ORDER AND CHAOS IN CONSERVATIVE AND IN REVERSIBLE SYSTEMS.

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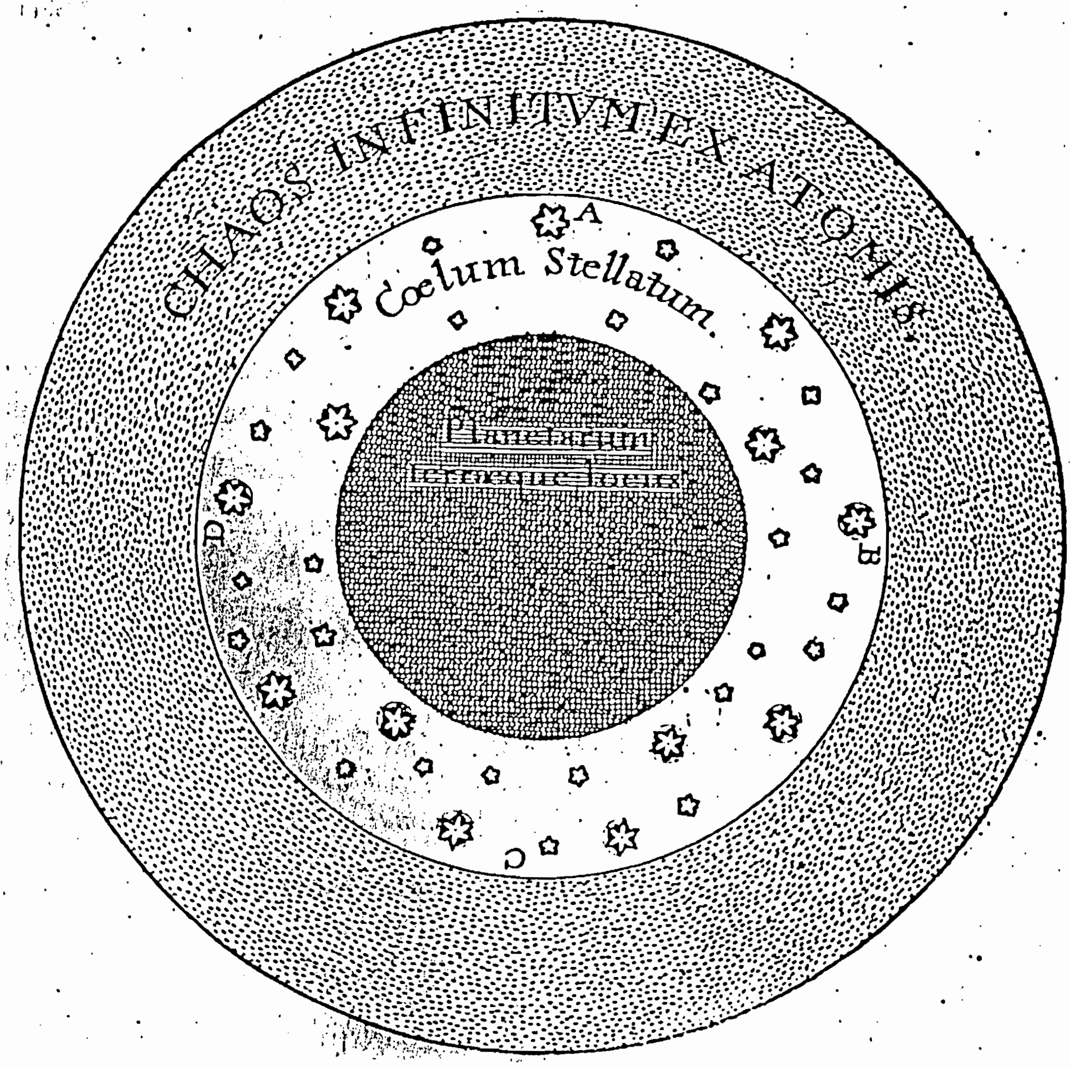
SYSTEMA ANTIQVORVM.



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1. INTRODUCTION

Dynamical systems where everything is ordered (integrable) and dynamical systems where everything is disordered (ergodic) are easiest to treat.

It has been shown, however, that most systems are neither integrable nor ergodic, and that most systems have both ordered and disordered regions in their phase space

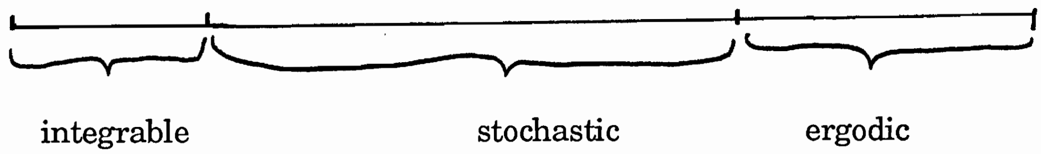


Fig. 1.

The first question one naturally asks is in what fraction of the phase space of a stochastic system is the motion ordered and in what fraction of phase space is the motion chaotic. This *quantitative* question is at present unanswered. The *qualitative* question of what phase space looks like in the ordered regions and in the chaotic regions has to a large extent been answered. This is the question we will be addressing in these lectures.

I will proceed from ordered systems to disordered ones. I'll start with integrable systems. Under perturbation these become stochastic. Finally I'll discuss an example of an ergodic system.

2. FROM CONTINUOUS TO DISCRETE TIME

Consider a particle with 2 degrees of freedom whose motion is described by Hamilton's equations:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \;,\; \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \;,\; i = 1,2. \tag{2.1}$$

This means that phase space is four-dimensional.

I'll now proceed to show how using the fact that phase space volume is conserved and that the energy is constant, the system can be reduced from a four-dimensional one with continuous time to a two-dimensional one with discrete time (Siegel and Moser, 1971, §22). The reader who is willing to accept on faith that this can be done may skip to the line below eq. (2.9).

At time $t = 0$ let there be an orbit that starts at $\vec{x}_0 = (q_1(0), q_2(0), p_1(0), p_2(0))$. Draw a 3D hyperplane through $\vec{x}_0$ (as an example we'll take the plane $p_2 = p_2(0)$), and assume the orbit intersects this plane again after a finite time. (This is true generally if the particle can't go off to infinity). Since it is impossible to draw in 4D I shall draw a 3D picture:

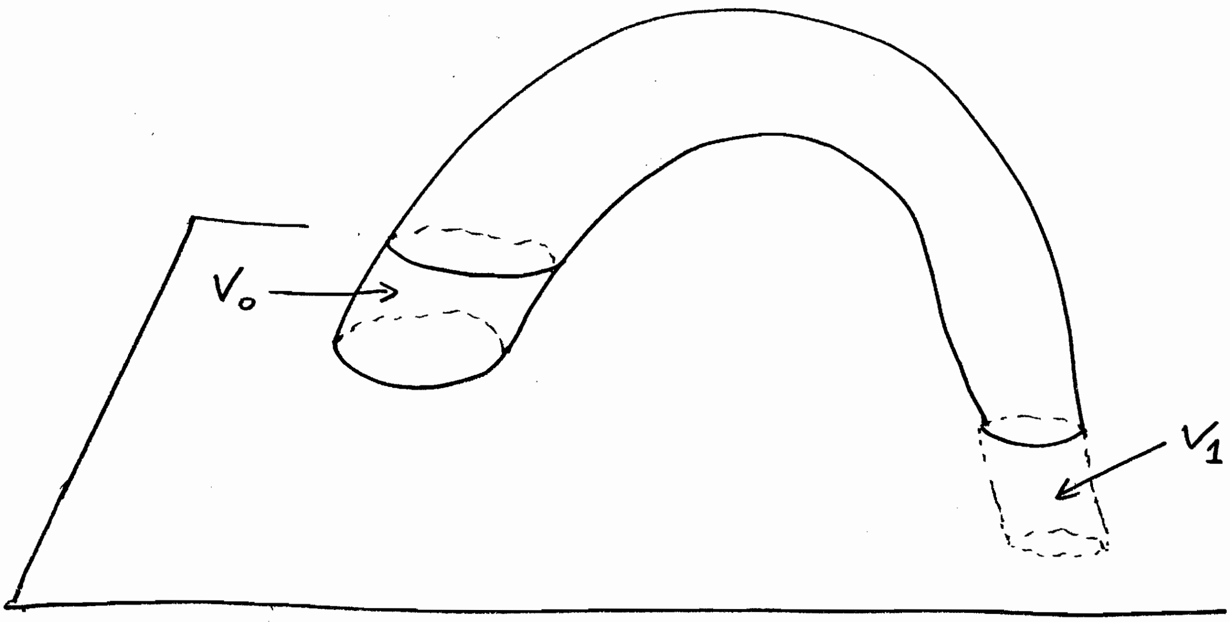


Fig. 2

This is called a Poincaré surface of section (Actually you usually want the plane to be intersected in the same sense. That is, however, also difficult to draw).

