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ON SOME LINEAR INTEGRAL EQUATIONS GENERATING SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Two types of linear inhomogeneous integral equations, which yield solutions of a broad class of nonlinear evolution equations, are investigated. One type is characterized by a two-fold integration with an arbitrary measure and contour over a complex variable k, and their complex conjugates, whereas the other one has a two-fold integration over one and the same contour. The inhomogeneous term, which may contain an arbitrary function of k, makes it possible to define a matrix structure on the solutions of the integral equations. The elements of these matrices are shown to obey a system of partial differential equations, the special form of which depends on the choice of the dispersion relation occurring in the integral equations. For special elements of the matrices closed partial differential equations can be derived, such as e.g. the nonlinear Schrödinger equation and the (real and complex) modified Korteweg–de Vries and sine-Gordon equations. The relations between the matrix elements are shown to lead to Miura transformations between the various partial differential equations.

1. Introduction

The study of nonlinear partial differential equations (PDE's), that are solvable by means of the inverse-scattering transform (IST) formalism, has become of great interest during the last decade. For many integrable PDE's the various ingredients of this formalism have been established and exact solutions, such as e.g. soliton solutions, have been found in a systematic way[1]). One of the underlying difficulties for this method is the choice of boundary conditions, which have to be taken into account from the very beginning. This feature very often obscures the fact that the crucial step of the inverse problem, i.e., the Gel'fand–Levitan–Marchenko equation, can in many respects be formulated in a way, which is independent of the choice of boundary conditions. A common feature of the nonlinear PDE's solvable via the IST is that solutions of these equations can be mapped onto a linear inhomogeneous integral equation and it is possible to explore these integral equations from a more general point of view[2,3]).

Recently, however, a new type of linear integral equation for the linearization of the Korteweg–de Vries equation (KdV) was proposed by Fokas and Ablowitz[4]), which enables one to extract complete information on the essential nonlinearity of the corresponding PDE, without having to go through the details of the inverse-scattering formalism, such as e.g. the choice of boundary conditions. The important feature is that the integral equation has a singular kernel and that it contains an integration with an arbitrary measure over a complex variable k, over an arbitrary contour in the complex k-plane. By choosing appropriate measures and contours, the various solutions of the KdV can be obtained directly from the integral equation.

As a first extension to the approach of Fokas and Ablowitz, one can take into consideration a more general inhomogeneous term in the integral equation, leading to a matrix structure, which can be inferred from the solutions. Taking into account the relations between the matrix elements, it has been shown that the integral equation of Fokas and Ablowitz provides solutions of the modified Korteweg–de Vries equation (MKdV) as well[5]). Secondly, one can take into consideration different types of integral equations. In refs. 5, 6 we have shown that the solutions of both the nonlinear Schrödinger equation (NLS) and the equation of motion for the classical Isotropic Heisenberg Spin Chain (IHSC) in the continuum limit can be found from one and the same linear integral equation. This has also been shown to be the case for the Boussinesq equation (BSQ) and the modified Boussinesq equation (MBSQ)[7]).

A major advantage of this way of treating nonlinear PDE's is that many relations between different PDE's become apparent in this context. As a first example, the Miura transformation connecting e.g. the KdV and the MKdV[8]), the NLS and the IHSC[9,10]), and the BSQ and the MBSQ[11]), can be derived systematically as a corollary from the integral equations, without having to rely on rather ad-hoc methods. Another example is the gauge equivalence between the Lax representations of the NLS and the IHSC[12]), which has been derived in a systematic way from the integral equation, cf. ref. 6. Similar gauge equivalences have also been discovered in the context of nonlinear σ-models[13]). Thirdly, the integral equations lead in a direct way to the associated linear eigenvalue problems for the PDE's under consideration, in which the solutions of the integral equation can be identified with the eigenvectors. Eliminating the potentials, which are the solutions of the PDE's, one can derive a nonlinear (integrable) PDE for the eigenvectors, cf. ref. 14. In this way the solutions of the classical Heisenberg spin chain with uniaxial anisotropy (AHSC) can be found from the linear integral equation for the NLS as well[15]).

The integral equation, which we have proposed in ref. 6 for the linearization of the nonlinear Schrödinger equation, has the following form

$$\phi_k^{(n)}(x,t) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')(l'-l)}\,\phi_l^{(n)}(x,t) = \frac{1}{k^n}\,\mathrm{e}^{\mathrm{i}(kx-k^2t)}, \quad (1.1)$$

where n is an integer labelling the different solutions. In eq. (1.1) C and C^* denote an arbitrary contour and its complex conjugate in the complex k-plane, and $\mathrm{d}\lambda(k)$ and $\mathrm{d}\lambda^*(k')$ are an arbitrary measure and its complex conjugate. By choosing different measures and contours we can find different solutions $\phi_k^{(n)}$.

The choice of measures and contours is restricted by two conditions:

(i) The contour C and the measure $\mathrm{d}\lambda(k)$ are to be chosen such that the kernel of the integral equation is properly defined and *regular*, in the sense that the solution $\phi_k^{(n)}$, for given measure and contour, is unique. This means e.g. that the homogeneous integral equation, i.e. the integral equation with the right-hand side replaced by zero, has only the zero solution.

(ii) The contour and the measure must be such that the differentiations with respect to x and t can be shifted through the integrals.

Taking into account these conditions, it has been possible to derive PDE's for the functions

$$\phi_{n,m}(x,t) = \int_C \mathrm{d}\lambda(k)\,\frac{1}{k^m}\,\phi_k^{(n)}(x,t), \quad (1.2)$$

defined with respect to the same measure and contour as in eq. (1.1). In particular we have shown that the function $\phi_{0,0}$ satisfies the NLS and that $\phi_{0,1} = \phi_{1,0}$ and $\phi_{1,1}$ satisfy the IHSC[6]). By making use of the factor $1/k^n$ in the inhomogeneous term of eq. (1.1), we have been able to obtain solutions of different PDE's from one and the same integral equation.

In eq. (1.1) we have chosen a very specific dispersion relation $\omega(k) = k^2$ for the time-dependence occurring in the exponentials. Many of the conclusions, however, which can be drawn from the integral equation, are independent of the choice of the dispersion relation. Thus an obvious generalization of the treatment in ref. 6 is provided, if we choose a more general dispersion. In this way it is possible to derive from the integral equation a broad class of nonlinear PDE's, which may have different dispersive behaviour. All the time-independent features, however, will remain the same for the equations of such a class.

The present paper is devoted to the study of two types of linear integral equations, namely

$$\text{I.}\quad \phi_k^{(n)}(x,t) \pm \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k(x,t)\rho^*_{l'}(x,t)}{(k-l')(l'-l)}\,\phi_l^{(n)}(x,t) = \frac{1}{k^n}\,\rho_k(x,t) \quad (1.3)$$

and

$$\text{II.}\quad v_k^{(n)}(x,t) + \int_C \mathrm{d}\lambda(l) \int_C \mathrm{d}\lambda(l') \frac{\rho_k(x,t)\rho_{l'}(x,t)}{(k+l')(l'+l)}\,v_l^{(n)}(x,t) = \frac{1}{k^n}\,\rho_k(x,t). \quad (1.4)$$

Eq. (1.3), which from now on will be denoted as type I, is an obvious generalization of eq. (1.1), the difference being that the exponentials in (1.1) are replaced by functions $\rho_k(x,t)$ proportional to $e^{i(kx-k^r t)}$ with the dispersion relation $\omega(k)=k^r$, r being an integer, and satisfying the linear differential equations

$$-i\partial_x\rho_k(x,t)=k\rho_k(x,t),\quad i\partial_t\rho_k(x,t)=k^r\rho_k(x,t),\quad (r\text{ integer}). \tag{1.5}$$

Eq. (1.4) is a new integral equation (type II), in which the integrations are performed twice over the same contour C. For both equations it is understood that the measure and the contour are chosen such that the conditions (i) and (ii), mentioned above, are satisfied.

The outline of this paper is as follows. In section 2 we derive a set of algebraic relations and differential equations for the quantities

$$\phi_{n,m}(x,t)=\int_C d\lambda(k)\frac{1}{k^m}\phi_k^{(n)}(x,t),\quad \psi_{n,m}(x,t)=\int_C d\lambda(k)\frac{1}{k^m}\psi_k^{(n)}(x,t), \tag{1.6}$$

where

$$\psi_k^{(n)}(x,t)=\int_{C^*} d\lambda^*(l')\frac{\rho_k(x,t)}{k-l'}\phi_{l'}^{(n)*}(x,t). \tag{1.7}$$

In section 3 the same is done for the integral equation of type II, for which we have

$$v_{n,m}(x,t)=\int_C d\lambda(k)\frac{1}{k^m}v_k^{(n)}(x,t),\quad w_{n,m}(x,t)=\int_C d\lambda(k)\frac{1}{k^m}w_k^{(n)}(x,t) \tag{1.8}$$

with

$$w_k^{(n)}(x,t)=\int_C d\lambda(l)\frac{\rho_k(x,t)}{k+l}v_l^{(n)}(x,t). \tag{1.9}$$

In section 4 these relations will be used, both for type I and type II, to give a general framework in order to derive, for all positive values of r, cf. (1.5), closed PDE's in terms of $\phi_{0,0}$ and $v_{0,0}$ only, and we present some explicit results for $r=2, 3, 4, 5$. In section 5 also PDE's for some other values of (n,m) are derived, together with the Miura transformations, which connect the solutions for different values of n and m. For negative values of r, it is more difficult to derive closed PDE's; some explicit results for the case $r=-1$ will be presented in section 6. Finally, in section 7 it is discussed how a reduction to a class of integral equations with only a single integration can be performed, leading e.g. to a generalization of the integral equation given by Fokas and Ablowitz[4]).

The treatment we give here provides a unifying framework for many partial differential equations. From the variety of equations, that can be described by one and the same linear integral equation, many conclusions can be drawn. The recent results on the connection between the conserved densities[16,17]), and the Lie-Bäcklund symmetries[18,19]) for the MKdV on the one hand, and the ones for the sine-Gordon equation on the other hand, can be regarded as a direct consequence of the integral equation. Furthermore, in a forthcoming publication we will show how Bäcklund transformations can be derived immediately from the integral equation by a singular transformation of measures[20]). Some of the connections we present here have also been found by Hirota[21]), who showed by a different method that several PDE's can be transformed into one and the same bilinear form. The relation between Hirota's method and our approach, however, is yet to be clarified. Moreover, it is not obvious that bilinearization implies exact integrability, cf. ref. 22.

2. Integral equation of type I; constitutive relations

The integral equation (1.3) can be written as a system of two coupled integral equations, i.e.,

$$\phi_k^{(n)}+\eta\int_{C^*} d\lambda^*(l')\frac{\rho_k\psi_{l'}^{(m)*}}{k-l'}=\frac{1}{k^n}\rho_k,\quad (\eta=\pm1), \tag{2.1a}$$

$$\psi_k^{(n)}-\int_{C^*} d\lambda^*(l')\frac{\rho_k\phi_{l'}^{(n)*}}{k-l'}=0. \tag{2.1b}$$

By taking the complex conjugate of (1.3) and using (2.1b), it is not difficult to show that the function $\psi_k^{(n)}$ also obeys an integral equation of type I, but with a different source term

$$\psi_k^{(n)}+\eta\int_C d\lambda(l)\int_{C^*} d\lambda^*(l')\frac{\rho_k\overset{*}{\rho}_{l'}}{(k-l')(l'-l)}\psi_l^{(n)}=\int_{C^*} d\lambda^*(l')\frac{\rho_k\overset{*}{\rho}_{l'}}{(k-l')l'^n}. \tag{2.2}$$

The functions $\phi_{n,m}$ and $\psi_{n,m}$, which will be investigated in this paper, can be regarded as elements of the infinite-dimensional matrices $\mathbf{\Phi}$ and $\mathbf{\Psi}$ resp., i.e.

$$(\mathbf{\Phi})_{n,m}=\phi_{n,m},\quad (\mathbf{\Psi})_{n,m}=\psi_{n,m}, \tag{2.3}$$

where n and m can have all integer values.

These matrices have simple symmetry properties, as can be seen as follows.

