) = n + n_i, n_i = z - i + 2, i = 3, 4, \dots, z - 1$
to reach the point $(\ell_z, m_z) = (\ell_0 - z + 1, m_0 + 1)$ with $n(\ell_z, m_z) = n + n_z, n_z = 2$;
- e) A final step to the left brings us to $(\ell_{z+1}, m_{z+1}) = (\ell_0 - z, m_0 + 1)$ with $n(\ell_{z+1}, m_{z+1}) = 1$.

The staircase is thus completed. See Figure 1 for an example.

To obtain the monodromy matrix \mathcal{L}_n we associate Lax matrices with the steps of the staircase in the following way:

- i) We associate with the first step from (ℓ_0, m_0) to (ℓ_1, m_1) the Lax matrix $S(n_0, n_1) = L_n$;
- ii) With the second step from (ℓ_1, m_1) to (ℓ_2, m_2) we associated the Lax matrix $S(n_1, n_2) = M_n^{-1}$;

- iii) With the $z - 2$ steps to the left from (ℓ_i, m_i) to (ℓ_{i+1}, m_{i+1}) with $i = 2, \dots, z - 1$ we associate the Lax matrices $S(n_i, n_{i+1}) = L_{n+z-i+1}$;
- iv) With the final step from (ℓ_z, m_z) to (ℓ_{z+1}, m_{z+1}) we associate the Lax matrix $S(n_z, n_{z+1}) = L_{n+1}$.

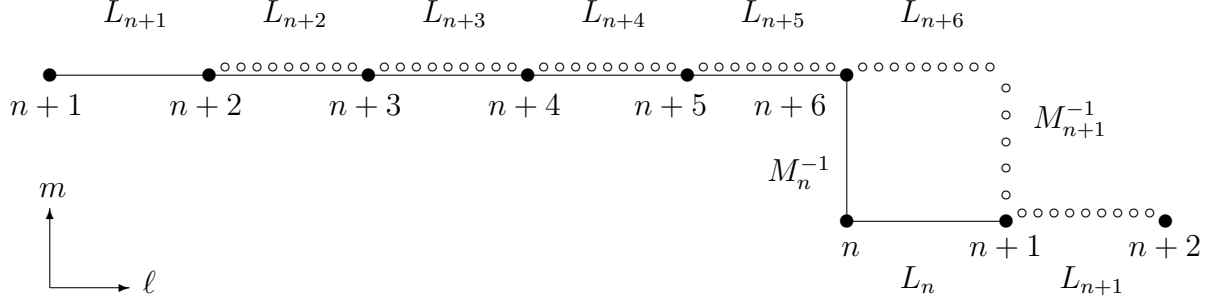


Figure 1: *Standard staircase* (—) and *shifted staircase* (○○○○○) under $n \rightarrow n + 1$, in the case $z_1 = 1, z_2 = 6$.

The monodromy matrix \mathcal{L}_n is the ordered product of Lax matrices along the standard staircase

$$\mathcal{L}_n = \prod_{i=0}^z S(n_i, n_{i+1}). \quad (45)$$

From the explicit $S(n_i, n_{i+1})$ we have the factorization property

$$\mathcal{L}_n = L_n M_n^{-1} A_n L_{n+1} \quad (46a)$$

with

$$A_n = L_{n+z-1} L_{n+z-2} \dots L_{n+2}. \quad (46b)$$

The shifted monodromy matrix

$$\mathcal{L}_{n+1} = \prod_{i=0}^z S(n_i + 1, n_{i+1} + 1), \quad (47a)$$

cf. Fig.(1), can be expressed as

$$\mathcal{L}_{n+1} = L_{n+1} M_{n+1}^{-1} L_{n+z} A_n \quad (47b)$$

and therefore, cf. (39)

$$\mathcal{L}_n L_{n+1}^{-1} - L_{n+1}^{-1} \mathcal{L}_{n+1} = (L_n M_n^{-1} - M_{n+1}^{-1} L_{n+z}) A_n = f_n N_n A_n. \quad (48)$$

Hence

$$\text{Trace } \mathcal{L}_n - \text{Trace } \mathcal{L}_{n+1} = f_n \text{Trace } N_n T_n \quad (49a)$$

with

$$T_n = A_n L_{n+1} = L_{n+z-1} L_{n+z-2} \dots L_{n+1}. \quad (49b)$$

The dual equation is given by

$$f_n^* = \text{Trace } N_n T_n = 0. \quad (50)$$

It is completely determined by the matrix $N_n(k)$ in eq.(39) and the matrices L in eq.(49b). Since $\text{Trace } \mathcal{L}_n$ is also an integral of the dual equation, we can use the difference to define a dual–dual equation $f_N^{**} = 0$ via $\text{Trace } \mathcal{L}_n - \text{Trace } \mathcal{L}_{n+1} = f_n^* f_n^{**}$. In comparing this to eq. (44) it is clear that the dual–dual equation $f_n^{**} = 0$ is just the original one $f_n = 0$, provided that the same integrals are used in the construction of the dual and the dual–dual equation.