Since phase space volume is conserved in a Hamiltonian system we have

$$\int_{V_0} dq_1\, dq_2\, dp_1\, dp_2 = \int_{V_1} dq_1\, dq_2\, dp_1\, dp_2 \tag{2.2}$$

We now change variables twice. The first change of variables is

$$p_2 \rightarrow t \tag{2.3}$$

and hence

$$dp_2 = \frac{dp_2}{dt}\, dt. \tag{2.4}$$

The second change of variables uses the fact that the Hamiltonian is constant along an orbit.

$$H(q_1, q_2, p_1, p_2) = E\,. \tag{2.5}$$

Changing variables

$$q_2 \rightarrow E \tag{2.6}$$

we have

$$\frac{\partial H}{\partial q_2}\, dq_2 = dE \tag{2.7}$$

Inserting (2.7) and (2.4) in (2.2) and choosing V_0 appropriately we get

$$\int_0^t \int_{E_0}^{E_1} \int_{S_0} \frac{\dfrac{dp_2}{dt}}{\dfrac{\partial H}{\partial q_2}}\, dq_1\, dp_1\, dE\, dt = \int_0^t \int_{E_0}^{E_1} \int_{S_1} \frac{\dfrac{dp_2}{\partial t}}{\dfrac{\partial H}{\partial q_2}}\, dq_1\, dp_1\, dE\, dt, \tag{2.8}$$

which, using Hamilton's equations can be reduced to

$$\int_{S_0} dq_1\, dp_1 = \int_{S_1} dq_1\, dp_1 \tag{2.9}$$

This defines an area-preserving mapping

$$M: \begin{aligned} q_1' &= F(q_1, p_1) \\ p_1' &= G(q_1, p_1) \end{aligned} \tag{2.10}$$

with Jacobian determinant equal to one:

$$\frac{\partial F}{\partial q_1}\frac{\partial G}{\partial p_1} - \frac{\partial F}{\partial p_1}\frac{\partial G}{\partial q_1} = 1 \tag{2.11}$$

Let me give an example:

$$\begin{aligned} x' &= y \\ y' &= -x + 2Cy + 2y^2 \end{aligned} \tag{2.12}$$

This is the so-called area-preserving Hénon mapping. Here C is an arbitrary parameter and I have used x, y instead of q_1, p_1. An equivalent way to write equation (2.12) is

$$\begin{aligned} x_{n+1} &= y_n \\ y_{n+1} &= -x_n + 2\,Cy_n + 2y_n^2 \end{aligned} \tag{2.13}$$

Two computer plots of some of the orbits of the Hénon map (at C = 0.24) are shown in Fig. 3. All the scattered dots in Fig. 3b belong to one single orbit. It is clear that there is both order and chaos in this stochastic mapping. Mappings like this exhibit all the properties of the original set of differential equations but are much easier to use in actual computations since there is no integration to be done. Hence one can easily experiment with these mappings on a pocket calculator or on a personal computer. For these reasons I will restrict the discussion in these lectures to 2D mappings.

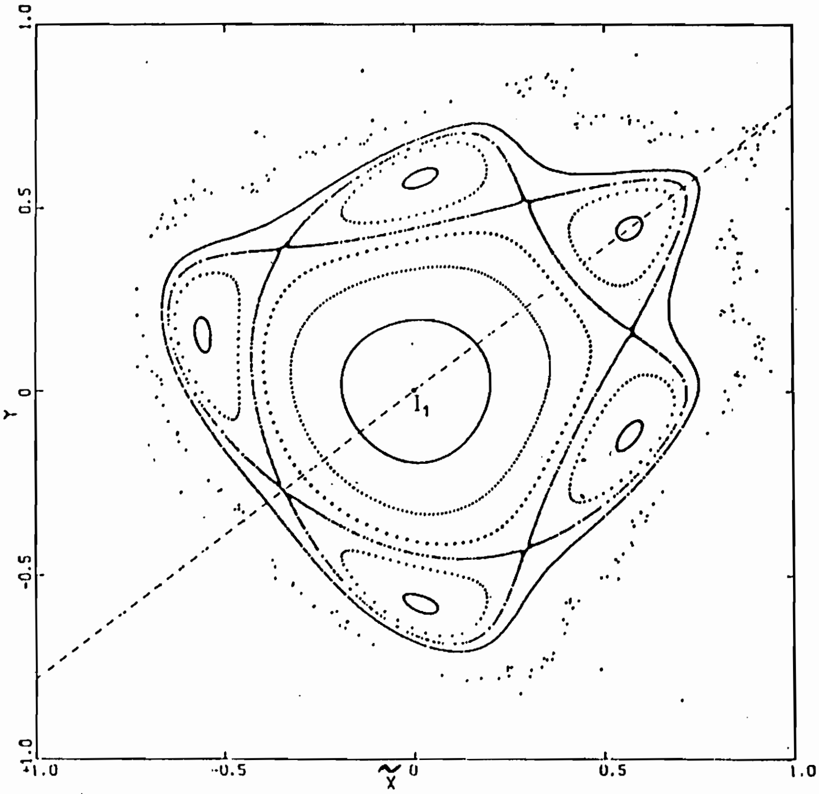

1.0
0.5
0
-0.5
-1.0
$\tilde{Y}$
-1.0
-0.5
0
0.5
1.0
$\tilde{X}$
I_1

Fig. 3a

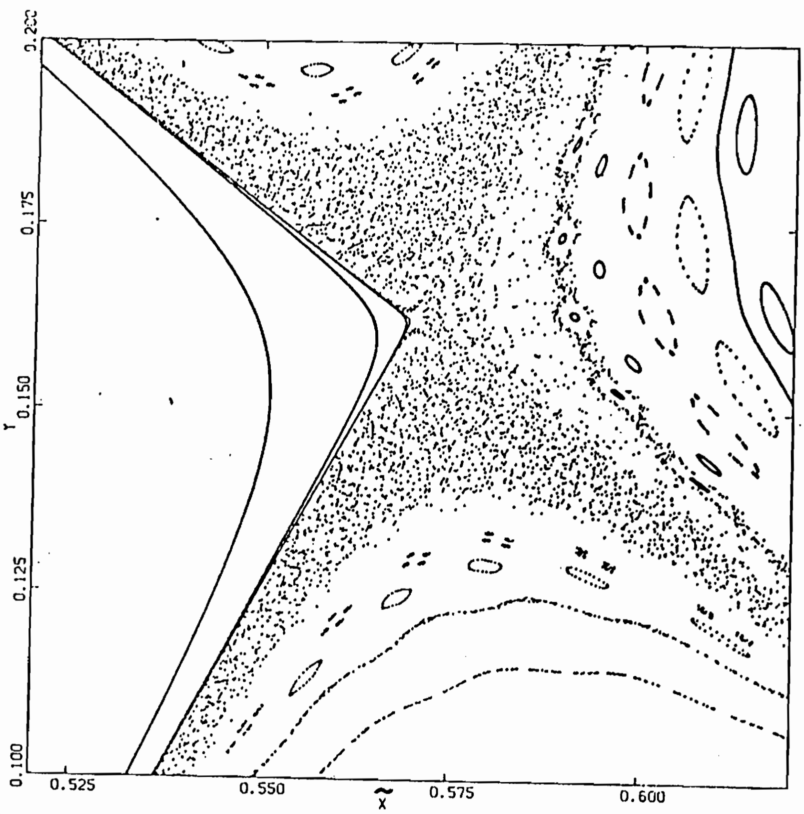

0.200
0.175
0.150
0.125
0.100
$\tilde{Y}$
0.525
0.550
0.575
0.600
$\tilde{X}$

Fig 3b

3. STABLE AND UNSTABLE FIXED POINTS. THE LINEAR CASE.

Consider an area-preserving mapping M

$$M: \quad \begin{aligned} x' &= F(x,y) \\ y' &= G(x,y) \end{aligned} \qquad (3.1)$$

Let us assume there is a fixed point $x' = x$, $y' = y$. Without loss of generality we take this fixed point to be at the origin, so $x' = x = 0$, $y' = y = 0$. We shall first examine the linear stability of this fixed point. Making a Taylor expansion of the mapping at the origin we then get

$$x' = a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + \ldots$$

$$y' = b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + \ldots \qquad (3.2)$$

From equation (2.11) we get that

$$\frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial x} = 1 \quad , \qquad (3.3)$$

so we know that

$$a_1 b_2 - a_2 b_1 = 1 \ . \qquad (3.4)$$

Defining the 2 x 2 matrix L by

$$L: = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \quad , \qquad (3.5)$$

we know that Det L = 1, trace L is real.