Eq. (1.6) together with (2.1a) can be rewritten as

$$\phi_{n,m} = \int_C \mathrm{d}\lambda(k)\, \frac{\phi_k^{(n)}}{\rho_k}\frac{1}{k^m}\rho_k = \int_C \mathrm{d}\lambda(k)\, \frac{\phi_k^{(n)}}{\rho_k}\left(\phi_k^{(m)} + \eta \int_{C^*} \mathrm{d}\lambda^*(l')\frac{\rho_k\psi_{l'}^{(m)*}}{k-l'}\right). \tag{2.4}$$

Interchanging the integrations over C and C^* and using (2.1b), yields the bilinear expression

$$\phi_{n,m} = \int_C \mathrm{d}\lambda(k)\, \frac{\phi_k^{(n)}\phi_k^{(m)}}{\rho_k} - \eta \int_{C^*} \mathrm{d}\lambda^*(l')\frac{\psi_{l'}^{(n)*}\psi_{l'}^{(m)*}}{\rho^*_{l'}}, \tag{2.5}$$

and from (2.5) it is immediately clear that

$$\mathbf{\Phi} = \mathbf{\Phi}^{\mathrm{T}}, \tag{2.6}$$

where the superscript T denotes the transposed matrix. In a similar way the bilinear expression for $\psi_{n,m}$ can be derived. From (1.6) and (2.1a) we have

$$\psi_{n,m} = \int_C \mathrm{d}\lambda(k)\, \frac{\psi_k^{(n)}}{\rho_k}\left(\phi_k^{(m)} + \eta \int_{C^*} \mathrm{d}\lambda^*(l')\frac{\rho_k\psi_{l'}^{(m)*}}{k-l'}\right). \tag{2.7}$$

Inserting (2.1b) in the first term of the right-hand side yields

$$\psi_{n,m} = \int_C \mathrm{d}\lambda(k) \int_{C^*} \mathrm{d}\lambda^*(l')\frac{\eta\psi_k^{(n)}\psi_{l'}^{(m)*} + \phi_k^{(m)}\phi_{l'}^{(n)*}}{k-l'}, \tag{2.8}$$

from which it is obvious that

$$\mathbf{\Psi} = -\mathbf{\Psi}^{\dagger}, \tag{2.9}$$

where $\mathbf{\Psi}^\dagger$ is the hermitean conjugate of $\mathbf{\Psi}$.

From eqs. (1.3) and (2.2), or equivalently (2.1a) and (2.1b) one can derive a set of algebraic relations connecting the different functions $\phi_k^{(n)}$ and $\psi_k^{(n)}$. For that purpose we consider the functions $k^p\phi_k^{(n)}(x,t)$, where p is an integer. Multiplying (1.3) by k^p, we have

$$\begin{aligned} k^p\phi_k^{(n)} &+ \eta \int_{C^*} \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l')\frac{\rho_k\rho^*_{l'}}{(k-l')(l'-l)}\, l^p\phi_l^{(n)} \\ &+ \eta \int_{C} \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l')\frac{\rho_k\rho^*_{l'}}{(k-l')(l'-l)}\left[(k^p - l'^p) + (l'^p - l^p)\right]\phi_l^{(n)} = \frac{1}{k^{n-p}}\rho_k. \end{aligned} \tag{2.10}$$

For positive p use can be made of the identity

$$k^p - l'^p = (k-l')\sum_{j=0}^{p-1} k^{p-1-j}l'^j, \quad (p>0), \tag{2.11}$$

in order to rewrite the last term on the left-hand side of (2.10) in the following way, using also eqs. (2.1b) and (1.6),

$$\begin{aligned} &\eta\sum_{j=0}^{p-1}\int_{C^*} \mathrm{d}\lambda^*(l')\left(k^{p-1-j}l'^j\rho_k\psi_{l'}^{(n)*} + \int_C \mathrm{d}\lambda(l)\, l'^{p-1-j}l^j\,\frac{\rho_k\rho^*_{l'}}{k-l'}\,\phi_l^{(n)}\right) \\ &= \eta\sum_{j=0}^{p-1}\left(\psi^*_{n,-j}k^{p-1-j}\rho_k + \phi_{n,-j}\int_{C^*} \mathrm{d}\lambda^*(l')\,\frac{\rho_k\rho^*_{l'}}{l'^{j+1-p}(k-l')}\right). \end{aligned} \tag{2.12}$$

Inserting (2.12) in (2.10) we find

$$\begin{aligned} &k^p\phi_k^{(n)} + \eta\int_C \mathrm{d}\lambda(l)\int_{C^*} \mathrm{d}\lambda^*(l')\,\frac{\rho_k\rho^*_{l'}}{(k-l')(l'-l)}\, l^p\phi_l^{(n)} \\ &= \frac{1}{k^{n-p}}\rho_k - \eta\sum_{j=0}^{p-1}\psi^*_{n,-j}k^{p-1-j}\rho_k - \eta\sum_{j=0}^{p-1}\phi_{n,-j}\int_{C^*} \mathrm{d}\lambda^*(l')\,\frac{\rho_k\rho^*_{l'}}{l'^{j+1-p}(k-l')}. \end{aligned} \tag{2.13}$$

Thus $k^p\phi_k^{(n)}$ obeys an integral equation of type I, but with a different source term. Comparing eq. (2.13) with eqs. (1.3) and (2.2) and using the regularity condition that the homogeneous integral equation has only the zero solution, we find the relation

$$k^p\phi_k^{(n)} = \phi_k^{(n-p)} - \eta\sum_{j=0}^{p-1}\left(\psi^*_{n,-j}\phi_k^{(j+1-p)} + \phi_{n,-j}\psi_k^{(j+1-p)}\right), \quad (p>0). \tag{2.14}$$

A similar relation can be derived for $\psi_k^{(n)}$. Multiplying (2.1b) with k^p, we have

$$k^p\psi_k^{(n)} = \int_{C^*} \mathrm{d}\lambda^*(l')\,(k^p - l'^p)\,\frac{\rho_k\phi_{l'}^{(n)*}}{k-l'} + \int_{C^*} \mathrm{d}\lambda^*(l')\frac{\rho_k l'^p\phi_{l'}^{(n)*}}{k-l'}. \tag{2.15}$$

Making use of (2.11) in the first term on the right-hand side of (2.15), and substituting (2.14) in the second term, eq. (2.15) can be rewritten as

$$\begin{aligned} k^p\psi_k^{(n)} &= \int_{C^*} \mathrm{d}\lambda^*(l')\frac{\rho_k\phi_{l'}^{(n-p)*}}{k-l'} + \sum_{j=0}^{p-1}\phi^*_{n,-j}k^{p-1-j}\rho_k \\ &\quad - \eta\sum_{j=0}^{p-1}\int_{C^*} \mathrm{d}\lambda^*(l')\,\frac{\rho_k}{k-l'}\left(\psi_{n,-j}\phi_{l'}^{(j+1-p)*} + \phi^*_{n,-j}\psi_{l'}^{(j+1-p)*}\right). \end{aligned} \tag{2.16}$$

Using (2.1a) and (2.1b), we immediately have

$$k^p \psi_k^{(n)} = \psi_k^{(n-p)} + \sum_{j=0}^{p-1} (\phi_{n,-j}^* \phi_k^{(j+1-p)} - \eta \psi_{n,-j} \psi_k^{(j+1-p)}), \quad (p > 0). \tag{2.17}$$

For $p < 0$, we have, instead of (2.11)

$$k^p - l'^p = -(k - l') \sum_{j=0}^{-p-1} k^{-j-1} l'^{j+p}, \quad (p < 0), \tag{2.18}$$

and this yields, by the same line of reasoning

$$k^p \phi_k^{(n)} = \phi_k^{(n-p)} + \eta \sum_{j=0}^{-p-1} (\psi_{n,-j-p}^* \phi_k^{(j+1)} + \phi_{n,-j-p} \psi_k^{(j+1)}), \quad (p < 0), \tag{2.19}$$

and

$$k^p \psi_k^{(n)} = \psi_k^{(n-p)} - \sum_{j=0}^{-p-1} (\phi_{n,-j-p}^* \phi_k^{(j+1)} - \eta \psi_{n,-j-p} \psi_k^{(j+1)}), \quad (p < 0). \tag{2.20}$$

Eqs. (2.14), (2.17), (2.19) and (2.20) can be written in a more compact way, using the matrices $\mathbf{\Phi}$ and $\mathbf{\Psi}$, cf. (2.3), and introducing the vectors $\boldsymbol{\phi}_k$ and $\boldsymbol{\psi}_k$ with components $\phi_k^{(n)}$ and $\psi_k^{(n)}$. We have the algebraic relations

$$k^p \boldsymbol{\phi}_k = \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\phi}_k - \eta \mathbf{\Psi}^* \cdot \mathbf{Q}_p \cdot \boldsymbol{\phi}_k - \eta \mathbf{\Phi} \cdot \mathbf{Q}_p \cdot \boldsymbol{\psi}_k, \tag{2.21a}$$

$$k^p \boldsymbol{\psi}_k = \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\psi}_k + \mathbf{\Phi}^* \cdot \mathbf{Q}_p \cdot \boldsymbol{\phi}_k - \eta \mathbf{\Psi} \cdot \mathbf{Q}_p \cdot \boldsymbol{\psi}_k. \tag{2.21b}$$

Here $\mathbf{J}^{\mathrm{T}}$ denotes the matrix that lowers the superscript of the components $\phi_k^{(n)}$ and $\psi_k^{(n)}$ by 1, or equivalently it is the transposed of a matrix $\mathbf{J}$, which is defined by

$$(\mathbf{J})_{n,m} = \delta_{m,n+1}, \quad (\mathbf{J}^{\mathrm{T}})_{n,m} = \delta_{n,m+1}. \tag{2.22}$$

The matrix $\mathbf{Q}_p$ in (2.21) is given by

$$\mathbf{Q}_p = \sum_{j=0}^{p-1} \mathbf{J}^j \cdot \mathbf{O} \cdot \mathbf{J}^{\mathrm{T}p-1-j}, \quad (p \geq 0), \tag{2.23a}$$

$$\mathbf{Q}_p = -\sum_{j=0}^{-p-1} \mathbf{J}^{p+j} \cdot \mathbf{O} \cdot \mathbf{J}^{\mathrm{T}-j-1}, \quad (p < 0), \tag{2.23b}$$

where the matrix $\mathbf{O}$ is defined by

$$(\mathbf{O})_{n,m} = \delta_{n,0}\delta_{m,0}, \tag{2.24}$$

and where we have identified $\mathbf{J}^{-1}$ and $\mathbf{J}^{\mathrm{T}}$.