5 Dual MKdV O Δ E in the case $z_1 = 1, z_2 = z$

We now specialize the discussion of the previous two sections to consider the dual O Δ E associated with the MKdV P Δ E $f_{\ell,m} = 0$, in the case that $z_1 = 1, z_2 = z, n = \ell + zm$. The dual equation associated with the more general reduction of the MKdV P Δ E with $z_2 > z_1 > 1$ can also be investigated. However, this is more complicated and will not be explicitly pursued in the present paper.

In the investigation of the dual equation of the $z_1 = 1, z_2 = z$ reduction of the MKdV P Δ E, one may anticipate that the number of parameters appearing in the dual equation will depend on the number of parameters in the original MKdV P Δ E, or in the Lax matrices occurring in the associated Lax representation. We therefore choose a rather general inhomogeneous setting in which the Lax matrices $L_{\ell,m}(k), M_{\ell,m}^{-1}(k)$ contain parameters $p_{\ell,m}, r_{\ell,m}, a_{\ell,m}, b_{\ell,m}, q_{\ell,m}, s_{\ell,m}, c_{\ell,m}, d_{\ell,m}$. The subscripts indicate a possible dependence of the parameters on the sites (ℓ, m) of the two dimensional lattice; albeit with such a dependence satisfying compatibility conditions to ensure that a consistent difference equation $f_{\ell,m} = 0$ follows from the Lax equation

$$L_{\ell,m}(k)M_{\ell,m}^{-1}(k) - M_{\ell+1,m}^{-1}(k)L_{\ell,m+1}(k) = 0, \quad (51)$$

i.e. eq. (39) with the right hand side replaced by 0.

To investigate the $z_1 = 1, z_2 = z$ reduction of the MKdV P Δ E with $u_{\ell,m}$ satisfying the periodicity constraint $u_{l,m} = u_{l-z,m+1} = u_n, n = l + zm$, we impose the conditions $\underline{p}_{l,m} = \underline{p}_{l-z,m+1} = \underline{p}_n$ for $\underline{p} = (p, r, a, b, q, s, c, d)$. That is, all parameters occurring in the Lax matrices satisfy the periodicity constraint also.

Hence we consider the Lax matrices

$$L_{\ell,m}(k) = \begin{pmatrix} p_n & -a_n u_{\ell+1,m} \\ \frac{-k^2 b_n}{u_{\ell,m}} & r_n \frac{u_{\ell+1,m}}{u_{\ell,m}} \end{pmatrix}, \quad M_{\ell,m}^{-1}(k) = \begin{pmatrix} s_n & d_n u_{\ell,m} \\ \frac{k^2 c_n}{u_{\ell,m+1}} & q_n \frac{u_{\ell,m}}{u_{\ell,m+1}} \end{pmatrix}. \quad (52)$$

The special case of (52): $p_{\ell,m} = p, r_{\ell,m} = r, q_{\ell,m} = q, s_{\ell,m} = s, a_{\ell,m} = b_{\ell,m} = c_{\ell,m} = d_{\ell,m} = 1$, was investigated in [1] in an investigation of four dimensional mappings

satisfying a specific symplectic structure (as a generalization of [6]). Working out the (1, 1) and (2, 2) elements of the condition (51) and insisting that the (1, 2) and (2, 1) elements of this condition lead to the same difference equation, we are led to the compatibility conditions

$$s_{n+1} = \frac{p_n}{p_{n+z}} s_n, \quad q_{n+1} = \frac{r_n}{r_{n+z}} q_n, \quad d_{n+1} = \frac{a_n}{b_{n+z}} c_n, \quad c_{n+1} = \frac{b_n}{a_{n+z}} d_n \quad (53)$$

and

$$k_n := k^2 \frac{a_n b_n}{p_n r_n} = k_{n+z}, \quad (54a)$$

or equivalently with (53)

$$l_n := \frac{c_n d_n}{q_n s_n} = l_{n+1} = l. \quad (54b)$$

Under these conditions, we have eq. (39) with

$$f_{\ell,m} = p_n d_n - a_n q_n \frac{u_{\ell+1,m}}{u_{\ell,m+1}} + \frac{u_{\ell+1,m+1}}{u_{\ell,m}} \left(a_{n+z} s_{n+1} - d_{n+1} r_{n+z} \frac{u_{\ell+1,m}}{u_{\ell,m+1}} \right). \quad (55a)$$