Let λ_1 and λ_2 be the eigenvalues of L

$$\lambda_1 \lambda_2 = 1$$

$$\lambda_1 + \lambda_2 = \text{real} \qquad (3.6)$$

Writing

$$\lambda_1 =: a + bi$$

$$\lambda_2 =: c + di \qquad (3.7)$$

equation (3.6) becomes

$$d = -b \tag{3.8a}$$

$$ac + b^2 = 1 \tag{3.8b}$$

$$b(c-a) = 0 \tag{3.8c}$$

So there are two distinct possibilities

$$a = \frac{1}{c}, b = d = 0 \tag{3.9}$$

or

$$a = c, \lambda_1 = \lambda_2^*, |\lambda_1| = |\lambda_2| = 1 \tag{3.10}$$

The first case (3.9) is called hyperbolic (for simplicity we will assume a > 0)

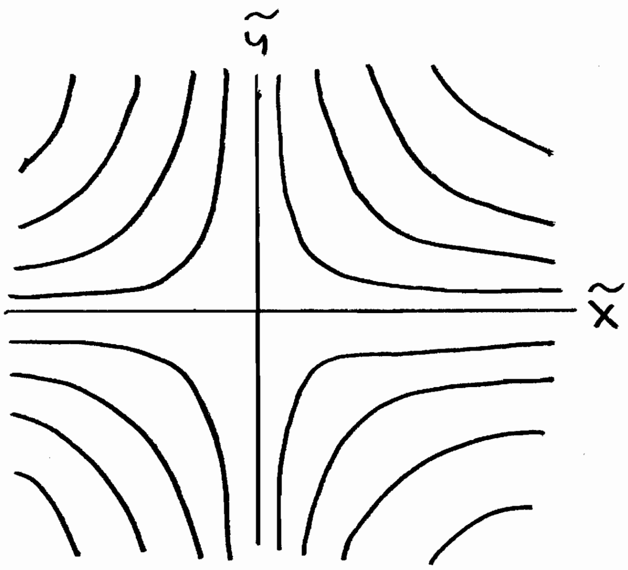


Figure 4

The second case (3.10) is called elliptic.

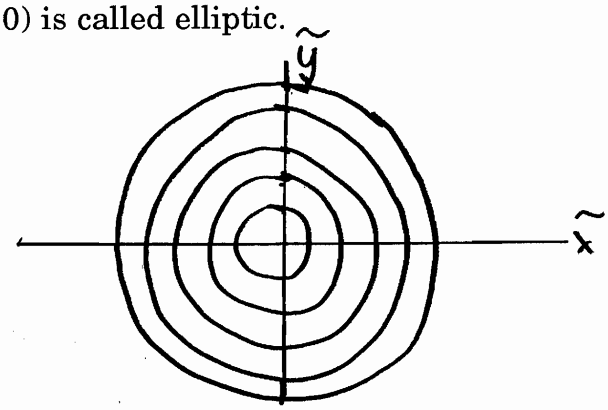


Figure 5

The hyperbolic case is **linearly** unstable. The elliptic case is **linearly** stable. What is the influence of the **nonlinear** terms in equation (3.2)? Loosely speaking an elliptic fixed point in a nonlinear system generally leads to order and a hyperbolic fixed point leads to chaos, as will be discussed in the next two sections.

Periodic orbits can be classified in the same way as fixed points. If e.g. an orbit returns to where it started after 4 iterations of the mapping then each point of this periodic orbit will be a fixed point of MMMM, also written M^4. M^4 is again area-preserving, so these fixed points are again either elliptic or hyperbolic. We therefore also speak of elliptic and hyperbolic periodic orbits. Near elliptic periodic orbits the motion will again be ordered, near hyperbolic periodic orbits the motion is chaotic.

In mechanics stability is what is sought after if you don't want your train to go off the track or the solar system to disintegrate. In these systems one wants to stay near elliptic orbits. In statistical mechanics on the contrary an assumption is made of molecular chaos or ergodicity so in those systems one wants as much hyperbolicity as possible.

4. STABLE FIXED POINTS. THE NONLINEAR CASE

4.1 Stable fixed points in integrable mappings.

Integrable mappings (McMillan, 1971; Quispel *et al* 1988a; 1988b) are mappings where each orbit lies on a closed curve. The motion on bounded curves is conjugate to a rotation. Integrable mappings are very much like linear ones. As an example of an integrable mapping consider Moser's twist map:

$$M_o:\quad \begin{aligned} x' &= x\cos\omega - y\sin\omega \\ y' &= x\sin\omega + y\cos\omega \end{aligned} \tag{4.1}$$

where $\omega = \omega(x^2+y^2)$. For simplicity we will assume that the derivative of ω is always unequal to zero. This is called a monotonic twist map. The mapping M_0 has a constant of the motion

$$x'^2 + y'^2 = x^2 + y^2 =: c \tag{4.2}$$

so motion takes place on circles.

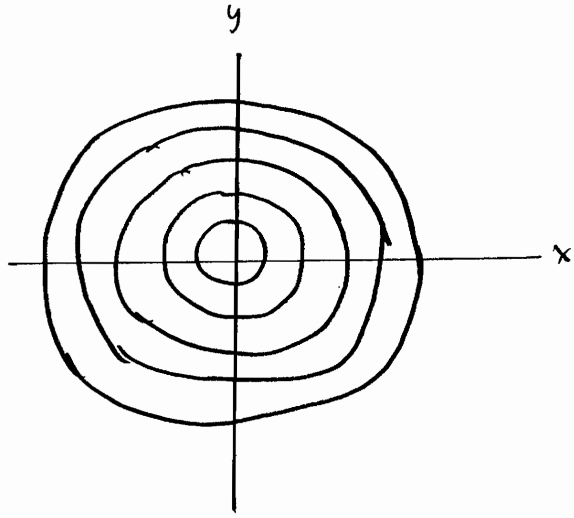


Fig. 6

Note: here and in the following circles may be replaced by smooth curves conjugate to circles as may be seen in the next example.

The second example of an integrable mapping is given by

$$\begin{aligned} x' &= y \\ y' &= -x + \frac{\frac{4}{3}y}{1+y^2} \end{aligned} \tag{4.3}$$

If you plot the orbits of this mapping on your computer (try it!) you will get a picture like figure 7.