Next we derive differential relations for the $\phi_k^{(n)}$ and $\psi_k^{(n)}$. For that purpose we suppose a linear differential equation for ρ_k of the form

$$-\mathrm{i}\partial \rho_k = k^p \rho_k, \tag{2.25}$$

where ∂ can be either ∂_x or $-\partial_t$, corresponding to $p = 1$ and $p = r$, respectively, cf. eq. (1.5). Applying $-\mathrm{i}\partial$ to the integral equation (1.3), we find

$$-\mathrm{i}\partial\phi_k^{(n)} + \eta \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \overset{*}{\rho}_{l'}}{(k-l')(l'-l)} (-\mathrm{i}\partial\phi_l^{(n)})$$
$$+ \eta \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \overset{*}{\rho}_{l'}}{(k-l')(l'-l)} (k^p - l'^p)\phi_l^{(n)} = \frac{1}{k^{n-p}} \rho_k. \tag{2.26}$$

Using again (2.11) for $p \geq 0$, and also eq. (2.1b), eq. (2.26) can be rewritten as

$$-\mathrm{i}\partial\phi_k^{(n)} + \eta \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \overset{*}{\rho}_{l'}}{(k-l')(l'-l)} (-\mathrm{i}\partial\phi_l^{(n)})$$
$$= \frac{1}{k^{n-p}} \rho_k - \eta \sum_{j=0}^{p-1} \psi_{n,-j}^* k^{p-1-j} \rho_k. \tag{2.27}$$

From (2.27) we see that $-\mathrm{i}\partial\phi_k^{(n)}$ obeys an integral equation of type I, and by comparing (2.27) with (1.3), and using the regularity condition, we have the relation

$$-\mathrm{i}\partial\phi_k^{(n)} = \phi_k^{(n-p)} - \eta \sum_{j=0}^{p-1} \psi_{n,-j}^* \phi_k^{(j+1-p)}, \quad (p \geq 0). \tag{2.28}$$

For $p < 0$ we can again use (2.18), from which it follows in the same way that

$$-\mathrm{i}\partial\phi_k^{(n)} = \phi_k^{(n-p)} + \eta \sum_{j=0}^{-p-1} \psi_{n,-j-p}^* \phi_k^{(j+1)}, \quad (p < 0). \tag{2.29}$$

The relations for $\psi_k^{(n)}$ can be obtained by applying $-\mathrm{i}\partial$ to (2.1b). We find

$$-\mathrm{i}\partial\psi_k^{(n)} = \int_{C^*} \mathrm{d}\lambda^*(l')\,(k^p - l'^p) \frac{\rho_k \phi_{l'}^{(n)*}}{k-l'} - \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k}{k-l'} (-l'^p + \mathrm{i}\partial)\phi_{l'}^{(n)*}. \tag{2.30}$$

For $p \geq 0$, using (2.11) in the first term and (2.14) and (2.28) in the second term of the right-hand side of (2.30), we find

$$-\mathrm{i}\partial\psi_k^{(n)} = \sum_{j=0}^{p-1} \left(\phi_{n,-j}^* k^{p-1-j} \rho_k - \eta \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k}{k-l'} \phi_{n,-j}^* \psi_{l'}^{(j+1-p)*} \right)$$
$$= \sum_{j=0}^{p-1} \phi_{n,-j}^* \phi_k^{(j+1-p)}, \quad (p \geq 0), \tag{2.31}$$

where the last step follows from eq. (2.1a). For $p < 0$, by similar manipulations, we obtain the result

$$-\mathrm{i}\partial\psi_k^{(n)} = -\sum_{j=0}^{-p-1} \phi^*_{n,-j-p}\phi_k^{(j+1)}, \quad (p<0). \tag{2.32}$$

Using the matrices $\mathbf{J}^{\mathrm{T}}$ and $\mathbf{Q}_p$, cf. (2.22) and (2.23), eqs. (2.28), (2.29), (2.31) and (2.32) can be rewritten as

$$-\mathrm{i}\partial\boldsymbol{\phi}_k = \mathbf{J}^{\mathrm{T}p}\cdot\boldsymbol{\phi}_k - \eta\boldsymbol{\Psi}^*\cdot\mathbf{Q}_p\cdot\boldsymbol{\phi}_k, \tag{2.33a}$$

$$-\mathrm{i}\partial\boldsymbol{\psi}_k = \boldsymbol{\Phi}^*\cdot\mathbf{Q}_p\cdot\boldsymbol{\phi}_k. \tag{2.33b}$$

Taking into account eq. (1.5) with $\partial=\partial_x$, $p=1$, or $\partial=-\partial_t$, $p=r$, eqs. (2.33a) and (2.33b) lead to the relations

$$\mathrm{i}\partial_t\boldsymbol{\phi}_k = \mathbf{J}^{\mathrm{T}r}\cdot\boldsymbol{\phi}_k - \eta\boldsymbol{\Psi}^*\cdot\mathbf{Q}_r\cdot\boldsymbol{\phi}_k, \tag{2.34a}$$

$$\mathrm{i}\partial_t\boldsymbol{\psi}_k = \boldsymbol{\Phi}^*\cdot\mathbf{Q}_r\cdot\boldsymbol{\phi}_k, \tag{2.34b}$$

$$-\mathrm{i}\partial_x\boldsymbol{\phi}_k = \mathbf{J}^{\mathrm{T}}\cdot\boldsymbol{\phi}_k - \eta\boldsymbol{\Psi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k, \tag{2.34c}$$

$$-\mathrm{i}\partial_x\boldsymbol{\psi}_k = \boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k. \tag{2.34d}$$

Eqs. (2.34a)–(2.34d) in combination with the algebraic relations (2.21a) and (2.21b) form the constitutive relations arising from the linear integral equation (1.3) with (1.5). In fact, (2.21) and (2.34) form the linear problem for the matrices $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$, i.e. we have a set of linear equations for the wave functions $\boldsymbol{\phi}_k$ and $\boldsymbol{\psi}_k$, of which the coefficients contain the potentials $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$. Note that the potentials $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ can be obtained explicitly from the wave functions $\boldsymbol{\phi}_k$ and $\boldsymbol{\psi}_k$ by an integration over the contour C, cf. (1.6) and (2.3).

Multiplying (2.21) and (2.34) by k^{-m} and integrating over the contour C, we can derive the algebraic relations and the PDE's containing only the matrices $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$. The result is:

$$\boldsymbol{\Phi}\cdot\mathbf{J}^p = \mathbf{J}^{\mathrm{T}p}\cdot\boldsymbol{\Phi} - \eta\boldsymbol{\Psi}^*\cdot\mathbf{Q}_p\cdot\boldsymbol{\Phi} - \eta\boldsymbol{\Phi}\cdot\mathbf{Q}_p\cdot\boldsymbol{\Psi}, \tag{2.35a}$$

$$\boldsymbol{\Psi}\cdot\mathbf{J}^p = \mathbf{J}^{\mathrm{T}p}\cdot\boldsymbol{\Psi} + \boldsymbol{\Phi}^*\cdot\mathbf{Q}_p\cdot\boldsymbol{\Phi} - \eta\boldsymbol{\Psi}\cdot\mathbf{Q}_p\cdot\boldsymbol{\Psi}, \tag{2.35b}$$

$$-\mathrm{i}\partial_x\boldsymbol{\Phi} = \mathbf{J}^{\mathrm{T}}\cdot\boldsymbol{\Phi} - \eta\boldsymbol{\Psi}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi}, \tag{2.35c}$$

$$-\mathrm{i}\partial_x\boldsymbol{\Psi} = \boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi}, \tag{2.35d}$$

$$\mathrm{i}\partial_t\boldsymbol{\Phi} = \mathbf{J}^{\mathrm{T}r}\cdot\boldsymbol{\Phi} - \eta\boldsymbol{\Psi}^*\cdot\mathbf{Q}_r\cdot\boldsymbol{\Phi}, \tag{2.35e}$$

$$\mathrm{i}\partial_t\boldsymbol{\Psi} = \boldsymbol{\Phi}^*\cdot\mathbf{Q}_r\cdot\boldsymbol{\Phi}. \tag{2.35f}$$

Eqs. (2.35c)–(2.35f) for fixed r form a system of coupled partial differential equations, which, in combination with the algebraic relations (2.35a), (2.35b) for integer p, is completely integrable in the sense that solutions can be found

from a linear integral equation, cf. (1.3), (1.6), (1.7), (2.3). (The linear integral equation may be regarded as the spectral decomposition of the associated linear eigenvalue problem.)

Remark. Instead of eq. (1.5) one can also consider a more general dispersion relation

$$\omega(k) = \sum_r \lambda_r k^r, \tag{2.36}$$

leading to

$$\mathrm{i}\partial_t\rho_k = \omega(k)\rho_k. \tag{2.37}$$

In this case (2.35e) and (2.35f) can be rewritten as

$$\mathrm{i}\partial_t\boldsymbol{\Phi} = \omega(\mathbf{J}^{\mathrm{T}})\cdot\boldsymbol{\Phi} - \eta\boldsymbol{\Psi}^*\cdot\mathbf{Q}\cdot\boldsymbol{\Phi}, \tag{2.38a}$$

$$\mathrm{i}\partial_t\boldsymbol{\Psi} = \boldsymbol{\Phi}^*\cdot\mathbf{Q}\cdot\boldsymbol{\Phi}, \tag{2.38b}$$

in which the matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \sum_r \lambda_r\mathbf{Q}_r. \tag{2.39}$$

3. Integral equation of type II; constitutive relations

The analysis, given in the preceding section for the integral equation of type I, can be done with only minor modifications for the integral equation of type II, given in eq. (1.4). As in section 2 we rewrite eq. (1.4) as a system of two coupled integral equations

$$v_k^{(n)} + \int_C \mathrm{d}\lambda(l)\,\frac{\rho_k w_l^{(n)}}{k+l} = \frac{1}{k^n}\rho_k, \tag{3.1a}$$

$$w_k^{(n)} - \int_C \mathrm{d}\lambda(l)\,\frac{\rho_k v_l^{(n)}}{k+l} = 0. \tag{3.1b}$$

Here it is not useful to introduce a quantity $\eta=\pm1$, since a factor η in front of the second term of the left-hand side of (1.4) can be included in the measure $\mathrm{d}\lambda(l)$. The integral equation for the quantity $w_k^{(n)}$ is again of type II, but has a different source term

$$w_k^{(n)} + \int_C \mathrm{d}\lambda(l)\int_C \mathrm{d}\lambda(l')\,\frac{\rho_k\rho_{l'}}{(k+l')(l'+l)}\,w_l^{(n)} = \int_C \mathrm{d}\lambda(l)\,\frac{\rho_k\rho_l}{(k+l)l^n}. \tag{3.2}$$

We define matrices $\mathbf{V}$ and $\mathbf{W}$ by

$$(\mathbf{V})_{n,m} = v_{n,m}, \quad (\mathbf{W})_{n,m} = w_{n,m}, \tag{3.3}$$

cf. (1.8). These matrices are symmetric, i.e.

$$\mathbf{V} = \mathbf{V}^{\mathrm{T}}, \quad \mathbf{W} = \mathbf{W}^{\mathrm{T}}, \tag{3.4}$$

as follows from the bilinear expressions

$$v_{n,m} = \int_C \mathrm{d}\lambda(k) \frac{1}{\rho_k} (v_k^{(n)} v_k^{(m)} + w_k^{(n)} w_k^{(m)}) \tag{3.5}$$

and

$$w_{n,m} = \int_C \mathrm{d}\lambda(k) \int_C \mathrm{d}\lambda(l) \frac{v_l^{(n)} v_k^{(m)} + w_k^{(n)} w_l^{(m)}}{k+l}, \tag{3.6}$$

which can be derived in an analogous way as (2.5) and (2.8).

The algebraic relations for $v_k^{(n)}$ and $w_k^{(n)}$ can be derived as follows. From (1.4) one has

$$k^p v_k^{(n)} + \int_C \mathrm{d}\lambda(l) \int_C \mathrm{d}\lambda(l') \frac{\rho_k \rho_{l'}}{(k+l')(l'+l)} l^p v_l^{(n)} + \int_C \mathrm{d}\lambda(l) \int_C \mathrm{d}\lambda(l') \frac{\rho_k \rho_{l'}}{(k+l')(l'+l)}$$
$$\times [(k^p - (-l')^p) + ((-l')^p - l^p)] v_l^{(n)} = \frac{1}{k^{n-p}} \rho_k. \tag{3.7}$$

We now use the relations

$$k^p - (-l')^p = (k+l') \sum_{j=0}^{p-1} k^{p-1-j} (-l')^j, \quad (p>0), \tag{3.8}$$

and

$$k^p - (-l')^p = -(k+l') \sum_{j=0}^{-p-1} k^{-j-1} (-l')^{j+p}, \quad (p<0), \tag{3.9}$$

Then eq. (3.7) can be rewritten as

$$k^p v_k^{(n)} + \int_C \mathrm{d}\lambda(l) \int_C \mathrm{d}\lambda(l') \frac{\rho_k \rho_{l'}}{(k+l')(l'+l)} l^p v_l^{(n)}$$
$$= \frac{1}{k^{n-p}} \rho_k - \sum_{j=0}^{p-1} (-1)^j \Bigg(w_{n,-j} k^{p-1-j} \rho_k$$
$$+ (-1)^p v_{n,-j} \int_C \mathrm{d}\lambda(l) \frac{\rho_k \rho_l}{(k+l) l^{j+1-p}} \Bigg), \tag{3.10}$$

for $p<0$, and

$$k^p v_k^{(n)} + \int_C \mathrm{d}\lambda(l) \int_C \mathrm{d}\lambda(l') \frac{\rho_k \rho_{l'}}{(k+l')(l'+l)} l^p v_l^{(n)}$$
$$= \frac{1}{k^{n-p}} \rho_k + (-1)^p \sum_{j=0}^{-p-1} (-1)^j \Bigg(w_{n,-j-p} k^{-j-1} \rho_k$$
$$+ (-1)^p v_{n,-j-p} \int_C \mathrm{d}\lambda(l) \frac{\rho_k \rho_l}{(k+l) l^{j+1}} \Bigg), \tag{3.11}$$

for $p<0$, cf. eq. (1.8). From eqs. (3.10) and (3.11), we can conclude, on account of the regularity condition, that

$$k^p v_k^{(n)} = v_k^{(n-p)} - \sum_{j=0}^{p-1} (-1)^j (w_{n,-j} v_k^{(j+1-p)} + (-1)^p v_{n,-j} w_k^{(j+1-p)}), \quad (p \geq 0), \tag{3.12}$$

and

$$k^p v_k^{(n)} = v_k^{(n-p)} + (-1)^p \sum_{j=0}^{-p-1} (-1)^j (w_{n,-j-p} v_k^{(j+1)} + (-1)^p v_{n,-j-p} w_k^{(j+1)}), \quad (p<0). \tag{3.13}$$