The equation $f_{\ell,m} = 0$ with $f_{\ell,m}$ given by (55a) is a general inhomogeneous version of the MKdV partial-difference equation of [1, 6] with parameters \underline{p} depending on the sites (ℓ, m) via the similarity variable n . Under the conditions (53),(54) for the parameters, it has the Lax representation (39) with the matrices $L_{\ell,m}(k)$ and $M_{\ell,m}^{-1}(k)$ given by (52) and the matrix $N_{\ell,m}(k)$ given by

$$N_{\ell,m}(k) = \begin{pmatrix} 0 & u_{\ell,m} \\ \frac{-k^2}{u_{\ell+1,m+1}} \frac{b_n}{a_{n+z}} \frac{p_{n+z}}{p_n} & 0 \end{pmatrix} \quad (55b)$$

In the travelling wave reduction $z_1 = 1, z_2 = z$, (55) reduces to the primary MKdV OΔE

$$f_n = 0 \quad (56a)$$

with

$$f_n = p_n d_n - a_n q_n \frac{u_{n+1}}{u_{n+z}} + \frac{u_{n+1+z}}{u_n} \left(a_{n+z} s_{n+1} - d_{n+1} r_{n+z} \frac{u_{n+1}}{u_{n+z}} \right) \quad (56b)$$

The dual equation follows from (50) and (55b) and is given by

$$f_n^* = 0, \quad (57a)$$

where

$$f_n^* = t_{21} u_n - \frac{b_n}{a_{n+z}} \frac{p_{n+z}}{p_n} \frac{k^2}{u_{n+z+1}} t_{12} \quad (57b)$$

and t_{12} and t_{21} are the off-diagonal elements of the matrix T_n defined by (49b).

Evaluating the off-diagonal elements of the matrix T_n it can be shown that the dual MKdV OΔE (57) can be expressed as a $(z-1)$ -dimensional mapping in terms of the reduced variables

$$W_{n+\nu} = \frac{r_{n+\nu}}{a_{n+\nu}} \frac{a_{n+\nu+1}}{p_{n+\nu+1}} \frac{u_{n+\nu+2}}{u_{n+\nu}}. \quad (58)$$

We have

$$\frac{W_{n+z-1}}{W_n} = k_n \frac{Z_{12}(W_{n+1}, W_{n+2}, \dots, W_{n+z-2})}{Z_{21}(W_{n+1}, W_{n+2}, \dots, W_{n+z-2})} \quad (59)$$

in which Z_{12} and Z_{21} are the off-diagonal elements of the matrix

$$\begin{aligned} Z(W_{n+1}, W_{n+2}, \dots, W_{n+z-2}) &= \begin{pmatrix} 1 & 1 \\ k_{n+z-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ k_{n+z-2}W_{n+z-2} & W_{n+z-2} \end{pmatrix} \\ &\cdot \begin{pmatrix} 1 & 1 \\ k_{n+z-3}W_{n+z-3} & W_{n+z-3} \end{pmatrix} \cdots \begin{pmatrix} 1 & 1 \\ k_{n+1}W_{n+1} & W_{n+1} \end{pmatrix}, \end{aligned} \quad (60)$$

cf. appendix A for some details of the derivation.

In eq (59) the numerator Z_{12} and denominator Z_{21} are multilinear functions containing all 2^{z-2} terms $W_{n+1}^{\mu_1} W_{n+2}^{\mu_2} \cdots W_{n+z-2}^{\mu_{z-2}}$, $\mu_1, \mu_2, \dots, \mu_{z-2} = 0, 1$, with coefficients depending on $k_{n+1}, k_{n+2}, \dots, k_{n+z-1}$.

For example for $z = 3, 4, 5$, we have respectively:

$$\begin{aligned} z = 3: \quad Z_{12}(W_{n+1}) &= 1 + W_{n+1} \\ Z_{21}(W_{n+1}) &= k_{n+2} + k_{n+1}W_{n+1} \\ z = 4: \quad Z_{12}(W_{n+1}, W_{n+2}) &= 1 + W_{n+1} + W_{n+2}(k_{n+2} + W_{n+1}) \\ Z_{21}(W_{n+1}, W_{n+2}) &= k_{n+3}(1 + k_{n+1}W_{n+1}) + W_{n+2}(k_{n+2} + k_{n+1}W_{n+1}) \\ z = 5: \quad Z_{12}(W_{n+1}, W_{n+2}, W_{n+3}) &= (1 + k_{n+3}W_{n+3})(1 + W_{n+1}) \\ &\quad + W_{n+2}(1 + W_{n+3})(k_{n+2} + W_{n+1}) \\ Z_{21}(W_{n+1}, W_{n+2}, W_{n+3}) &= (k_{n+4} + k_{n+3}W_{n+3})(1 + k_{n+1}W_{n+1}) \\ &\quad + W_{n+2}(k_{n+4} + W_{n+3})(k_{n+2} + k_{n+1}W_{n+1}). \end{aligned} \quad (61)$$