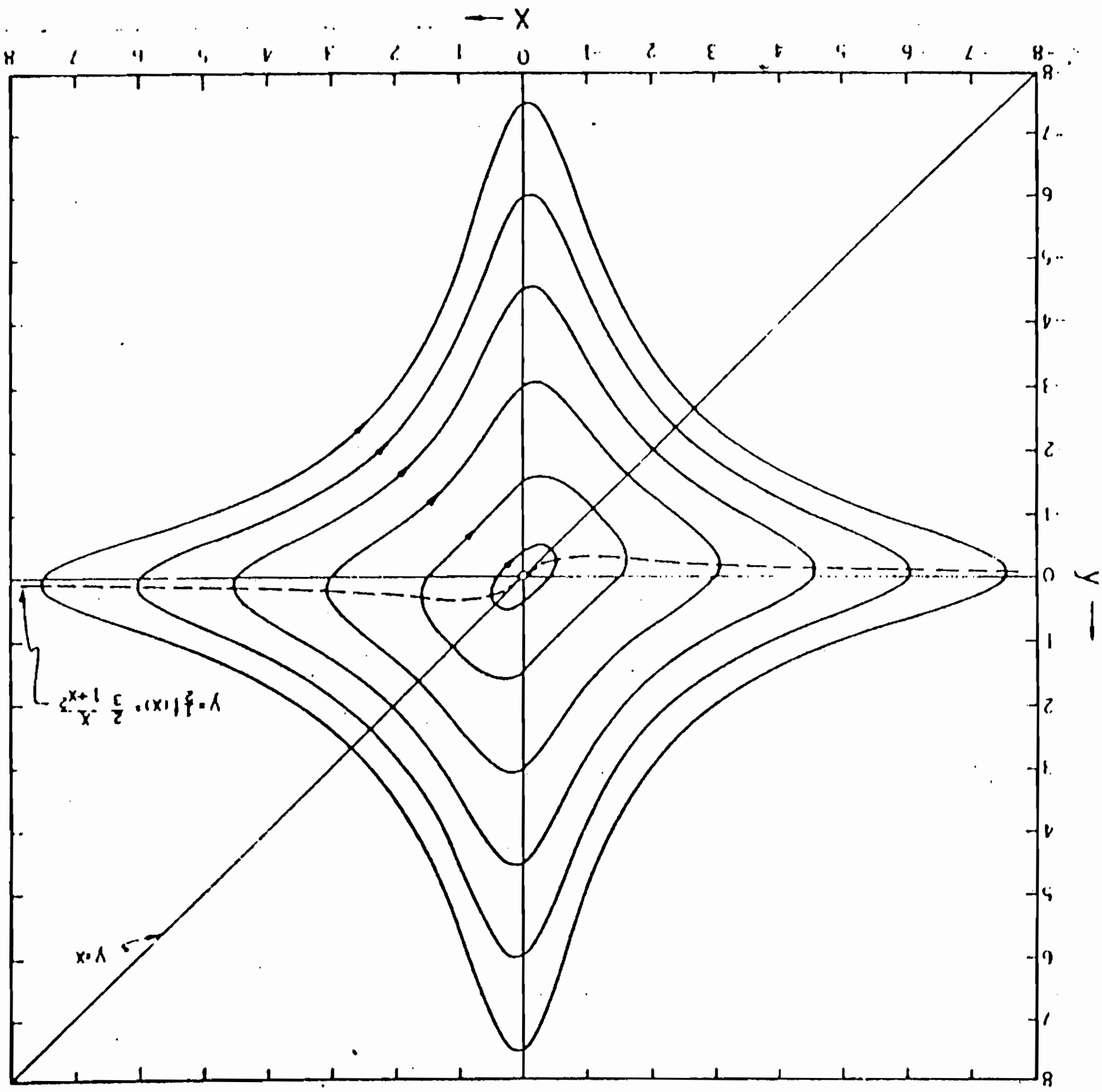


Fig.7

The invariant giving the closed curves in this case is

$$x^2y^2 + x^2 + y^2 - \frac{4}{3}xy =: c \qquad (4.4)$$

4.2 Stable fixed points in stochastic mappings.

What happens to an integrable mapping under a (area-preserving) perturbation?

Loosely speaking a torus has three possibilities:

1. It persists.
2. It develops holes, *i.e.* becomes a Cantor set.
3. It breaks completely forming an even number of periodic orbits.

To understand the perturbed system we must first understand the unperturbed system (4.1)

$$x' = x\cos\omega(x^2+y^2) - y\sin\omega(x^2+y^2)$$

$$y' = x\sin\omega(x^2+y^2) + y\cos\omega(x^2+y^2) \qquad (4.5)$$

If $\omega(x^2+y^2)$ is a rational fraction of 2π the torus corresponding to that radius is filled with periodic orbits. If *e.g.*, $\omega(x^2+y^2)/2\pi = 3/5$ all points on the corresponding torus return to where they began after 5 iterations, having wound around the origin 3 times. For this reason $\omega/2\pi$ is called the winding number. Not far from there, there may be a torus such that

$\omega/2\pi = -\frac{1}{2} + \frac{1}{2}\sqrt{5} = .61803398\ldots$. This torus is filled with orbits that never return to where they started. Each orbit fills the torus densely. These orbits are called quasiperiodic.

What happens to these tori under perturbation? The KAM-theorem (sub-section 4.2.1.) will tell us that (under small perturbations) the "sufficiently irrational" tori persist. The Aubry-Mather theorem (subsection 4.2.2) will tell us that the "less irrational" circles either persist or break to form a Cantor set (sometimes called a Cantorus for obvious reasons). Finally the Poincaré-Birkhoff theorem (subsection 4.2.3) will tell us that the periodic tori are destroyed completely by resonance, leaving an even number of periodic orbits (usually 2). All perturbed orbits retain there original winding number.

4.2.1. Small denominators and the KAM-theorem.

Let's rewrite the integrable mappping (4.1) as follows:

$$M_0 : \begin{array}{l} \theta' = \theta + \omega(r) \\ r' = r \end{array} \tag{4.6}$$

We change variables again

$$x := \theta$$

$$y := \omega(r)$$

(Store in your memory that x is an angle. These x and y are of course not the same variables as in (4.1)). Under an ε-perturbation (4.6) then becomes (Siegel and Moser (1971) p.226):

$$M_\varepsilon : \begin{array}{l} x' = x + y + f(x,y;\varepsilon) \\ y' = y + g(x, y; \varepsilon) \end{array} \tag{4.7}$$

Under what conditions will this perturbed system retain an invariant curve given in parametrised form by

$$x = \xi + u(\xi;\varepsilon)$$

$$y = v(\xi;\varepsilon) \tag{4.8}$$

such that the motion on the curve is given by

$$\xi' = \xi + \omega \tag{4.9}$$

(Store in your memory that ξ is an angle)

Plugging all this in (4.7) we get

$$\xi' + u(\xi';\varepsilon) = \xi + u(\xi;\varepsilon) + v(\xi;\varepsilon) + f\left(\xi + u(\xi;\varepsilon),\, v(\xi;\varepsilon);\, \varepsilon\right) \tag{4.10}$$
$$v(\xi';\varepsilon) = v(\xi;\varepsilon) + g\left(\xi + u(\xi;\varepsilon),\, v(\xi;\varepsilon);\, \varepsilon\right)$$

Now expand in powers of ε:

$$u(\xi;\varepsilon) =: \sum_{n=1}^{\infty} \varepsilon^n u_n(\xi)$$
$$v(\xi;\varepsilon) =: \omega + \sum_{n=1}^{\infty} \varepsilon^n v_n(\xi) \tag{4.11}$$
$$f\left(\xi + u(\xi;\varepsilon),\, v(\xi;\varepsilon);\, \varepsilon\right) =: \sum_{n=1}^{\infty} \varepsilon^n F_n(\xi)$$
$$g\left(\xi + u(\xi;\varepsilon),\, v(\xi;\varepsilon);\, \varepsilon\right) =: \sum_{n=1}^{\infty} \varepsilon^n G_n(\xi)$$

This yields

$$u_n(\xi+\omega) = u_n(\xi) + v_n(\xi) + F_n(\xi)$$
$$v_n(\xi+\omega) = v_n(\xi) + G_n(\xi) \tag{4.12}$$

Since ξ is an angle all these functions are periodic in ξ .

We therefore expand in Fourier series:

$$u_n(\xi) =: \sum_{k=0,\pm 1,\pm 2 \ldots} \hat{u}_{n,k}\, e^{ik\xi} \tag{4.13}$$

and analogously $v_n, F_n, G_n \to \hat{v}_{n,k}, \hat{F}_{n,k}, \hat{G}_{n,k}$.