Introducing the matrix $\mathbf{R}_p$ by

$$\mathbf{R}_p = \sum_{j=0}^{p-1} \mathbf{J}^j \cdot \mathbf{O} \cdot (-\mathbf{J}^{\mathrm{T}})^{p-1-j}, \quad (p \geq 0), \tag{3.14a}$$

and

$$\mathbf{R}_p = -\sum_{j=0}^{-p-1} \mathbf{J}^{j+p} \cdot \mathbf{O} \cdot (-\mathbf{J}^{\mathrm{T}})^{-j-1}, \quad (p<0), \tag{3.14b}$$

eqs. (3.12) and (3.13) can be combined to

$$k^p \boldsymbol{v}_k = \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{v}_k + (-1)^p \mathbf{W} \cdot \mathbf{R}_p \cdot \boldsymbol{v}_k + \mathbf{V} \cdot \mathbf{R}_p \cdot \boldsymbol{w}_k, \tag{3.15}$$

for all integer p, where $\boldsymbol{v}_k$ and $\boldsymbol{w}_k$ are vectors with components $v_k^{(n)}$ and $w_k^{(n)}$.

The algebraic relations for $w_k^{(n)}$ can be derived multiplying (3.1b) by k^p, i.e.

$$k^p w_k^{(n)} = \int_C \mathrm{d}\lambda(l) (k^p - (-l)^p) \frac{\rho_k v_l^{(n)}}{k+l} + (-1)^p \int_C \mathrm{d}\lambda(l) \frac{\rho_k l^p v_l^{(n)}}{k+l}. \tag{3.16}$$

Using (3.8) and (3.12) for $p \geq 0$, and (3.9) and (3.13) for $p<0$, we find

$$k^p w_k^{(n)} = (-1)^p w_k^{(n-p)} + \sum_{j=0}^{p-1} (-1)^j (v_{n,-j} v_k^{(j+1-p)} - (-1)^p w_{n,-j} w_k^{(j+1-p)}), \quad (p \geq 0), \tag{3.17}$$

and

$$k^p w_k^{(n)} = (-1)^p w_k^{(n-p)} + (-1)^p \sum_{j=0}^{-p-1} (-1)^j(-v_{n,-j-p} v_k^{(j+1)} + (-1)^p w_{n,-j-p} w_k^{(j+1)}), \quad (p<0). \quad (3.18)$$

In matrix notation eqs. (3.17) and (3.18) yield

$$k^p w_k = (-\mathbf{J}^{\mathrm{T}})^p \cdot w_k + \mathbf{W} \cdot \mathbf{R}_p \cdot w_k - (-1)^p \mathbf{V} \cdot \mathbf{R}_p \cdot v_k, \quad (3.19)$$

for all integer p.

In order to derive differential relations for $v_k^{(n)}$ and $w_k^{(n)}$, we can take again (2.25), but now we must assume p to be odd, in order that $\mathrm{i}\partial(\rho_k \rho_{l'})$ contains a factor $(k+l')$. Applying $-\mathrm{i}\partial$ to the integral equation (1.4), we have

$$-\mathrm{i}\partial v_k^{(n)} + \int_C \mathrm{d}\lambda(l) \int_C \mathrm{d}\lambda(l') \frac{\rho_k \rho_{l'}}{(k+l')(l'+l)} (-\mathrm{i}\partial v_l^{(n)})$$
$$+ \int_C \mathrm{d}\lambda(l) \int_C \mathrm{d}\lambda(l') \frac{\rho_k \rho_{l'}}{(k+l')(l'+l)} (k^p + l'^p) v_l^{(n)} = \frac{1}{k^{n-p}} \rho_k. \quad (3.20)$$

For odd values of p, we can use eq. (3.8) or (3.9) in the last term of the left-hand side of (3.20). Applying (3.1b) and the regularity condition for the integral equation (1.4), we obtain

$$-\mathrm{i}\partial v_k^{(n)} = v_k^{(n-p)} - \sum_{j=0}^{p-1} (-1)^j w_{n,-j} v_k^{(j+1-p)}, \quad (p \text{ odd}, p>0), \quad (3.21)$$

and

$$-\mathrm{i}\partial v_k^{(n)} = v_k^{(n-p)} - \sum_{j=0}^{-p-1} (-1)^j w_{n,-j-p} v_k^{(j+1)}, \quad (p \text{ odd}, p<0), \quad (3.22)$$

and eqs. (3.21) and (3.22) can be combined to

$$-\mathrm{i}\partial v_k = \mathbf{J}^{\mathrm{T}p} \cdot v_k - \mathbf{W} \cdot \mathbf{R}_p \cdot v_k, \quad (p \text{ odd}), \quad (3.23)$$

for odd values of p. Applying $-\mathrm{i}\partial$ to eq. (3.1b) we can derive an equation for $-\mathrm{i}\partial w_k^{(n)}$, which yields

$$-\mathrm{i}\partial w_k = \mathbf{V} \cdot \mathbf{R}_p \cdot v_k, \quad (p \text{ odd}). \quad (3.24)$$

Taking into account eq. (1.5) for the factors $\rho_k(x,t)$, we have the relations

$$\mathrm{i}\partial_t v_k = \mathbf{J}^{\mathrm{T}r} \cdot v_k - \mathbf{W} \cdot \mathbf{R}_r \cdot v_k, \quad (r \text{ odd}), \quad (3.25a)$$

$$\mathrm{i}\partial_t w_k = \mathbf{V} \cdot \mathbf{R}_r \cdot v_k, \quad (r \text{ odd}), \quad (3.25b)$$

$$-\mathrm{i}\partial_x v_k = \mathbf{J}^{\mathrm{T}} \cdot v_k - \mathbf{W} \cdot \mathbf{O} \cdot v_k, \quad (3.25c)$$

$$-\mathrm{i}\partial_x w_k = \mathbf{V} \cdot \mathbf{O} \cdot v_k, \quad (3.25d)$$

which, in combination with the algebraic relations (3.15) and (3.19), form the constitutive relations of the linear problem associated with the integral equation (1.4).

Multiplying (3.15), (3.19) and (3.25) by k^{-m} and integrating over the contour C, we obtain the equations

$$\mathbf{V} \cdot \mathbf{J}^p = \mathbf{J}^{\mathrm{T}p} \cdot \mathbf{V} + (-1)^p \mathbf{W} \cdot \mathbf{R}_p \cdot \mathbf{V} + \mathbf{V} \cdot \mathbf{R}_p \cdot \mathbf{W}, \quad (3.26a)$$

$$\mathbf{W} \cdot \mathbf{J}^p = (-\mathbf{J}^{\mathrm{T}})^p \cdot \mathbf{W} + \mathbf{W} \cdot \mathbf{R}_p \cdot \mathbf{W} - (-1)^p \mathbf{V} \cdot \mathbf{R}_p \cdot \mathbf{V}, \quad (3.26b)$$

$$-\mathrm{i}\partial_x \mathbf{V} = \mathbf{J}^{\mathrm{T}} \cdot \mathbf{V} - \mathbf{W} \cdot \mathbf{O} \cdot \mathbf{V}, \quad (3.26c)$$

$$-\mathrm{i}\partial_x \mathbf{W} = \mathbf{V} \cdot \mathbf{O} \cdot \mathbf{V}, \quad (3.26d)$$

$$\mathrm{i}\partial_t \mathbf{V} = \mathbf{J}^{\mathrm{T}r} \cdot \mathbf{V} - \mathbf{W} \cdot \mathbf{R}_r \cdot \mathbf{V}, \quad (r \text{ odd}), \quad (3.26e)$$

$$\mathrm{i}\partial_t \mathbf{W} = \mathbf{V} \cdot \mathbf{R}_r \cdot \mathbf{V}, \quad (r \text{ odd}). \quad (3.26f)$$

Again, a more general dispersion relation can be taken into account. Choosing,

$$\mathrm{i}\partial_t \rho_k = \omega(k) \rho_k, \quad (3.27)$$

where $\omega(k)$ is an odd meromorphic function of k, i.e.

$$\omega(k) = \sum_r \lambda_r k^r, \quad (\lambda_r = 0, \text{ if } r \text{ even}), \quad (3.28)$$

we find

$$\mathrm{i}\partial_t \mathbf{V} = \omega(\mathbf{J}^{\mathrm{T}}) \cdot \mathbf{V} - \mathbf{W} \cdot \mathbf{R} \cdot \mathbf{V}, \quad (3.29a)$$

and

$$\mathrm{i}\partial_t \mathbf{W} = \mathbf{V} \cdot \mathbf{R} \cdot \mathbf{V}, \quad (3.29b)$$

where

$$\mathbf{R} = \sum_r \lambda_r \mathbf{R}_r, \quad (\lambda_r = 0, \text{ if } r \text{ even}). \quad (3.30)$$

4. Partial differential equations

In this section we present, for positive values of r, a method for deriving matrix partial differential equations in terms of only $\mathbf{\Phi}$, in the case of the integral equation of type I, and in terms of only $\mathbf{V}$, in the case of the integral equation of type II, without involving the matrices $\mathbf{J}$ or $\mathbf{J}^{\mathrm{T}}$. From the form of these matrix PDE's, one immediately has partial differential equations containing only the $(0,0)$ components of the matrices $\mathbf{\Phi}$ and $\mathbf{V}$, i.e. the functions $\phi_{0,0}$ and $v_{0,0}$.

4.1. *Matrix PDE's for type I*

In order to get rid of the matrices $\mathbf{J}^{\mathrm{T}}$, we first derive a recursive relation for the quantity

$$\boldsymbol{F}_k^{(p)} \equiv \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\phi}_k - \eta \boldsymbol{\Psi}^* \cdot \mathbf{Q}_p \cdot \boldsymbol{\phi}_k = k^p \boldsymbol{\phi}_k + \eta \boldsymbol{\Phi} \cdot \mathbf{Q}_p \cdot \boldsymbol{\psi}_k, \tag{4.1}$$

cf. (2.21a), occurring in the right-hand side of (2.33a).