6 Integrability of dual MKdV

To prove that the dual MKdV O Δ E is integrable we first construct two integrals $\mathcal{J}_s, \mathcal{J}_q$ and two 2-integrals³ $\mathcal{J}_{c_n}, \mathcal{J}_{d_n}$ satisfying $\mathcal{J}_{c_{n+1}} = \mathcal{J}_{d_n}$, $\mathcal{J}_{d_{n+1}} = \mathcal{J}_{c_n}$.

From eqns. (46a), (49b), (50), it is clear that the parameters $\{s_n, c_n, d_n, q_n\}$ in the matrix M_n^{-1} in Trace \mathcal{L}_n dissociate from the parameters p_n, a_n, b_n, r_n occurring in the matrices L and the dual equation. To investigate the integrals of the dual equation it is therefore worthwhile to consider the decomposition

$$\text{Trace } \mathcal{L}_n = \mathcal{J}_{s_n} + \mathcal{J}_{c_n} + \mathcal{J}_{d_n} + \mathcal{J}_{q_n} \quad (62)$$

³More generally, k -integrals of a mapping are defined as integrals of the k^{th} iterate of the mapping [19], here corresponding to the shift $n \rightarrow n + k$

in which $\mathcal{J}_{s_n}, \mathcal{J}_{c_n}, \mathcal{J}_{d_n}, \mathcal{J}_{q_n}$ are terms arising from elements s_n, c_n, d_n, q_n of the matrix M_n^{-1} in $\text{Trace } \mathcal{L}_n = \text{Trace } L_n M_n^{-1} T_n$

Here using (52), the \mathcal{J} 's can be expressed in terms of the matrix elements t_{ij} of T_n .

$$\begin{pmatrix} \frac{\mathcal{J}_{s_n}}{s_n} & \frac{\mathcal{J}_{c_n}}{k^2 c_n} \\ \frac{\mathcal{J}_{d_n}}{d_n} & \frac{\mathcal{J}_{q_n}}{q_n} \end{pmatrix} = \begin{pmatrix} t_{11} & \frac{t_{12}}{u_n} \\ t_{21} u_n & t_{22} \end{pmatrix} \cdot \begin{pmatrix} p_n & -a_n \frac{u_{n+1}}{u_{n+z}} \\ -k^2 b_n & r_n \frac{u_{n+1}}{u_{n+z}} \end{pmatrix}. \quad (63)$$

Introducing

$$J_{s_n} = \mathcal{J}_{s_n}/s_n, \quad J_{c_n} = \mathcal{J}_{c_n}/c_n, \quad J_{d_n} = \mathcal{J}_{d_n}/d_n, \quad J_{q_n} = \mathcal{J}_{q_n}/q_n \quad (64)$$

and using eq. (53) to express $s_{n+1}, q_{n+1}, d_{n+1}, c_{n+1}$ in terms of s_n, q_n, c_n, d_n , we find from (49a) and (50) that if the dual equation is satisfied then

$$\begin{aligned} 0 = \text{Trace } \mathcal{L}_n - \text{Trace } \mathcal{L}_{n+1} &= s_n \left\{ J_{s_n} - \frac{p_n}{p_{n+z}} J_{s_{n+1}} \right\} + q_n \left\{ J_{q_n} - \frac{r_n}{r_{n+z}} J_{q_{n+1}} \right\} + \\ &+ c_n \left\{ J_{c_n} - \frac{a_n}{b_{n+z}} J_{d_{n+1}} \right\} + d_n \left\{ J_{d_n} - \frac{b_n}{a_{n+z}} J_{c_{n+1}} \right\}. \end{aligned} \quad (65)$$

Since the matrices $L_{n+1}, L_{n+2}, \dots, L_{n+z-1}$ and N_n occurring in the dual equation do not contain any of the parameters s_n, q_n, c_n, d_n , eq. (65) must be valid independently of the choice of s_n, q_n, c_n, d_n . This means that

$$J_{s_n} = \frac{p_n}{p_{n+z}} J_{s_{n+1}}, \quad J_{q_n} = \frac{r_n}{r_{n+z}} J_{q_{n+1}}, \quad J_{c_n} = \frac{a_n}{b_{n+z}} J_{d_{n+1}}, \quad J_{d_n} = \frac{b_n}{a_{n+z}} J_{c_{n+1}} \quad (66)$$

and hence, using again (53) and (64),

$$\mathcal{J}_{s_n} = \mathcal{J}_{s_{n+1}} := \mathcal{J}_s, \quad \mathcal{J}_{q_n} = \mathcal{J}_{q_{n+1}} := \mathcal{J}_q, \quad \mathcal{J}_{c_n} = \mathcal{J}_{d_{n+1}}, \quad \mathcal{J}_{d_n} = \mathcal{J}_{c_{n+1}}. \quad (67)$$