We obtain

$$\Sigma\, \hat{u}_{n,k}\, e^{ik\xi + ik\omega} = \Sigma\, \hat{u}_{n,k}\, e^{ik\xi} + \Sigma\, \hat{v}_{n,k}\, e^{ik\xi} + \Sigma\, \hat{F}_{n,k}\, e^{ik\xi} \tag{4.14}$$
$$\Sigma\, \hat{v}_{n,k}\, e^{ik\xi + ik\omega} = \Sigma\, \hat{v}_{n,k}\, e^{ik\xi} + \Sigma\, \hat{G}_{n,k}\, e^{ik\xi}$$

So equating term by term we get

$$\hat{u}_{n,k}\left(e^{ik\omega} - 1\right) = \hat{v}_{n,k} + \hat{F}_{n,k} \tag{4.15}$$
$$\hat{v}_{n,k}\left(e^{ik\omega} - 1\right) = \hat{G}_{n,k}$$

We impose $\hat{G}_{n,o} = 0$ and then get

$$\hat{v}_{n,k} = \frac{\hat{G}_{n,k}}{e^{ik\omega} - 1} \; ; \; \hat{u}_{n,k} = \frac{\hat{F}_{n,k}}{e^{ik\omega} - 1} + \frac{\hat{G}_{n,k}}{\left(e^{ik\omega} - 1\right)^2} \tag{4.16}$$

Hence

$$v_n(\xi) = \sum_{k=\pm 1, \pm 2 \ldots} \frac{\hat{G}_{n,k}}{e^{ik\omega} - 1} e^{ik\xi} \tag{4.17}$$

$$u_n(\xi) = \sum_{k=0, \pm 1, \pm 2 \ldots} \frac{\hat{F}_{n,k}}{e^{ik\omega} - 1} e^{ik\xi} + \frac{\hat{G}_{nk}}{\left(e^{ik\omega} - 1\right)^2} e^{ik\xi}$$

It is clear that if ω is a rational fraction of 2π these sums diverge in general. This means that there is *no* invariant curve with rational winding number $\frac{\omega}{2\pi}$ in the perturbed system (4.7).

Intermezzo:

Equation (4.6) is equivalent to

$$\begin{aligned} \theta' &= \theta + 2\pi \frac{\omega_1}{\omega_2} \\ r' &= r \end{aligned} \tag{4.18}$$

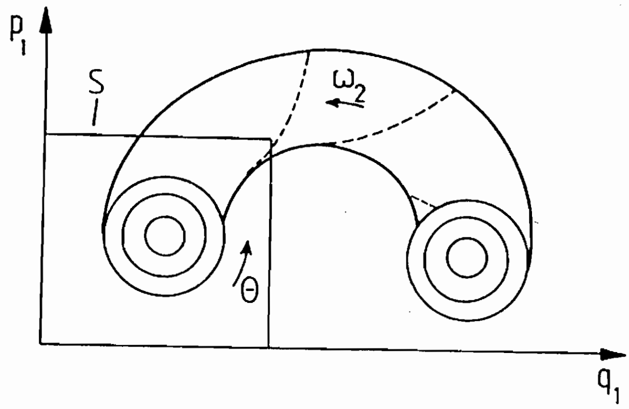


Figure 8

So when $\frac{\omega}{2\pi}$ is (ir) rational it corresponds to $\frac{\omega_1}{\omega_2}$ (ir) rational. For this reason $\frac{\omega}{2\pi}$ is rational is called a resonance. But even if ω is irrational the sums in (4.17) may diverge. This is the famous problem of "small denominators" that troubled celestial mechanics for centuries. The basic questions that arise are:

1. Under what conditions on ω do the sums in (4.17) converge?
2. What happens for those irrational winding numbers where the conditions under 1) are not met?

The answer to question 2) is given by the Aubry-Mather theorem which will be discussed in the next subsection. Question one is answered by the KAM-theorem (see *e.g.* Moser, 1986b):

There is a function $\gamma(\varepsilon)$ such that there exists a smooth closed curve invariant under M_ε with winding number $\omega/2\pi$ provided

$$\left|\frac{\omega}{2\pi} - \frac{p}{q}\right| \geq \gamma(\varepsilon)\,|q|^{-2.5} \quad \text{for all integers } p,\, q \neq 0,$$

$$|\varepsilon| \leq \varepsilon_0(\omega)\ , \tag{4.19}$$

where γ is a number whose precise value is not given by the theorem but which tends to zero with the perturbation ε. (This is the so-called **global** version of the KAM theorem). If we take a rotation interval $0 \leq \frac{\omega}{2\pi} \leq 1$ as an example an upper bound for the excluded volume is

$$\gamma(\varepsilon)\left[1 + 2\sum_{q=2}^{\infty} \frac{q-1}{q^{2.5}}\right] = \gamma(\varepsilon)\, 3.54176\ldots$$

This excluded volume goes to zero with ε (see Fig. 9)

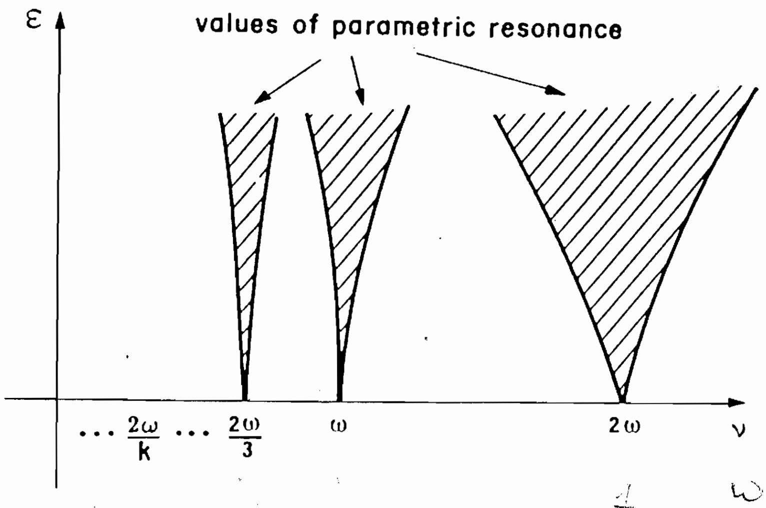


Figure 9

As an example consider the motion of an asteroid around the sun, perturbed by the motion of Jupiter, as shown in Fig. 10. This three-body problem is a famous example of a stochastic system, and we expect that the asteroid motion becomes unstable if the ratio of the frequency of the asteroid motion ω and the angular frequency of Jupiter ω_J becomes rational. Fig. 11 illustrates that, in fact, gaps occur in the asteroid distribution for rational ω/ω_J. On the other hand, the existence of stable asteroid orbits ($f \neq 0$) can be considered as a confirmation of the KAM theorem.

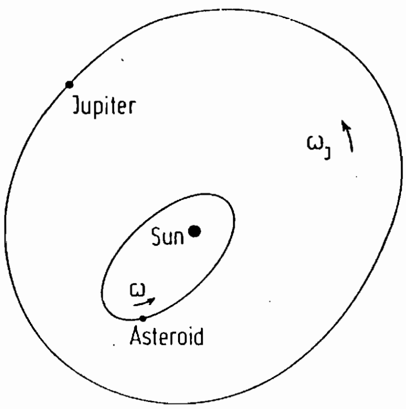
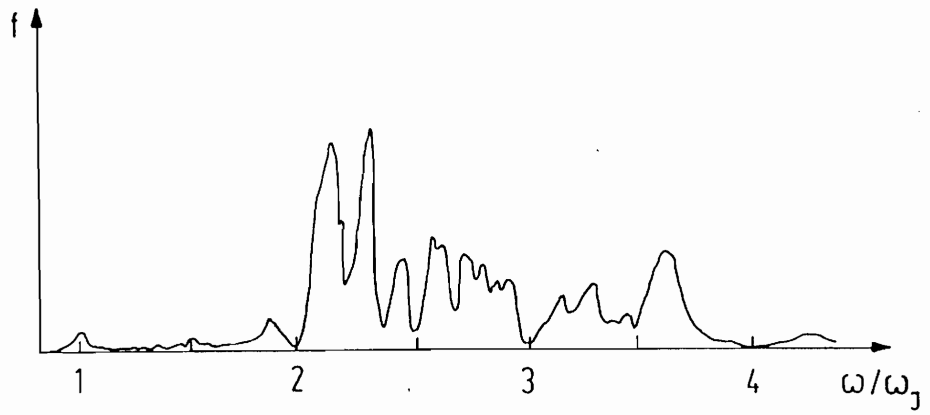


Figure 10



Fraction f of asteroids in the belt between Mars and Jupiter as a function of ω/ω_J (after Berry, 1978).