Using (2.34c) and the recursion relation

$$\mathbf{Q}_p = \mathbf{J} \cdot \mathbf{Q}_{p-1} + \mathbf{O} \cdot \mathbf{J}^{\mathrm{T}p-1} = \mathbf{Q}_{p-1} \cdot \mathbf{J}^{\mathrm{T}} + \mathbf{J}^{p-1} \cdot \mathbf{O}, \tag{4.2}$$

cf. (2.23a), (2.23b), which is valid for positive, as well as negative values of p, we obtain

$$\boldsymbol{F}_k^{(p)} = \mathbf{J}^{\mathrm{T}p-1} \cdot (-\mathrm{i}\partial_x \boldsymbol{\phi}_k + \eta \boldsymbol{\Psi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k) - \eta \boldsymbol{\Psi}^* \cdot (\mathbf{J}^{p-1} \cdot \mathbf{O} + \mathbf{Q}_{p-1} \cdot \mathbf{J}^{\mathrm{T}}) \cdot \boldsymbol{\phi}_k. \tag{4.3}$$

We rewrite eq. (4.3), using also (2.35d), as

$$\begin{aligned} \boldsymbol{F}_k^{(p)} = & -\mathrm{i}\partial_x(\mathbf{J}^{\mathrm{T}p-1} \cdot \boldsymbol{\phi}_k - \eta \boldsymbol{\Psi}^* \cdot \mathbf{Q}_{p-1} \cdot \boldsymbol{\phi}_k) - \mathrm{i}\eta\partial_x(\boldsymbol{\Psi}^* \cdot \mathbf{Q}_{p-1} \cdot \boldsymbol{\phi}_k) \\ & + \eta(\mathbf{J}^{\mathrm{T}p-1} \cdot \boldsymbol{\Psi}^* - \boldsymbol{\Psi}^* \cdot \mathbf{J}^{p-1}) \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k \\ & - \eta \boldsymbol{\Psi}^* \cdot \mathbf{Q}_{p-1} \cdot (-\mathrm{i}\partial_x \boldsymbol{\phi}_k + \eta \boldsymbol{\Psi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k) \\ = & -\mathrm{i}\partial_x \boldsymbol{F}_k^{(p-1)} - \eta \mathbf{G}^{(p-1)*} \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k - \eta \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{G}_k^{(p-1)}, \end{aligned} \tag{4.4}$$

where we have introduced the vector

$$\boldsymbol{G}_k^{(p)} \equiv -\mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\psi}_k + k^p \boldsymbol{\psi}_k + \eta \boldsymbol{\Psi} \cdot \mathbf{Q}_p \cdot \boldsymbol{\psi}_k = \boldsymbol{\Phi}^* \cdot \mathbf{Q}_p \cdot \boldsymbol{\phi}_k, \tag{4.5}$$

cf. (2.21b), which occurs in (2.34b), and the matrix

$$\mathbf{G}^{(p)} \equiv -\mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\Psi} + \boldsymbol{\Psi} \cdot \mathbf{J}^p + \eta \boldsymbol{\Psi} \cdot \mathbf{Q}_p \cdot \boldsymbol{\Psi} = \boldsymbol{\Phi}^* \cdot \mathbf{Q}_p \cdot \boldsymbol{\Phi}, \tag{4.6}$$

cf. (2.35b), (2.35f), which can be obtained from (4.5), after an integration over the contour C.

Next we express $\boldsymbol{G}_k^{(p)}$ in terms of $\boldsymbol{F}_k^{(p)}$. Differentiating (4.5) with respect to x, we find, using (2.34d) and (2.35d),

$$-\mathrm{i}\partial_x \boldsymbol{G}_k^{(p)} = (k^p - \mathbf{J}^{\mathrm{T}p} + \eta \boldsymbol{\Psi} \cdot \mathbf{Q}_p) \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k + \eta \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\Phi} \cdot \mathbf{Q}_p \cdot \boldsymbol{\Psi}_k. \tag{4.7}$$

Inserting in eq. (4.7), eq. (4.1), and the matrix

$$\mathbf{F}^{(p)} = \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\Phi} - \eta \boldsymbol{\Psi}^* \cdot \mathbf{Q}_p \cdot \boldsymbol{\Phi}, \tag{4.8}$$

which can be obtained from (4.1), after an integration over the contour C, we find

$$-\mathrm{i}\partial_x \boldsymbol{G}_k^{(p)} = \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{F}_k^{(p)} - \mathbf{F}^{(p)*} \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k. \tag{4.9}$$

Eq. (4.9) can be formally solved and inserted in (4.4), which leads to

$$\boldsymbol{F}_k^{(p)} = \Omega[\boldsymbol{F}_k^{(p-1)}], \tag{4.10}$$

where the action of the operator Ω on an arbitrary vector $\boldsymbol{a}_k$, with components $a_k^{(n)}$, is defined as

$$\begin{aligned} \Omega[\boldsymbol{a}_k] = & -\mathrm{i}\partial_x \boldsymbol{a}_k - \mathrm{i}\eta \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \partial_x^{-1}(\boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{a}_k - \mathbf{A}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k) \\ & + \mathrm{i}\eta \partial_x^{-1}(\boldsymbol{\Phi} \cdot \mathbf{O} \cdot \mathbf{A}^* - \mathbf{A} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k, \end{aligned} \tag{4.11}$$

and the elements of the associated matrix $\mathbf{A}$ are given by

$$(\mathbf{A})_{n,m} = a_{n,m}, \quad a_{n,m} = \int_C \mathrm{d}\lambda(k) \frac{1}{k^m} a_k^{(n)}. \tag{4.12}$$

In eq. (4.11) ∂_x^{-1} denotes an integration over x, in which the integration constant has yet to be determined. In appendix A, it will be shown that no undetermined integration constants appear in the right-hand side of (4.10). For every $p > 0$, the vector $\boldsymbol{F}_k^{(p)} = -\mathrm{i}\partial \boldsymbol{\phi}_k$ can be evaluated using

$$\boldsymbol{F}_k^{(p)} = \Omega^p[\boldsymbol{\phi}_k] \tag{4.13}$$

and no integration constant will occur in the final result.

The linear problem corresponding to (2.21) and (2.33), for positive r, i.e. $\omega(k) = k^r$, $(r > 0)$ and $\mathrm{i}\partial_t \rho_k = k^r \rho_k$ in (1.5), can be expressed as

$$\mathrm{i}\partial_t \boldsymbol{\phi}_k = \boldsymbol{F}_k^{(r)} = \Omega^r[\boldsymbol{\phi}_k], \tag{4.14a}$$

$$\mathrm{i}\partial_t \boldsymbol{\psi}_k = \boldsymbol{G}_k^{(r)} = \mathrm{i}\partial_x^{-1}(\boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \Omega^r[\boldsymbol{\phi}_k] - \Omega^r[\boldsymbol{\Phi}]^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k), \tag{4.14b}$$

$$-\mathrm{i}\partial_x \boldsymbol{\phi}_k = k\boldsymbol{\phi}_k + \eta \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\psi}_k, \tag{4.14c}$$

$$-\mathrm{i}\partial_x \boldsymbol{\psi}_k = \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k, \tag{4.14d}$$

in which the action of the operator Ω on an arbitrary matrix $\mathbf{A}$ is defined as

$$\begin{aligned} \Omega[\mathbf{A}] = & -\mathrm{i}\partial_x \mathbf{A} - \mathrm{i}\eta \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \partial_x^{-1}(\boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \mathbf{A} - \mathbf{A}^* \cdot \mathbf{O} \cdot \boldsymbol{\Phi}) \\ & + \mathrm{i}\eta \partial_x^{-1}(\boldsymbol{\Phi} \cdot \mathbf{O} \cdot \mathbf{A}^* - \mathbf{A} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot \boldsymbol{\Phi}). \end{aligned} \tag{4.15}$$

The corresponding PDE can be expressed as

$$\mathrm{i}\partial_t \boldsymbol{\Phi} = \Omega^r[\boldsymbol{\Phi}], \tag{4.16}$$

as follows by integrating (4.14a) over the contour C. Furthermore, from the form of the operator Ω or from the relations (A.5), (A.6) and (4.4) it is obvious that the (0, 0) element of the matrix relation (4.16) leads to a closed partial differential equation, containing only $\phi_{0,0}$. The corresponding linear problem is given by the $n = 0$ component of (4.14a)–(4.14d).

In order to obtain the PDE for $\phi_{0,0}$ for a specific value of r in an explicit form, one first has to evaluate the vector $\boldsymbol{F}_k^{(r)} = \Omega^r[\boldsymbol{\phi}_k]$. The vectors $\boldsymbol{F}_k^{(r)}$ and $\boldsymbol{G}_k^{(r)}$, for positive r, can be found, either by using (4.10) a number of times in such a way that the "integrations" ∂_x^{-1} can be cancelled in every step, or by applying the recursion relations (A.5) and (A.6), in combination with (4.4). The results for $r = 1, 2, 3, 4, 5$ are presented in appendix B. The PDE's for $\phi_{0,0}$ for $r = 2, 3, 4, 5$ can be found, integrating (B.2)–(B.5) over the contour C, and inserting the result in (4.16). We find

$$r=2, \quad \mathrm{i}\partial_t\phi_{0,0} + \partial_x^2\phi_{0,0} + 2\eta|\phi_{0,0}|^2\phi_{0,0} = 0, \tag{4.17}$$

$$r=3, \quad \partial_t\phi_{0,0} - \partial_x^3\phi_{0,0} - 6\eta|\phi_{0,0}|^2\partial_x\phi_{0,0} = 0, \tag{4.18}$$

$$\begin{aligned} r=4, \quad & \mathrm{i}\partial_t\phi_{0,0} - \partial_x^4\phi_{0,0} - \eta(8|\phi_{0,0}|^2\partial_x^2\phi_{0,0} + 6\phi_{0,0}^*(\partial_x\phi_{0,0})^2 \\ & + 2\phi_{0,0}^2\partial_x^2\phi_{0,0}^* + 4|\partial_x\phi_{0,0}|^2\phi_{0,0}) + 6|\phi_{0,0}|^4\phi_{0,0} = 0, \end{aligned} \tag{4.19}$$

$$\begin{aligned} r=5, \quad & \partial_t\phi_{0,0} + \partial_x^5\phi_{0,0} + 10\eta[|\phi_{0,0}|^2\partial_x^3\phi_{0,0} + (\partial_x^2\phi_{0,0})(\partial_x\phi_{0,0}^*)\phi_{0,0} \\ & + \phi_{0,0}(\partial_x^2\phi_{0,0}^*)(\partial_x\phi_{0,0}) + 2\phi_{0,0}^*(\partial_x\phi_{0,0})\partial_x^2\phi_{0,0} + |\partial_x\phi_{0,0}|^2\partial_x\phi_{0,0}] \\ & + 30|\phi_{0,0}|^4\partial_x\phi_{0,0} = 0. \end{aligned} \tag{4.20}$$

Eq. (4.17), for $r = 2$, is the nonlinear Schrödinger equation (NLS); eq. (4.18), for $r = 3$, may be called the complex modified Korteweg–de Vries equation, whereas eqs. (4.19) and (4.20), for $r = 4$ and $r = 5$, respectively, have to our knowledge not been given before in the literature.

Remarks

(i) By applying the operator Ω^p to the functions $\phi_{n,m}$ extra factors k^p and l'^p are introduced in the integrands of the bilinear expression (2.5). In fact, using the property

$$\int_C \mathrm{d}\lambda(k)\frac{1}{\rho_k}\phi_k^{(n)}\psi_k^{(m)} + \int_{C^*} \mathrm{d}\lambda^*(l')\frac{1}{\rho_{l'}^*}\psi_{l'}^{(n)*}\phi_{l'}^{(m)*} = 0, \tag{4.21}$$

and eqs. (2.33) and (2.21a), one can show that

$$-\mathrm{i}\partial\phi_{n,m} = \int_C \mathrm{d}\lambda(k)\,k^p\frac{1}{\rho_k}\phi_k^{(n)}\phi_k^{(m)} - \eta\int_{C^*}\mathrm{d}\lambda^*(l')l'^p\frac{1}{\rho_{l'}^*}\psi_{l'}^{(n)*}\psi_{l'}^{(m)*}, \tag{4.22}$$

and furthermore we have $-\mathrm{i}\partial\boldsymbol{\Phi} = \Omega^p[\boldsymbol{\Phi}]$. Operators Ω were already formulated in a different context in ref. 23, see also ref. 24.