From eq. (63) we can solve the off-diagonal elements t_{12}, t_{21} of the matrix T_n to obtain

$$\frac{1}{u_n^2} \frac{t_{12}}{t_{21}} = \frac{a_n \frac{\mathcal{J}_{s_n}}{s_n} \frac{u_{n+1}}{u_{n+z}} + \frac{p_n}{k^2} \frac{\mathcal{J}_{c_n}}{c_n}}{k^2 b_n \frac{\mathcal{J}_{q_n}}{q_n} + r_n \frac{\mathcal{J}_{d_n}}{d_n} \frac{u_{n+1}}{u_{n+z}}}. \quad (68)$$

Inserting this in the dual MKdV (57), including the factor a_{n+z}/b_n in the denominator and the factor p_{n+z}/p_n in the numerator and using (53) yields

$$\frac{u_{n+z+1}}{u_n} = \frac{k^2 a_n \frac{\mathcal{J}_{s_n}}{s_{n+1}} \frac{u_{n+1}}{u_{n+z}} + p_{n+z} \frac{\mathcal{J}_{c_n}}{c_n}}{k^2 a_{n+z} \frac{\mathcal{J}_{q_n}}{q_n} + r_n \frac{\mathcal{J}_{d_n}}{c_{n+1}} \frac{u_{n+1}}{u_{n+z}}}. \quad (69)$$

Eq. (69) is very similar to the MKdV O Δ E (56). To prove its integrability we rewrite it as follows, recalling (54b):

$$\frac{u_{n+z+1}}{u_n} = \frac{k^2 a_n q_n \mathcal{J}_{s_n} \frac{u_{n+1}}{u_{n+z}} + p_n d_n \mathfrak{l}^{-1} \mathcal{J}_{c_n}}{k^2 a_{n+z} s_{n+1} \mathcal{J}_{q_n} + r_{n+z} d_{n+1} \mathfrak{l}^{-1} \mathcal{J}_{d_n} \frac{u_{n+1}}{u_{n+z}}}. \quad (70)$$

Eq. (70) is the travelling wave reduction $u_{\ell,m} \rightarrow u_{\ell+zm} = u_n$ of the P Δ E

$$\frac{u_{\ell+1,m+1}}{u_{\ell,m}} = \frac{a_n q_n^* \frac{u_{\ell+1,m}}{u_{\ell,m+1}} - p_n d_n^*}{a_{n+z} s_{n+1}^* - r_{n+z} d_{n+1}^* \frac{u_{\ell+1,m}}{u_{\ell,m+1}}} \quad (71a)$$

with

$$\begin{aligned} q_n^* &= k^2 q_n \mathcal{J}_s, \quad d_n^* = -d_n \mathfrak{l}^{-1} \mathcal{J}_{c_n} \\ s_{n+1}^* &= k^2 s_{n+1} \mathcal{J}_q, \quad d_{n+1}^* = -d_{n+1} \mathfrak{l}^{-1} \mathcal{J}_{d_n}. \end{aligned} \quad (71b)$$

Since (71a) is identical to (55a), it follows that in order for (71a) to be integrable via a Lax representation, we must impose the compatibility conditions

$$\frac{s_{n+1}^*}{s_n^*} = \frac{p_n}{p_{n+z}}, \quad \frac{q_{n+1}^*}{q_n^*} = \frac{r_n}{r_{n+z}}, \quad \frac{d_{n+1}^*}{c_n^*} = \frac{a_n}{b_{n+z}}, \quad \frac{c_{n+1}^*}{d_n^*} = \frac{b_n}{a_{n+z}}. \quad (72)$$