Figure 11

4.2.2 The Aubry-Mather Theorem

Before we tackle the Aubry-Mather theorem I will digress a little on the subject of Cantor sets. An example of a *Cantor set* is given by the following construction: Take the interval [0,1] and remove the open "middle third", *i.e.* the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. Next remove the central $\frac{1}{9}$, then $\frac{1}{27}$ etc., of each piece. Continuing ad infinitum we obtain a *Cantor set*.: Note that the length (measure) L of this *Cantor set* is finite:

$$L = \prod_{n=1}^{\infty}\left(1 - \frac{1}{3^n}\right) \approx .28006\ldots$$

The *Hausdorff dimension* of this *Cantor set* is one. (*Cantor sets* with nonzero length are sometimes called fat).

Now the Aubry-Mather theorem roughly says that for any irrational winding number there is either a smooth invariant torus with that number or an invariant *Cantor set* with that rotation. Moreover, this *Cantor set* is a subset of a closed curve that is Lipschitz continuous.

The picture that emerges combining the results in this and the previous section is that very near an integrable system almost all irrational tori persist. But that as the perturbation perameter in ε is increased, more and more of them develop holes and become cantori.

The Aubry-Mather theorem has been proved only in D = 2. It does however, not require the system to be close to integrable.

4.4.3 The Poincaré-Birkhoff theorem

In 1912 Poincaré enunciated the following theorem but succeeded in giving a proof only in certain special cases.

The general case was proved by Birkhoff a year later (Birkhoff, 1913).

THEOREM: Let us suppose that a continuous one-to-one area-preserving transformation T takes the ring R formed by concentric circles C_a and C_b of radii a and b respectively, ($a > b > 0$), into itself in such a way as to advance the points of C_a in a positive sense and the points of C_b in the negative sense. *Then there are at least two invariant points*.

Using this theorem it is easy to show that of the infinitely many periodic orbits of a given period in an integrable system at least 2 remain under perturbation. One of these will be elliptic, the other hyperbolic. The proof goes as follows:

Consider a circle with rational winding number $\frac{\omega}{2\pi} = p/q$ in an integrable mapping M_0. The KAM-theorem tells us that the circles with radius a and b with $\frac{\omega(a)}{2\pi}$ an irrational number greater than $\frac{\omega}{2\pi}$ and $\frac{\omega(b)}{2\pi}$ an irrational number smaller than $\frac{\omega}{2\pi}$ (where ω(a) and ω(b) do not lie in the "excluded volume") persist in the mapping M_ε.

Now take for $T = M_\varepsilon^q$ then the above theorem says T has 2 fixed points. This implies that M_ε has 2 orbits of period p/q. □

5. UNSTABLE FIXED POINTS. THE NONLINEAR CASE

5.1 Unstable Fixed Points in Integrable Mappings

Remember that in the ~~linear iteration~~ linearisation about an unstable fixed point there is one eigenvalue greater than one and one eigenvalue smaller than one.

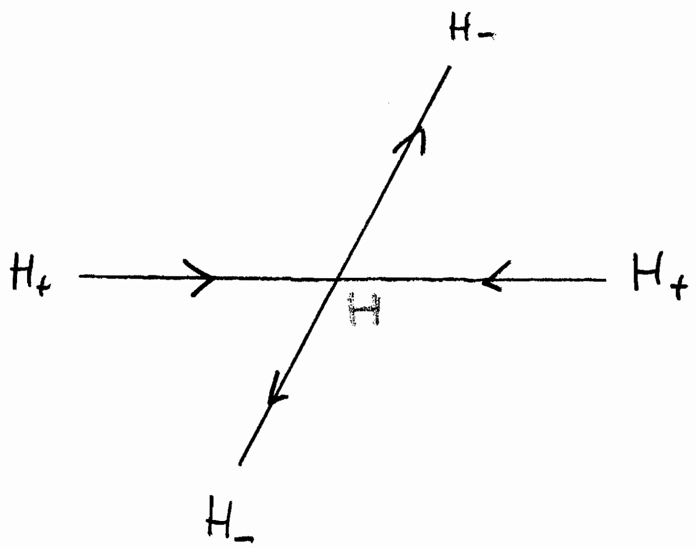


Figure 12.

At any hyperbolic point H, four invariant curves meet. Two of these are ingoing curves H_+ and the other two are outgoing curves H_-. A point X lies on H_+ if it arrives at H after infinitely many iterations of the mapping M, *i.e.,*

$$M^n X \to H \text{ as } n \to \infty$$

if X is on H^+. Similarly, a point X is on H^- if it was at H infinitely many iterations ago i.e.

$$M^{-n} X \to H \text{ as } \quad n \to \infty$$

if X is on H^-.

What happens as we follow the arcs H_+ and H_- away from H?

In an integrable nonlinear mapping there are three possibilites.

1. H^+ (H^-) goes off to infinity.
2. H^+ meets H^- of the same fixed point and coincides with it exactly (homoclinic case).
3. H^+ meets H^- of another fixed part and coincides with it exactly (heteroclinic case).

Examples of 2 and 3 (and 1) are seen in Figs. 13 and 14 respectively.

5.2 Unstable Fixed Points in Stochastic Mappings

It is not very surprising that homoclinic and heteroclinic separatrices do not persist under perturbation. So what happens generically? One thing that cannot happen is for any arc, say H_+, to intersect itself. However, it can and does happen that the H_+ and H_- that exactly meet in the integrable case now intersect. If there is one intersection like this then there must be infinitely many and we are led to the complicated picture sketched in Fig. 15.

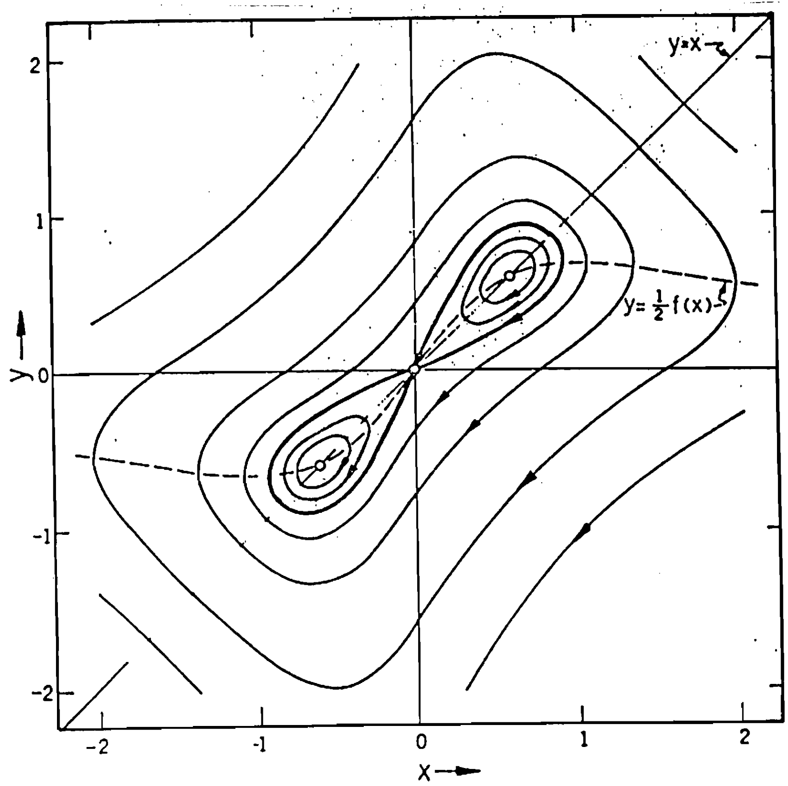
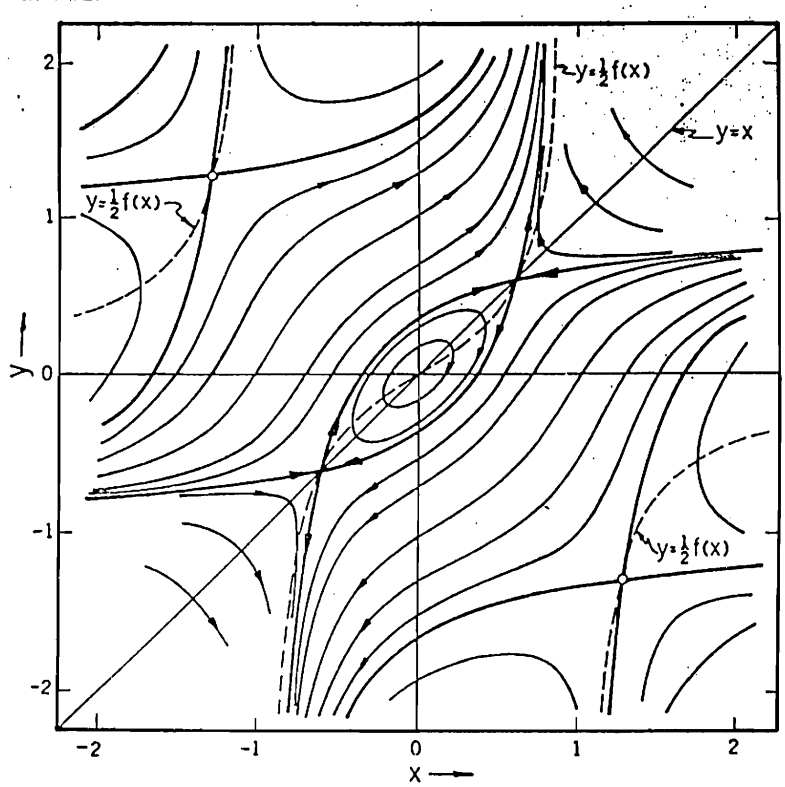