(ii) As was noted at the end of section 2, it is also possible to choose a more general dispersion relation $\omega(k) = \Sigma_r\lambda_r k^r$ in the integral equation, cf. eq. (2.38). Taking into account only positive powers of k, i.e. $\lambda_r = 0$ for $r < 0$, we obtain PDE's of the form

$$\mathrm{i}\partial_t\boldsymbol{\Phi} = \sum_{r>0}\lambda_r\Omega^r[\boldsymbol{\Phi}]. \tag{4.23}$$

All PDE's of the form (4.23) are completely integrable in the sense that solutions can be found from the linear integral equation (1.3) with the dispersion relation $\omega(k) = \Sigma_r\lambda_r k^r$. The (0, 0) element of (4.23) leads to a closed PDE for $\phi_{0,0}$, the solutions of which can be obtained from (1.3) and (1.6) with $n = m = 0$. An example of such a PDE is Hirota's equation[25]), which can be obtained, taking $\lambda_r = 0$ for $r \neq 2, 3$, i.e.

$$\mathrm{i}\partial_t\phi_{0,0} + \lambda_2\partial_x^2\phi_{0,0} - \mathrm{i}\lambda_3\partial_x^3\phi_{0,0} = -2\eta\lambda_2|\phi_{0,0}|^2\phi_{0,0} + 6\mathrm{i}\lambda_3\eta|\phi_{0,0}|^2\partial_x\phi_{0,0}. \tag{4.24}$$

4.2. *Matrix PDE's for type II*

The elimination of the matrices $\mathbf{J}^{\mathrm{T}}$ in the case of the integral equation of type II, proceeds in a similar way as in the previous subsection. We define the quantities

$$\boldsymbol{Y}_k^{(p)} \equiv \mathbf{J}^{\mathrm{T}p}\cdot v_k + (-1)^p\mathbf{W}\cdot\mathbf{R}_p\cdot v_k = k^p v_k - \mathbf{V}\cdot\mathbf{R}_p\cdot w_k, \tag{4.25}$$

and

$$\boldsymbol{Z}_k^{(p)} \equiv k^p w_k - (-\mathbf{J}^{\mathrm{T}})^p\cdot w_k - \mathbf{W}\cdot\mathbf{R}_p\cdot w_k = -(-1)^p\mathbf{V}\cdot\mathbf{R}_p\cdot v_k, \tag{4.26}$$

for all integer p, cf. (3.15) and (3.19). Then, for all odd r, we have the equations

$$\mathrm{i}\partial_t v_k = \boldsymbol{Y}_k^{(r)}, \tag{4.27a}$$

$$\mathrm{i}\partial_t w_k = \boldsymbol{Z}_k^{(r)}. \tag{4.27b}$$

In an analogous way as before we can derive the recursion relation

$$\boldsymbol{Y}_k^{(p)} = -\mathrm{i}\partial_x\boldsymbol{Y}_k^{(p-1)} + \mathbf{V}\cdot\mathbf{O}\cdot\boldsymbol{Z}_k^{(p-1)} - (-1)^{p-1}\mathbf{Z}^{(p-1)}\cdot\mathbf{O}\cdot v_k \tag{4.28}$$

and the relation

$$-\mathrm{i}\partial_x\boldsymbol{Z}_k^{(p)} = -(-1)^p\mathbf{Y}^{(p)}\cdot\mathbf{O}\cdot v_k + \mathbf{V}\cdot\mathbf{O}\cdot\boldsymbol{Y}_k^{(p)}, \tag{4.29}$$

where

$$\mathbf{Y}^{(p)} = \mathbf{J}^{\mathrm{T}p}\cdot\mathbf{V} + (-1)^p\mathbf{W}\cdot\mathbf{R}_p\cdot\mathbf{V}, \tag{4.30}$$

and

$$\mathbf{Z}^{(p)} = \mathbf{W}\cdot\mathbf{J}^p - (-\mathbf{J}^{\mathrm{T}})^p\cdot\mathbf{W} - \mathbf{W}\cdot\mathbf{R}_p\cdot\mathbf{W}. \tag{4.31}$$

In the derivation of (4.28) and (4.29) we have made use of the recursion relation

$$\mathbf{R}_{p+1} = \mathbf{J}\cdot\mathbf{R}_p + \mathbf{O}\cdot(-\mathbf{J}^{\mathrm{T}})^p = -\mathbf{R}_p\cdot\mathbf{J}^{\mathrm{T}} + \mathbf{J}^p\cdot\mathbf{O}, \tag{4.32}$$

and eqs. (3.25c), (3.25d), (3.26d), (4.25), (4.26), (4.30) and (4.31).

From eqs. (4.28) and (4.29) it can be inferred that

$$\mathbf{Y}_k^{(p)} = \begin{cases}\Omega^+[\mathbf{Y}_k^{(p-1)}], & (p \text{ even}),\\ \Omega^-[\mathbf{Y}_k^{(p-1)}], & (p \text{ odd}),\end{cases} \tag{4.33}$$

where

$$\Omega^\pm[\boldsymbol{a}_k] = -\mathrm{i}\partial_x\boldsymbol{a}_k + \mathrm{i}\mathbf{V}\cdot\mathbf{O}\cdot\partial_x^{-1}(\mathbf{V}\cdot\mathbf{O}\cdot\boldsymbol{a}_k \pm \mathbf{A}\cdot\mathbf{O}\cdot\boldsymbol{v}_k)$$
$$\pm\,\mathrm{i}\partial_x^{-1}(\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{A}\pm\mathbf{A}\cdot\mathbf{O}\cdot\mathbf{V})\cdot\mathbf{O}\cdot\boldsymbol{v}_k, \tag{4.34}$$

and, in a similar way as before it can be shown that the integrations ∂_x^{-1} do not yield undetermined constants in (4.33), cf. appendix A.

The linear problem, corresponding to (3.15), (3.19) and (3.25), can be expressed as, cf. (4.27a) and (4.27b),

$$\mathrm{i}\partial_t\boldsymbol{v}_k = \mathbf{Y}_k^{(r)} = (\Omega^-\Omega^+)^{(r-1)/2}\Omega^-[\boldsymbol{v}_k], \quad (r \text{ odd}), \tag{4.35a}$$

$$\mathrm{i}\partial_t\boldsymbol{w}_k = \mathbf{Z}_k^{(r)} = \mathrm{i}\partial_x^{-1}(\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{Y}_k^{(r)} + \mathbf{Y}^{(r)}\cdot\mathbf{O}\cdot\boldsymbol{v}_k), \quad (r \text{ odd}), \tag{4.35b}$$

$$-\mathrm{i}\partial_x\boldsymbol{v}_k = k\boldsymbol{v}_k - \mathbf{V}\cdot\mathbf{O}\cdot\boldsymbol{w}_k, \tag{4.35c}$$

$$-\mathrm{i}\partial_x\boldsymbol{w}_k = \mathbf{V}\cdot\mathbf{O}\cdot\boldsymbol{v}_k, \tag{4.35d}$$

and we have the matrix PDE

$$\mathrm{i}\partial_t\mathbf{V} = (\Omega^-\Omega^+)^{(r-1)/2}\Omega^-[\mathbf{V}], \quad (r \text{ odd}), \tag{4.36}$$

in which

$$\Omega^\pm[\mathbf{A}] = -\mathrm{i}\partial_x\mathbf{A} + \mathrm{i}\mathbf{V}\cdot\mathbf{O}\cdot\partial_x^{-1}(\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{A}\pm\mathbf{A}\cdot\mathbf{O}\cdot\mathbf{V})$$
$$\pm\,\mathrm{i}\partial_x^{-1}(\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{A}\pm\mathbf{A}\cdot\mathbf{O}\cdot\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}, \tag{4.37}$$

for an arbitrary $\mathbf{A}$. From the form of the operators Ω^+, Ω^-, or from the recursion relations (A.8), (A.9) and (4.29), it is clear that the (0, 0) element of (4.36) yields a closed PDE containing only $v_{0,0}$, and the $n = 0$ component of (4.35a)–(4.35d) is the associated linear problem.

As an example we write down the PDE's obeyed by $v_{0,0}$, for the values $r = 3, 5$. Using the explicit expressions for $\mathbf{Y}_k^{(3)}$ and $\mathbf{Y}_k^{(5)}$, given in eqs. (C.2a) and (C.3a) of appendix C, we find

$$r = 3, \quad \partial_t v_{0,0} - \partial_x^3 v_{0,0} = -6v_{0,0}^2\partial_x v_{0,0}, \tag{4.38}$$

$$r = 5, \quad \partial_t v_{0,0} + \partial_x^5 v_{0,0} = 10[v_{0,0}^2\partial_x^3 v_{0,0} + 4v_{0,0}(\partial_x v_{0,0})\partial_x^2 v_{0,0}$$
$$+ (\partial_x v_{0,0})^3] - 30v_{0,0}^4\partial_x v_{0,0}. \tag{4.39}$$

Eq. (4.38) is the modified Korteweg–de Vries equation (MKdV) and (4.39) is a higher-order MKdV type of equation. (Note that (4.20) may be regarded as a complex version of (4.39)). As in the previous subsection one may also consider linear combinations of PDE's, corresponding to a dispersion relation $\omega(k) = \Sigma_r \lambda_r k^r$, $r > 0$, but now $\omega(k)$ must be an odd function of k.

5. Miura transformations

In section 4 matrix partial differential equations for positive r were derived, from which immediately PDE's for the (0, 0) element of $\mathbf{\Phi}$ and $\mathbf{V}$ could be deduced. In this section we will consider the possibility of deriving closed equations for other elements of the matrices $\mathbf{\Phi}$ and $\mathbf{V}$. One could attempt to derive these equations by applying $\mathbf{J}$ and $\mathbf{J}^{\mathrm{T}}$ a number of times to the integral equation, and solving the matrices $\mathbf{O}\cdot\mathbf{\Phi}\cdot\mathbf{O}$ and $\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}$ via the relations (2.35) and (3.26). For general $\phi_{n,m}$, cf. (1.6), or $v_{n,m}$, cf. (1.8), this procedure is quite intricate and it is not obvious that a closed PDE containing only one specific $\phi_{n,m}$ or $v_{n,m}$ can be derived. In this section we shall derive explicit PDE's for the (0, 1), (1, 0) and (1, 1) elements, restricting ourselves to the choice of sign $\eta = 1$ in eq. (1.3). This is relatively easy, because the algebraic relations (2.35a), (2.35b), or (3.26a), (3.26b), reduce in this case to relations containing only two matrix elements, rather than three, which is the usual situation.

5.1. *Partial differential equation for $\phi_{1,0}$ (type I)*

Taking the (1, 1) element of (2.35b) for $\eta = 1$, $p = 1$, we have

$$\psi_{1,0} = \psi_{0,1} + \phi^*_{1,0}\phi_{0,1} - \psi_{1,0}\psi_{0,1}, \tag{5.1}$$

and hence, cf. (2.6) and (2.9),

$$|1 - \psi_{1,0}|^2 + |\phi_{1,0}|^2 = 1. \tag{5.2}$$

Taking the (1, 0) element of eqs. (2.35c) and (2.35d), with $\eta = 1$, we can express $\phi_{0,0}$ in terms of $\phi_{1,0}$ and $\psi_{1,0}$, i.e.

$$-\mathrm{i}\partial_x\phi_{1,0} = (1 - \psi^*_{1,0})\phi_{0,0}, \tag{5.3a}$$

$$-\mathrm{i}\partial_x\psi_{1,0} = \phi^*_{1,0}\phi_{0,0}, \tag{5.3b}$$

and eliminating $\phi_{0,0}$, we find

$$(1 - \psi_{1,0})\partial_x\psi^*_{1,0} = \phi_{1,0}\partial_x\phi^*_{1,0}. \tag{5.4}$$

Differentiating (5.3a) with respect to x and using (5.2), we have

$$-\partial_x^2\phi_{1,0}=\frac{\mathrm{i}(1-\psi_{1,0})\partial_x\psi^*_{\bar{1},0}}{1-|\phi_{1,0}|^2}(-\mathrm{i}\partial_x\phi_{1,0})+(1-\psi^*_{\bar{1},0})\phi_{0,0}(-\mathrm{i}\partial_x \ln \phi_{0,0}), \tag{5.5}$$

and from (5.3a) and (5.4) we obtain an explicit expression for $\partial_x \ln \phi_{0,0}$ in terms of $\phi_{1,0}$, i.e.

$$\partial_x \ln \phi_{0,0}=\frac{\partial_x^2\phi_{1,0}}{\partial_x\phi_{1,0}}+\frac{\phi_{1,0}\partial_x\phi^*_{\bar{1},0}}{1-|\phi_{1,0}|^2}. \tag{5.6}$$

An explicit expression for $\partial_t \ln \phi_{0,0}$ in terms of $\phi_{1,0}$, can be found using the PDE for $\phi_{0,0}$ and expressing the terms in the right-hand side in terms of $\partial_x \ln \phi_{0,0}$, as given by (5.6), and of

$$|\phi_{0,0}|^2=\frac{|\partial_x\phi_{1,0}|^2}{1-|\phi_{1,0}|^2}, \tag{5.7}$$

cf. (5.3a) and (5.2). For $\phi_{0,0}$ we may choose any PDE of the type (4.16), but as an example we shall restrict ourselves to Hirota's equation[25]), with $\eta=1$, given in eq. (4.24). From (4.24) $\partial_t \ln \phi_{0,0}$ can be expressed as

$$\begin{aligned}\mathrm{i}\partial_t \ln \phi_{0,0}&=-\lambda_2(y''+y'^2)+\mathrm{i}\lambda_3(y'''+3y''y'+y'^3)\\&\quad+\frac{|\partial_x\phi_{1,0}|^2}{1-|\phi_{1,0}|^2}(-2\lambda_2+6\mathrm{i}\lambda_3 y'),\end{aligned} \tag{5.8}$$

where we have used the abbreviation $y=\ln \phi_{0,0}$, and the primes denote differentiations with respect to x.