Defining

$$c_n^* = -c_n \mathcal{J}_{d_n} \mathfrak{l}^{-1} \quad (73)$$

and using (53) and (67) it follows with (71b) that this condition is indeed satisfied. Comparing (71) and (72) with (55a) and (53) it follows that the matrices

$$L_{\ell,m}(k^*) = \begin{pmatrix} p_n & -a_n \frac{u_{\ell+1,m}}{u_{\ell,m}} \\ \frac{-k^{*2} b_n}{u_{\ell,m}} & r_n \frac{u_{\ell+1,m}}{u_{\ell,m}} \end{pmatrix} \quad (74a)$$

and

$$M_{\ell,m}^{*-1}(k^*) = \begin{pmatrix} s_n^* & d_n^* \frac{u_{\ell,m}}{u_{\ell,m+1}} \\ \frac{k^{*2} c_n^*}{u_{\ell,m+1}} & q_n^* \frac{u_{\ell,m}}{u_{\ell,m+1}} \end{pmatrix}, \quad (74b)$$

in which k^* is a new spectral parameter, satisfy the relation

$$\begin{aligned} &L_{\ell,m}(k^*) M_{\ell,m}^{*-1}(k^*) - M_{\ell+1,m}^{*-1}(k^*) L_{\ell,m+1}(k^*) = \\ &= \left(p_n d_n^* - a_n q_n^* \frac{u_{\ell+1,m}}{u_{\ell,m+1}} + \frac{u_{\ell+1,m+1}}{u_{\ell,m}} \left(a_{n+z} s_{n+1}^* - d_{n+1}^* r_{n+z} \frac{u_{\ell+1,m}}{u_{\ell,m+1}} \right) \right) N_{\ell,m}(k^*). \end{aligned} \quad (75)$$

This means that eq. (71) has a Lax representation in terms of the matrices $L_{\ell,m}(k^*)$ and $M_{\ell,m}^{*-1}(k^*)$ and this can be used to evaluate the monodromy matrix \mathcal{L}_n^* for the travelling wave reduction (70) built up from these matrices. From Trace \mathcal{L}_n^* , one can obtain additional integrals of the dual MKdV O Δ E for larger values of z .

The parameters $\mathcal{J}_s, \mathcal{J}_{c_n}, \mathcal{J}_{d_n}$ and \mathcal{J}_q can be determined specifying the initial conditions for $u_n, u_{n+1}, \dots, u_{n+z}$ on a standard staircase. Because of this, eq. (29) will be called the *level-set dependent* (LSD) MKdV O Δ E.

The term *level-set dependent* relates to a concept introduced in [3, 4]. Given two O Δ Es, with O Δ E (2) possessing integrals, we will say

$$O\Delta E(2) = LSD \ O\Delta E(1), \quad (76)$$

if O Δ E (2) acts on the level sets of its integrals as O Δ E (1). The idea is that an initial condition for O Δ E (2) determines the values of the integrals of O Δ E (2). Then the ensuing motion can be described by a simpler (and possibly integrable) O Δ E (1). This is completely analogous to the two dimensional setting of [3, 4] in which it was shown that each QRT map actually acts as a generalised McMillan map on the level sets of the QRT integral. Here we have shown that the *dual* MKdV map acts as a MKdV map on the level sets of the integrals \mathcal{J}_s , \mathcal{J}_q and the 2-integrals $\mathcal{J}_{c_n} = \mathcal{J}_{d_{n+1}}$ and $\mathcal{J}_{d_n} = \mathcal{J}_{c_{n+1}}$.

The explicit expressions for \mathcal{J}_s , \mathcal{J}_q , \mathcal{J}_{c_n} and \mathcal{J}_{d_n} are given by ⁴

$$\begin{aligned}\mathcal{J}_s &= S(Z_{11} + k_n W_n Z_{12}) \\ \mathcal{J}_q &= \frac{Q(Z_{21} + W_n Z_{22})}{W_n W_{n+1} \cdots W_{n+z-2}} \\ \mathcal{J}_{c_n} &= -k^2 C_n S(Z_{11} + W_n Z_{12}) \frac{u_{n+1}}{u_{n+z}} \\ \mathcal{J}_{d_n} &= -C_n^{-1} \iota Q \frac{(Z_{21} + k_n W_n Z_{22})}{W_n W_{n+1} \cdots W_{n+z-2}} \frac{u_{n+z}}{u_{n+1}},\end{aligned}\tag{77}$$

in which the $W_{n+\nu}$ have been defined by (58) and $Z_{\alpha\beta} = Z_{\alpha\beta}(W_{n+1}, W_{n+2}, \dots, W_{n+z-2})$ are matrix elements of the matrix Z defined by (60).

The parameter C_n is defined by

$$C_n = \frac{c_n a_n}{s_n p_n}\tag{78}$$

and the constants Q and S by

$$\begin{aligned}S_n &= p_n p_{n+1} \cdots p_{n+z-1} s_n = S \quad (S_{n+1}/S_n = 1) \\ Q_n &= r_n r_{n+1} \cdots r_{n+z-1} q_n = Q \quad (Q_{n+1}/Q_n = 1)\end{aligned}\tag{79}$$

Some details of the derivation of (77) are presented in appendix B.