Fig. 13



The transformation $x' = y$, $y' = -x + f(y)$, with $f(y) = 2ky/(1 - y^2)$, has the family of invariant curves $x^2y^2 - x^2 - y^2 + 2kxy = \text{const}$. Some members of this family, for the case $k = 0.64$, are shown.

Fig. 14

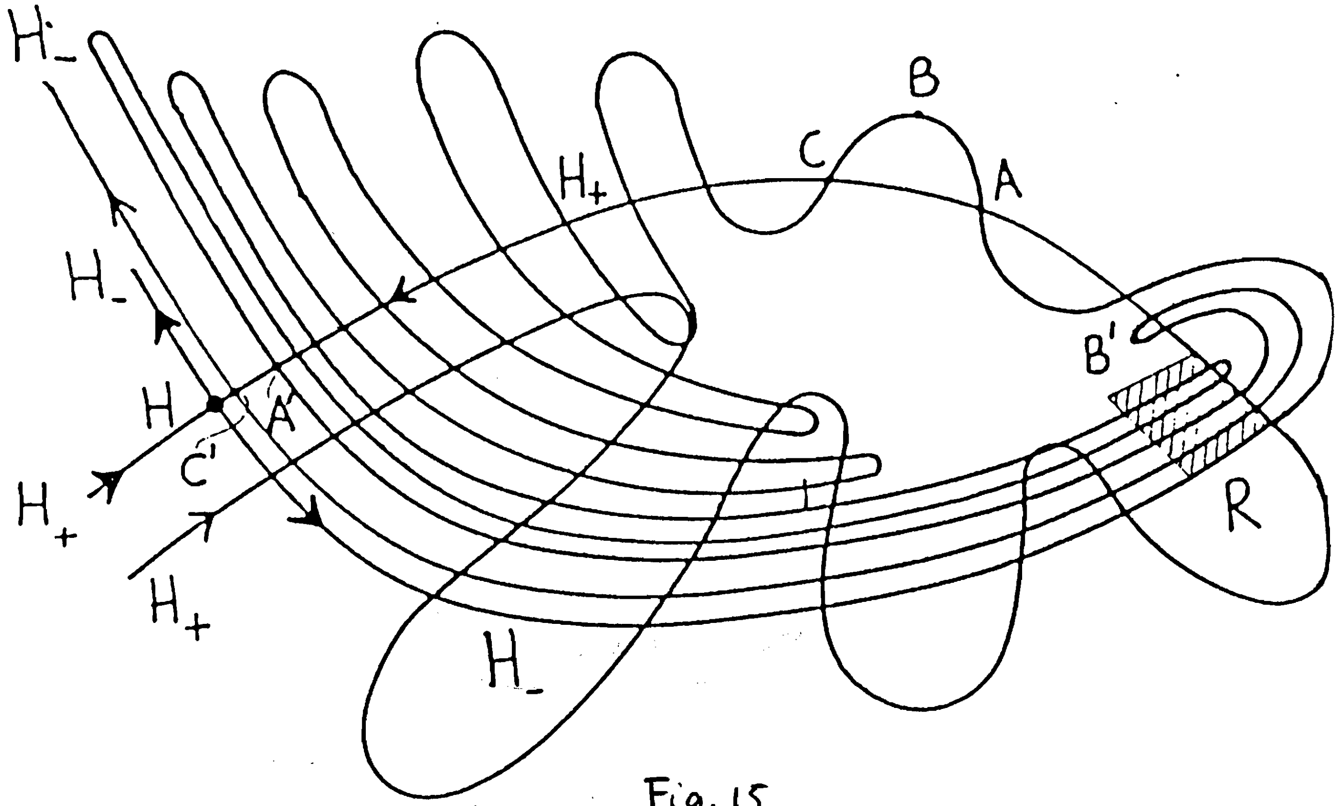


Fig. 15

5.3 Unstable Fixed Points in Ergodic Mappings.

In previous sections, we linked the origin of chaotic motion with the existence of hyperbolic fixed points. In order for statistical mechanics to be applicable we want the system to be very chaotic, in fact, ergodic, for then space averages are equal to time averages.

A sufficient (but not necessary) condition for a system to be ergodic is that it is a so-called Anosov System as can be seen in Figure 16.

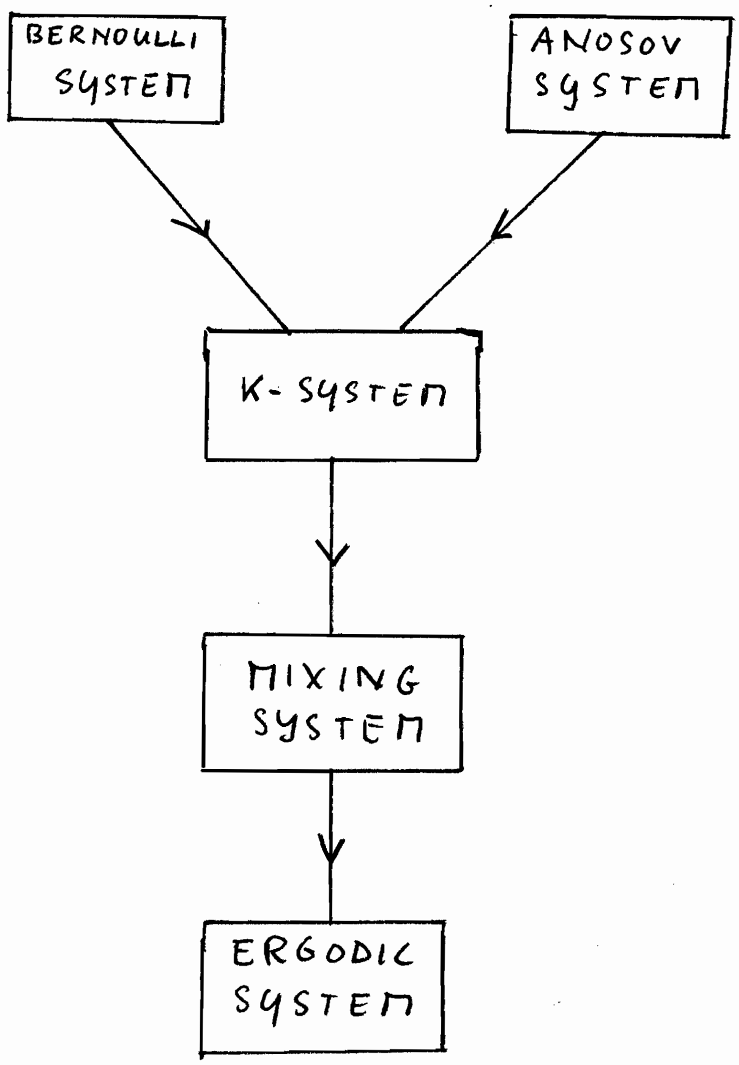


Figure 16

K-system → positive Kolmogorov-entropy

Mixing System → approach equilibrium

Ergodic system → time average equals phase space average.

As an example of an Anosov system consider the "cat-map"

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix}, \bmod \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{where } M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

This is a linear map, so it is rather exceptional. I will show that all periodic orbits of this map are hyperbolic and that the map has positive Kolmogorov-entropy.