The function $\phi_{0,0}$ can be expressed in terms of $\phi_{1,0}$, using (5.6)–(5.8) and the relation

$$\phi_{0,0}=\frac{|\partial_x\phi_{1,0}|}{(1-|\phi_{1,0}|^2)^{1/2}}\exp\left[\mathrm{i}\int_\Gamma\left\{\mathrm{d}l_x\,\partial_x \operatorname{Im}\ln \phi_{0,0}+\mathrm{d}l_t\,\partial_t \operatorname{Im}\ln \phi_{0,0}\right\}\right], \tag{5.9}$$

where Γ is an arbitrary curve in the (x,t)-plane connecting the points $(0,0)$ and (x,t), and $(\mathrm{d}l_x,\mathrm{d}l_t)$ is an infinitesimal two-dimensional vector tangent to Γ (According to Stokes' theorem the right-hand side of (5.9) is independent of the choice of Γ).

The PDE for $\phi_{1,0}$ can be derived, taking the $(1,0)$ element of (4.23) with $\lambda_r=0$ for $r\neq 2,3$, and using (B.2a) and (B.3a) with $\eta=1$, given in appendix B. We have

$$\begin{aligned}\mathrm{i}\partial_t\phi_{1,0}&=-\lambda_2\partial_x^2\phi_{1,0}+\mathrm{i}\lambda_3\partial_x^3\phi_{1,0}-2\lambda_2|\phi_{0,0}|^2\phi_{1,0}+3\mathrm{i}\lambda_3\phi_{1,0}\phi^*_{\bar{0},0}\partial_x\phi_{0,0}\\&\quad+3\mathrm{i}\lambda_3|\phi_{0,0}|^2\partial_x\phi_{1,0}.\end{aligned} \tag{5.10}$$

Inserting (5.6) and (5.7), eq. (5.10) can be rewritten as

$$\begin{aligned}\mathrm{i}\partial_t\phi_{1,0}&=-\lambda_2\partial_x^2\phi_{1,0}+\mathrm{i}\lambda_3\partial_x^3\phi_{1,0}\\&\quad+\frac{-2\lambda_2\phi_{1,0}|\partial_x\phi_{1,0}|^2+3\mathrm{i}\lambda_3|\partial_x\phi_{1,0}|^2\partial_x\phi_{1,0}+3\mathrm{i}\lambda_3\phi_{1,0}(\partial_x\phi^*_{\bar{1},0})\partial_x^2\phi_{1,0}}{1-|\phi_{1,0}|^2}\\&\quad+\frac{3\mathrm{i}\lambda_3\phi_{1,0}^2|\partial_x\phi_{1,0}|^2\partial_x\phi^*_{\bar{1},0}}{(1-|\phi_{1,0}|^2)^2}.\end{aligned} \tag{5.11}$$

Eq. (5.11) is an integrable PDE, since its solutions can be obtained from the linear integral equation (1.3) with the dispersion relation $\omega(k)=\lambda_2k^2+\lambda_3k^3$, and eq. (5.9) provides the Miura transformation mapping an arbitrary solution of (5.11) on a solution of Hirota's equation (5.8), (4.24).

The special case $\lambda_3=0$ of eq. (5.11) has been studied extensively in ref. 6, where it was also proved that this equation is equivalent to the equation of motion for the classical Isotropic Heisenberg Spin Chain (IHSC). Eq. (5.11) for $\lambda_3\neq 0$ has, to our knowledge, not been given before in the literature.

5.2. *Partial differential equation for $\phi_{1,1}$ (type I)*

We now proceed with the derivation of a PDE for $\phi_{1,1}$. From the relations (2.35c) and (2.35d) with $\eta=1$, we have, taking the $(1,1)$ element

$$-\mathrm{i}\partial_x\phi_{1,1}=(1-\psi^*_{\bar{1},0})\phi_{1,0}, \tag{5.12a}$$

$$-\mathrm{i}\partial_x\psi_{1,1}=|\phi_{1,0}|^2. \tag{5.12b}$$

Eq. (5.12a), together with (5.2), implies that

$$|\phi_{1,0}|^2=\tfrac{1}{2}\pm\tfrac{1}{2}(1-4|\partial_x\phi_{1,1}|^2)^{1/2}. \tag{5.13}$$

By differentiating (5.12a) with respect to x, and using (5.4), (5.2) and (5.12a), we have

$$\begin{aligned}-\partial_x^2\phi_{1,1}&=-\mathrm{i}(1-\psi^*_{\bar{1},0})\partial_x\phi_{1,0}+\frac{\mathrm{i}\phi_{1,0}\partial_x|\phi_{1,0}|^2-\mathrm{i}|\phi_{1,0}|^2\partial_x\phi_{1,0}}{1-\psi_{1,0}}\\&=\frac{\partial_x\phi_{1,1}}{1-|\phi_{1,0}|^2}(\partial_x|\phi_{1,0}|^2-\partial_x \ln \phi_{1,0}),\end{aligned} \tag{5.14}$$

so that

$$\partial_x \ln \phi_{1,0}=\tfrac{1}{2}[1\mp(1-4|\partial_x\phi_{1,1}|^2)^{1/2}]\frac{\partial_x^2\phi_{1,1}}{\partial_x\phi_{1,1}}\mp\frac{\partial_x|\partial_x\phi_{1,1}|^2}{(1-4|\partial_x\phi_{1,1}|^2)^{1/2}}. \tag{5.15}$$

The relation for $\partial_t \ln \phi_{1,0}$, in terms of $\phi_{1,1}$, can be inferred from (5.15), (5.13) and

$$\mathrm{i}\partial_t \ln \phi_{1,0} = -\lambda_2(z'' + z'^2) + \mathrm{i}\lambda_3(z''' + 3z''z' + z'^3) + \frac{|\phi_{1,0}|^2}{1-|\phi_{1,0}|^2}(-2\lambda_2|z'|^2 + 3\mathrm{i}\lambda_3 z'^*(z'' + 2z'^2)) + \frac{|\phi_{1,0}|^4}{(1-|\phi_{1,0}|^2)^2} 3\mathrm{i}\lambda_3 z'^*|z'|^2, \tag{5.16}$$

with $z \equiv \ln \phi_{1,0}$, which follows directly from (5.11). The function $\phi_{1,0}$ can be expressed in terms of $\phi_{1,1}$, using (5.13), (5.15), (5.16) and the relation

$$\phi_{1,0} = [\tfrac{1}{2} \pm \tfrac{1}{2}(1 - |4\partial_x\phi_{1,1}|^2)^{1/2}]^{1/2} \times \exp\left[\mathrm{i}\int_\Gamma \{\mathrm{d}l_x \operatorname{Im} \partial_x \ln \phi_{1,0} + \mathrm{d}l_t \operatorname{Im} \partial_t \ln \phi_{1,0}\}\right], \tag{5.17}$$

cf. also (5.9) for the meaning of Γ, $\mathrm{d}l_x$ and $\mathrm{d}l_t$.

The PDE for $\phi_{1,1}$ can be derived taking the (1,1) element of (4.23), and using the expressions (B2a), (B.3a), given in appendix B. We have

$$\mathrm{i}\partial_t\phi_{1,1} = -\lambda_2(\partial_x^2\phi_{1,1} + 2\phi_{1,0}^2\phi_{0,0}^*) + \mathrm{i}\lambda_3[\partial_x^3\phi_{1,1} + 6(\partial_x \ln \phi_{1,0})\phi_{1,0}^2\phi_{0,0}^*]. \tag{5.18}$$

Using (5.13), (5.15) and the relation

$$\phi_{1,0}^2\phi_{0,0}^* = (\partial_x\phi_{1,1})\partial_x|\phi_{1,0}|^2 - |\phi_{1,0}|^2\partial_x^2\phi_{1,1}, \tag{5.19}$$

which can be inferred from (5.3a), (5.2), (5.12a) and (5.14), in (5.18), we obtain the PDE for $\phi_{1,1}$, i.e.

$$\mathrm{i}\partial_t\phi_{1,1} = \mp\lambda_2[(\partial_x\phi_{1,1})\partial_x(1-4|\partial_x\phi_{1,1}|^2)^{1/2} - (\partial_x^2\phi_{1,1})(1-4|\partial_x\phi_{1,1}|^2)^{1/2}] + \mathrm{i}\lambda_3\left[\partial_x^3\phi_{1,1} + 6|\partial_x^2\phi_{1,1}|^2\partial_x\phi_{1,1} + \frac{6(\partial_x|\partial_x\phi_{1,1}|^2)^2}{1-4|\partial_x\phi_{1,1}|^2}\partial_x\phi_{1,1}\right]. \tag{5.20}$$

Eq. (5.20) is an integrable PDE in the sense that its solutions can be obtained from the linear integral equation (1.3) with $\omega(k) = \lambda_2 k^2 + \lambda_3 k^3$ and $n = 1$.

Introducing a real 3-dimensional vector $\boldsymbol{S} = (S^x, S^y, S^z)$ with $\boldsymbol{S}\cdot\boldsymbol{S} = 1$, and identifying

$$S^+ \equiv S^x + \mathrm{i}S^y = -2\mathrm{i}\partial_x\phi_{1,1}, \text{ and } S^z = \mp(1-4|\partial_x\phi_{1,1}|^2)^{1/2}, \tag{5.21}$$

eq. (5.20) can be differentiated to give

$$\partial_t\boldsymbol{S} = \lambda_2\boldsymbol{S}\times\partial_x^2\boldsymbol{S} + \lambda_3\partial_x[\partial_x^2\boldsymbol{S} + \tfrac{3}{2}\boldsymbol{S}(\partial_x\boldsymbol{S})\cdot\partial_x\boldsymbol{S}]. \tag{5.22}$$

Eq. (5.22) has been given before by Papanicolaou[26]) and reduces in the case $\lambda_3 = 0$ to the IHSC. The Miura transformations (5.17) and (5.9) provide a mapping of the solutions of the Papanicolaou equation (5.22) on solutions of the Hirota equation (4.24). (A related mapping was obtained in ref. 26, by extending the line of reasoning given in ref. 9 for the special case $\lambda_3 = 0$.)

5.3. *Integral equation of type II*

So far we only considered Miura transformations that could be derived from the relations (2.35) for the integral equation of type I. About the integral equation of type II we can be very short. In fact, taking the (1, 0) and (1, 1) elements of (3.26c) and (3.26d), we have

$$-\mathrm{i}\partial_x v_{1,0} = (1 - w_{1,0})v_{0,0}, \tag{5.23a}$$

$$-\mathrm{i}\partial_x w_{1,0} = v_{1,0}v_{0,0}, \tag{5.23b}$$

$$-\mathrm{i}\partial_x v_{1,1} = (1 - w_{1,0})v_{1,0}, \tag{5.23c}$$

$$-\mathrm{i}\partial_x w_{1,1} = v_{1,0}^2, \tag{5.23d}$$

together with the algebraic relation

$$(1 - w_{1,0})^2 + v_{1,0}^2 = 1, \tag{5.24}$$

cf. (3.26b). From (5.23a) and (5.24), $v_{0,0}$ can be solved, giving

$$v_{0,0} = \frac{-\mathrm{i}\partial_x v_{1,0}}{(1 - v_{1,0}^2)^{1/2}} = -\mathrm{i}\partial_x(\arcsin v_{1,0}), \tag{5.25}$$

and from (5.23c) and (5.24) we have

$$v_{1,0}^2 = \tfrac{1}{2} \pm \tfrac{1}{2}(1 + 4(\partial_x v_{1,1})^2)^{1/2}. \tag{5.26}$$

Eqs. (5.25) and (5.26) imply that in the case of the integral equation of type II, the transformations from $v_{1,0}$ to $v_{0,0}$ and from $v_{1,1}$ to $v_{1,0}$ become trivial, so that the PDE's for $v_{1,0}$ and $v_{1,1}$ are equivalent to the one for $v_{0,0}$.