In summary, we have considered the dual MKdV O Δ E associated with the $z_1 = 1, z_2 = z$ similarity reduction of the MKdV P Δ E. These considerations can be extended to more general similarity reductions $z_1 > 1$. Also there, the dual MKdV O Δ E can be expressed as a generalized MKdV with parameters \mathcal{J} depending on the initial conditions but in the general case we don't anticipate relations like (67) ensuring the existence of a Lax representation and consequently the integrability of the dual MKdV O Δ E. The present treatment can also be adapted to the SG case with only minor variations.

7 Concluding remarks

We have studied the dual MKdV O Δ E associated with the $z_1 = 1, z_2 = z$ similarity reduction of the MKdV P Δ E. This dual equation is an integrable system, since it is the

⁴Note that \mathcal{J}_s and \mathcal{J}_q depend only on the variables W_n .

LSD version of the MKdV O Δ E. In this LSD version, there appear the integrals $\mathcal{J}_s, \mathcal{J}_q$ and the 2-integrals $\mathcal{J}_{c_n} = \mathcal{J}_{d_{n+1}}$ and $\mathcal{J}_{d_n} = \mathcal{J}_{c_{n+1}}$ of the dual MKdV which follow from the trace of the monodromy matrix. The dual MKdV has a Lax representation that is obtained from the Lax representation (52) of the MKdV P Δ E using the substitutions (71b) which involve the integrals $\mathcal{J}_s, \mathcal{J}_q$ and the 2-integrals $\mathcal{J}_{c_n}, \mathcal{J}_{d_n}$, and introduces a new spectral parameter. The (reduced variable) dual MKdV (59)-(60) contains the parameters k_n, \dots, k_{n+z-1} , defined by (54a). In the homogeneous case, this set of parameters reduces only to the single parameter k . However, it is worthwhile to note that the dual equation has been obtained using the trace of the monodromy matrix

$$\text{Trace } T_n = \sum_m k^{2m} I_m \quad (80)$$

in which the integrals I_m of the original MKdV O Δ E are the coefficients of different powers of the spectral parameter. Rather than considering the special case of (80), one can also start from an arbitrary linear combination of integrals

$$I = \sum_m \alpha_m I_m \quad (81)$$

and the dual MKdV in the homogeneous case can be inferred from (59)-(60) by replacing k^{2m} by the parameter α_m . In this way, the family of dual MKdV mappings for increasing z will contain an increasing number of parameters α_m , but the existence of a Lax representation is not obvious in this more general case.

Finally, it may be noted that the dual equation (59)-(60) has been obtained taking into account only the integrals of the MKdV O Δ E that appear in the trace of the monodromy matrix. However, in special cases, the MKdV O Δ E $f_n = 0$ of (56) may have additional integrals which can be inferred from the work of Hydon [2] or by considering homogeneous expressions in the variables u of various degrees. This would then lead to a dual equation in the variables u with more parameters, similar to the case of the motivating example of Section 2. But in that case the Lax representation (74) and LSD description (70) are not valid anymore and the integrability of the dual equation is open.

We conclude with some remarks:

- The considerations in this paper can be applied to the $z_1 = 1, z_2 = z$ reductions of other P Δ E's as well. For instance, for the reduction of the integrable SG P Δ E, the dual equation can be expressed as a LSD SG O Δ E and a Lax representation is obtained by simple substitutions from the Lax representation of the SG O Δ E. Also the dual of the $z_1 = 1, z_2 = z$ reduction of the KdV P Δ E can be investigated in a similar way.
- The treatment in this paper can also be extended to more general reductions with $z_1 > 1$ of the MKdV P Δ E. The dual equation can be obtained from the trace of the monodromy matrix and one can show that the dual equation can be regarded as an extended version of the MKdV O Δ E as given by (70) with

the coefficients \mathcal{J}_{s_n} , \mathcal{J}_{q_n} , \mathcal{J}_{c_n} and \mathcal{J}_{d_n} explicitly given in terms of the variables u_n . In this general case, however, the reasoning leading to a condition of the type (66) is not applicable anymore and there is no direct relation between the coefficients \mathcal{J}_{s_n} , \mathcal{J}_{q_n} , \mathcal{J}_{c_n} and \mathcal{J}_{d_n} and the integrals of the dual MKdV. So in this case the extended MKdV equation (70) cannot be interpreted as an LSD MKdV equation and also a Lax representation like (75) does not apply. Hence, at this stage, the integrability of the dual MKdV equation for $z_1 > 1$ has not been proved. It is interesting to note, however, that the $z_1 = 2$, $z_2 = 3$ reduction of the MKdV P Δ E leads to a dual equation which is a four-dimensional mapping that is superintegrable by the existence of two integrals $I_n = I_{n+1}$, $J_n = J_{n+1}$ and two coupled 2-integrals $G_n = H_{n+1}$, $H_n = G_{n+1}$. We hope to investigate the integrability of the more general cases in the near future.