To study the periodic orbits let's start with period 1:

$$\begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix}$$

It follows that $x = y = 0$. The eigenvalues at this fixed point are the eigenvalues of M, *i.e.*.

$$\lambda_1 = \frac{3}{2} + \frac{1}{2}\sqrt{5} \qquad \text{and } \lambda_2 = \lambda_1^{-1} \; .$$

H^+ and H^- fill the torus densely.

The period n orbits are given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = M^n \begin{pmatrix} x \\ y \end{pmatrix} \qquad \bmod \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and their eigenvalues are λ_1^n and λ_1^{-n}. This shows that **all** periodic orbits are hyperbolic.

To calculate the Kolmogorov entropy of this map we first need to define Lyapunov exponents. The Lyapunov exponents of an orbit are related to the eigenvalues of the Jacobian matrix.

$$J_n = \begin{pmatrix} \dfrac{\partial x_n}{\partial x_o} & \dfrac{\partial x_n}{\partial y_o} \\[2ex] \dfrac{\partial y_n}{\partial x_o} & \dfrac{\partial y_n}{\partial y_o} \end{pmatrix}$$

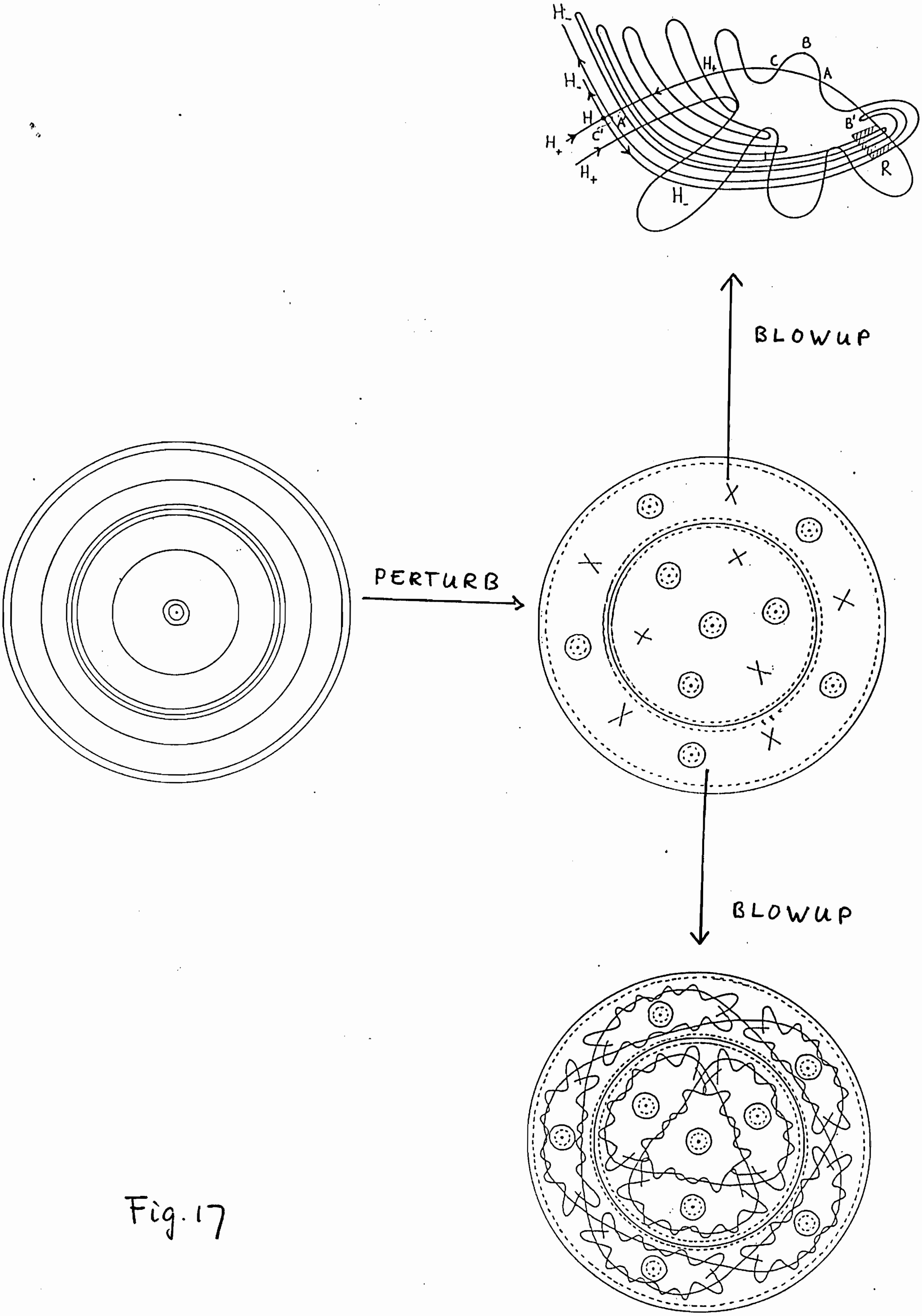


Fig. 17

If J_n has eigenvalues $\lambda_{1,n}$ and $\lambda_{2,n}$ then the Lyapunov exponents are

$$\lim_{n\to\infty} \frac{1}{n} \ln\left|\lambda_{1,n}\right| \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \ln\left|\lambda_{2,n}\right|$$

The Kolmogorov entropy is then given by the sum of the phase space averages of the positive Lyapunov exponents.

In our case taking the phase space average is simple since the Lyapunov exponent is constant. The limit $n\to\infty$ is also simple since $\frac{1}{n}\ln\left|\lambda_{1,n}\right|$ doesn't depend on n. Hence we get

$$\text{Kolmogorov entropy} = \ln\left(\frac{3}{2}+\frac{1}{2}\sqrt{5}\right)$$

This checks with the fact that the system is mixing.

6. PUTTING IT ALL TOGETHER

Having discussed what happens to stable and unstable fixed points in integrable systems under small perturbations we can now put the pieces together.

However, before we do this I must tell you the **local** version of the KAM theorem. (See e.g. Sevryuk (1986) p.4 and p. 147). This version says that near a generic elliptic fixed point of a **nonintegrable** mapping the KAM theorem also holds, where the small parameter ε now represents the distance from the fixed point. The Aubry-Mather and Poincaré-Birkhoff theorems also hold locally near any fixed point so we get all phenomena repeating on a smaller and smaller scale. This is indicated in Fig. 17.

$$r=1,\ \theta = 0,\ \frac{2\pi}{3},\ \pi \text{ or } \frac{4\pi}{3}\ .$$

This is an illustration of the Poincaré-Birkhoff theorem. The circle r=1 in the integrable mapping (8.1) breaks into an even number of period 1 orbits.

At $r = 1,\ \theta = \dfrac{2\pi}{3}$ the Jacobian is

$$\frac{1-\frac{1}{2}\sqrt{3}\ \varepsilon}{1+\frac{1}{2}\sqrt{3}\ \varepsilon}$$

at $r = 1,\ \theta = \dfrac{4\pi}{3}$ it is

$$\frac{1+\frac{1}{2}\sqrt{3}\ \varepsilon}{1-\frac{1}{2}\sqrt{3}\ \varepsilon}$$

So at these fixed points area is not preserved!

Appendix of Mathematical Definitions (Devaney (1986))

Cantor set

A set is a Cantor set if it is a closed, totally disconnected and perfect subset of a closed interval. A set is toally disconnected if it contains no intervals; a set is perfect if every point in it is an accumulation point or *limit point* of other points in the set.

Limit point

Let $S \subset \mathbb{R}$. A point $x \varepsilon \mathbb{R}$ is a limit point of S if there is a sequence of points $x_n \subset S$ converging to x.

Closed Set

Let $S \subset \mathbb{R}$. S is a closed set if it contains all of its *limit points*

Closure

Let $S \subset \mathbb{R}$. The closure of S consists of all points in S together with all *limit points* of S.

Dense

A subset U of $S \subset \mathbb{R}$ is dense if the *closure* of U is S.

Hausdorff dimension

If for a set of points in d dimensions the number $N(\ell)$ of d-spheres of diameter ℓ needed to cover the set increases like

$$N(\ell) \propto \ell^{-D} \text{ for } \ell \rightarrow 0$$

then D is called the *Hausdorff dimension* of the set.