- From (76), it follows that each orbit of O Δ E (2) corresponds to an orbit of O Δ E (1) restricted to a particular intersection of level sets of the integrals of O Δ E (2). This shows, that considered “orbit-by-orbit” O Δ E (2) is solvable, and in that sense integrable, if O Δ E (1) is. For example, if O Δ E (2) is a rational map, each of its orbits will have vanishing algebraic entropy, thus passing this test. However, on the face of it, LSD integrability does not imply symplecticity or Liouville integrability of O Δ E (2), as these refer to global structures of O Δ E (2) by which orbits are assembled together.

Appendix A

To prove eq. (59) we first express the Lax matrix $L_{n+\nu}(k)$, cf. (52), as

$$L_{n+\nu}(k) = p_{n+\nu} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{p_{n+\nu+1}}{a_{n+\nu+1}} \frac{1}{u_{n+\nu+2}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ k_{n+\nu} W_{n+\nu} & W_{n+\nu} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -\frac{a_{n+\nu}}{p_{n+\nu}} u_{n+\nu+1} \end{pmatrix} \quad (\text{A1})$$

Then from (49b) we have

$$T_n = p_{n+1} p_{n+2} \cdots p_{n+z-1} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{p_{n+z}}{a_{n+z}} \frac{1}{u_{n+z+1}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ k_{n+z-1} W_{n+z-1} & W_{n+z-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ k_{n+z-2} W_{n+z-2} & W_{n+z-2} \end{pmatrix} \cdots \cdot \begin{pmatrix} 1 & 1 \\ k_{n+1} W_{n+1} & W_{n+1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -\frac{a_{n+1}}{p_{n+1}} u_{n+2} \end{pmatrix} \quad (\text{A2})$$

which can be expressed as

$$T_n = p_{n+1} \cdots p_{n+z-1} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{r_{n+z-1}}{a_{n+z-1}} \frac{1}{u_{n+z-1}} \end{pmatrix} \cdot Z \cdot \begin{pmatrix} 1 & 0 \\ 0 & -\frac{a_{n+1}}{p_{n+1}} u_{n+2} \end{pmatrix} \quad (\text{A3})$$

in which the matrix Z has been defined by (60).

From (A3) and (57) we then obtain

$$u_{n+z+1} = \frac{k^2 b_n}{a_{n+z}} \frac{p_{n+z}}{p_n} \frac{Z_{12}}{Z_{21}} \frac{a_{n+1}}{p_{n+1}} \frac{a_{n+z-1}}{r_{n+z-1}} \frac{u_{n+2}}{u_n} \frac{u_{n+z-1}}{u_n} \quad (\text{A4})$$

which with (54a) and (58) yields (59).

Appendix B

From eq. (63) and eq. (A3) we obtain the relations

$$\begin{aligned} \mathcal{J}_{s_n} &= S \left(Z_{11} + k_n \frac{r_n}{a_n} \frac{a_{n+1}}{p_{n+1}} \frac{u_{n+2}}{u_n} Z_{12} \right) \\ \mathcal{J}_{q_n} &= S \frac{q_n}{s_n} \frac{a_n}{p_n} \frac{r_{n+z-1}}{a_{n+z-1}} \frac{u_n u_{n+1}}{u_{n+z-1} u_{n+z}} \left(Z_{21} + \frac{r_n}{a_n} \frac{a_{n+1}}{p_{n+1}} \frac{u_{n+2}}{u_n} Z_{22} \right) \\ \mathcal{J}_{c_n} &= -k^2 S \frac{c_n a_n}{s_n p_n} \left(Z_{11} + \frac{r_n}{a_n} \frac{a_{n+1}}{p_{n+1}} \frac{u_{n+2}}{u_n} Z_{12} \right) \frac{u_{n+1}}{u_{n+z}} \\ \mathcal{J}_{d_n} &= -S \frac{d_n}{s_n} \frac{r_{n+z-1}}{a_{n+z-1}} \frac{u_n}{u_{n+z-1}} \left(Z_{21} + k_n \frac{r_n}{a_n} \frac{a_{n+1}}{p_{n+1}} \frac{u_{n+2}}{u_n} Z_{22} \right) \end{aligned} \quad (\text{B1})$$

and eqs. (77) can be derived using (58) and the relation

$$\frac{u_n u_{n+1}}{u_{n+z-1} u_{n+z}} = \frac{a_{n+z-1}}{a_n} \frac{r_n r_{n+1} \cdots r_{n+z-2}}{p_{n+1} p_{n+2} \cdots p_{n+z-1}} \frac{1}{W_n W_{n+1} \cdots W_{n+z-2}} \quad (\text{B2})$$

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