6. Complex sine-Gordon equation

Although the relations (2.35), which were derived in section 2 for the integral equation of type I, and the relations (3.26), which were derived in section 3 for the integral equation of type II, are valid for all integer r, the method for finding partial differential equations containing only $\phi_{0,0}$, which we developed in section 4, is restricted to positive values of r. For negative r values the situation is more complicated. For the special case $r = -1$, however, it is possible to derive closed PDE's for the functions $\phi_{0,0}$, $\phi_{0,1} = \phi_{1,0}$ and $\phi_{1,1}$, as will be shown in this section. In appendix D, we shall give a method to derive three coupled equations in $\phi_{0,0}$, $\phi_{1,0}$ and $\phi_{1,1}$, which applies to all negative r values.

6.1. *Integral equation of type I*

In the case $r=-1$, taking into account that $\mathbf{Q}_{-1}=-\mathbf{J}^{\mathrm{T}}\cdot\mathbf{O}\cdot\mathbf{J}$, cf. eq. (2.23), we find from eqs. (2.34a) and (2.34c), taking $\eta=1$,

$$\partial_x\partial_t\boldsymbol{\phi}_k=(\mathbf{J}+\mathbf{\Psi}^*\cdot\mathbf{J}^{\mathrm{T}}\cdot\mathbf{O}\cdot\mathbf{J})\cdot(\mathbf{J}^{\mathrm{T}}\cdot\boldsymbol{\phi}_k-\mathbf{\Psi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k)-\mathrm{i}(\partial_x\mathbf{\Psi}^*)\cdot\mathbf{J}^{\mathrm{T}}\cdot\mathbf{O}\cdot\mathbf{J}\cdot\boldsymbol{\phi}_k, \tag{6.1}$$

which, in view of (2.35b) and (2.35d), can be rewritten as

$$\partial_x\partial_t\boldsymbol{\phi}_k=\boldsymbol{\phi}_k+\mathbf{\Phi}\cdot\mathbf{O}\cdot\mathbf{\Phi}^*\cdot\mathbf{Q}_{-1}\cdot\boldsymbol{\phi}_k+\mathbf{\Phi}\cdot\mathbf{Q}_{-1}\cdot\mathbf{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k. \tag{6.2}$$

From eq. (6.2) we immediately obtain three coupled PDE's containing only $\phi_{0,0}$, $\phi_{1,0}$ and $\phi_{1,1}$, viz.

$$\partial_x\partial_t\phi_{0,0}=(1-2|\phi_{1,0}|^2)\phi_{0,0}, \tag{6.3a}$$

$$\partial_x\partial_t\phi_{1,0}=(1-|\phi_{1,0}|^2)\phi_{1,0}-\phi_{0,0}\phi^*_{1,0}\phi_{1,1}, \tag{6.3b}$$

$$\partial_x\partial_t\phi_{1,1}=(1-2|\phi_{1,0}|^2)\phi_{1,1}. \tag{6.3c}$$

Also for other negative r, it is possible to derive three coupled equations, as will be shown in appendix D. For $r=-1$, it is rather easy to derive closed PDE's containing only $\phi_{0,0}$, $\phi_{1,0}$ and $\phi_{1,1}$.

In fact, taking the (0, 0) element of (2.35e) for $r=-1$, we have

$$\mathrm{i}\partial_t\phi_{0,0}=(1-\psi_{1,0})\phi_{1,0}, \tag{6.4}$$

which, in combination with the algebraic relation (5.2), yields

$$|\phi_{1,0}|^2=\tfrac{1}{2}\pm\tfrac{1}{2}(1-4|\partial_t\phi_{0,0}|^2)^{1/2}. \tag{6.5}$$

Inserting (6.5) in (6.3a) and differentiating the result with respect to t, we obtain

$$\chi=\mp\partial_t\frac{\partial_x\chi}{(1-|\chi|^2)^{1/2}}, \tag{6.6}$$

with

$$\chi\equiv2\partial_t\phi_{0,0}. \tag{6.7}$$

Eq. (6.6) can be called the complex sine-Gordon equation, since for real $\chi\equiv\sin\theta$ it reduces to $\partial_x\partial_t\theta=\mp\sin\theta$. Eq. (6.6) has been given before in eq. (3.5) of ref. 27, where it has been inferred from the equations for the reduced nonlinear $O(4)$ σ-model, derived by Pohlmeyer[28]) and Lund and Regge[29]). A bilinearization of (6.6) was given in ref. 21, while the inverse scattering scheme was formulated in ref. 30. (In our approach, the Gel'fand–Levitan equation can be obtained directly from the linear integral equation, as shown in appendix A of ref. 6 by a line of reasoning, which is independent of the value of r.)

Using eq. (5.3a) for $\phi_{0,0}$ and the equation

$$\phi_{1,1}=\frac{\mathrm{i}\partial_t\phi_{1,0}}{1-\psi_{1,0}}, \tag{6.8}$$

which follows from the (1, 0) element of (2.34e) for $r=-1$, in combination with the algebraic relation (5.2), eq. (6.3b) leads to

$$\partial_x\partial_t\phi_{1,0}+\frac{(\partial_x\phi_{1,0})(\partial_t\phi_{1,0})}{1-|\phi_{1,0}|^2}\phi^*_{1,0}=(1-|\phi_{1,0}|^2)\phi_{1,0}. \tag{6.9}$$

Eq. (6.9) was for the first time given by Getmanov[31]) and can be shown to be related to the reduced nonlinear $O(4)$ σ-model, cf. ref. 27.

Finally, inserting (5.13) into (6.3c), we obtain

$$\partial_x\partial_t\phi_{1,1}=\mp(1-4|\partial_x\phi_{1,1}|^2)^{1/2}\phi_{1,1}, \tag{6.10}$$

which, after differentiating with respect to x, leads to the complex sine-Gordon equation (6.6) with $\chi=2\partial_x\phi_{1,1}$, and $\partial_x\leftrightarrow\partial_t$.

Remarks

(1) From eqs. (2.34) and (2.21), for $r=-1$, one can derive the Lax representations[27]) for the complex sine-Gordon equation. Some details of the derivation will be given in appendix E.

(ii) In appendix D, it will be shown, that for all negative values of r, one can derive three coupled equations in $\phi_{0,0}$, $\phi_{0,1}=\phi_{1,0}$ and $\phi_{1,1}$.

6.2. *Integral equation of type II*

From the integral equation of type II, using $\mathbf{R}_{-1}=\mathbf{J}^{\mathrm{T}}\cdot\mathbf{O}\cdot\mathbf{J}$, cf. (3.14), we find from (3.26a) and (3.26b)

$$\partial_x\partial_t\boldsymbol{v}_k=(\mathbf{J}-\mathbf{W}\cdot\mathbf{J}^{\mathrm{T}}\cdot\mathbf{O}\cdot\mathbf{J})\cdot(\mathbf{J}^{\mathrm{T}}\cdot\boldsymbol{v}_k-\mathbf{W}\cdot\mathbf{O}\cdot\boldsymbol{v}_k)+\mathrm{i}(\partial_x\mathbf{W})\cdot(\mathbf{J}^{\mathrm{T}}\cdot\mathbf{O}\cdot\mathbf{J})\cdot\boldsymbol{v}_k, \tag{6.11}$$

which, in view of (3.26b) and (3.26d) can be rewritten as

$$\partial_x\partial_t\boldsymbol{v}_k=\boldsymbol{v}_k-\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{J}^{\mathrm{T}}\cdot\mathbf{O}\cdot\mathbf{J}\cdot\boldsymbol{v}_k-\mathbf{V}\cdot\mathbf{J}^{\mathrm{T}}\cdot\mathbf{O}\cdot\mathbf{J}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\boldsymbol{v}_k. \tag{6.12}$$

Multiplying (6.12) by k^{-m}, and integrating over the contour C, and taking the (0, 0) element of the resulting matrix equation, we have

$$\partial_x\partial_t v_{0,0}=v_{0,0}(1-2v^2_{1,0}). \tag{6.13}$$

From the (0, 0) element of (3.26e) we have

$$i\partial_t v_{0,0} = v_{1,0}(1 - w_{1,0}), \tag{6.14}$$

which, in combination with the algebraic relation

$$v_{1,0}^2 + (1 - w_{1,0})^2 = 1, \tag{6.15}$$

yields

$$v_{1,0}^2 = \tfrac{1}{2} \pm \tfrac{1}{2}[1 + 4(\partial_t v_{0,0})^2]^{1/2}. \tag{6.16}$$

Inserting (6.16) in (6.13), we obtain a closed PDE for $v_{0,0}$, viz.

$$\partial_x\partial_t v_{0,0} = \mp v_{0,0}[1 + 4(\partial_t v_{0,0})^2]^{1/2}, \tag{6.17}$$

which, after substituting

$$u = \arcsin 2i\partial_t v_{0,0}, \tag{6.18}$$

becomes the sine-Gordon equation

$$\partial_x\partial_t u = \mp \sin u. \tag{6.19}$$

In a similar way as before one may derive closed PDE's for $v_{1,0}$ and $v_{1,1}$. We shall not write down the result, because the Miura transformations (5.25) and (5.26) are trivial, so that the PDE's for $v_{1,0}$ and $v_{1,1}$ are equivalent to the sine-Gordon equation.

7. The Korteweg–de Vries equation and the modified nonlinear Schrödinger equation

7.1. *Reduction of the integral equation of type II*

In this section we will show how the integral equation of type II reduces to another type of integral equation with only one integration, instead of a two-fold integration. For that purpose we consider eqs. (3.1), cf. eq. (1.4), and introduce the quantity

$$u_k^{(n)} = v_k^{(n)} - i w_k^{(n)}. \tag{7.1}$$

Then eqs. (3.1a) and (3.1b) can be combined into a single integral equation, i.e.

$$u_k^{(n)} + i\rho_k \int_C d\lambda(l)\, \frac{u_l^{(n)}}{k + l} = \frac{1}{k^n}\, \rho_k. \tag{7.2}$$

Eq. (7.2) can be regarded as a generalization of the integral equation, given by Fokas and Ablowitz[4]) for $n = 0$, cf. also ref. 5.

The relations for $u_k^{(n)}$ can be found by combining the relations for the quantities $v_k^{(n)}$ and $w_k^{(n)}$, which we have given in section 3. From eqs. (3.15) and (3.19), we find

$$k^p \boldsymbol{u}_k = \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{u}_k + i\mathbf{U} \cdot \mathbf{R}_p \cdot \boldsymbol{u}_k, \quad (p \text{ even}), \tag{7.3}$$

for all even integers p, where $\boldsymbol{u}_k$ denotes the vector with components $u_k^{(n)}$ and the matrix $\mathbf{U}$ is defined by

$$(\mathbf{U})_{n,m} = u_{n,m}, \quad u_{n,m} = \int_C d\lambda(k)\, \frac{1}{k^m}\, u_k^{(n)}, \tag{7.4}$$

and from (7.1) we find

$$\mathbf{U} = \mathbf{V} - i\mathbf{W} = \mathbf{U}^{\mathrm{T}}, \tag{7.5}$$

cf. also (3.4). From eqs. (3.23) and (3.24) we obtain

$$-i\partial \boldsymbol{u}_k = \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{v}_k - i\mathbf{U} \cdot \mathbf{R}_p \cdot \boldsymbol{v}_k, \quad (p \text{ odd}), \tag{7.6}$$

for odd integer p. Making use of (3.15) and (3.19), but now for odd p-values, we have

$$k^p \boldsymbol{u}_k = 2\mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{v}_k - \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{u}_k + i\mathbf{U} \cdot \mathbf{R}_p \cdot \boldsymbol{u}_k - 2i\mathbf{U} \cdot \mathbf{R}_p \cdot \boldsymbol{v}_k, \quad (p \text{ odd}), \tag{7.7}$$

leading, in combination with (7.6) to

$$(k^p + 2i\partial)\boldsymbol{u}_k = -\mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{u}_k + i\mathbf{U} \cdot \mathbf{R}_p \cdot \boldsymbol{u}_k, \quad (p \text{ odd}). \tag{7.8}$$

Taking into account eq. (1.5) for the factors $\rho_k(x, t)$, eq. (7.8) can be expressed as

$$(k + 2i\partial_x)\boldsymbol{u}_k = -\mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{u}_k + i\mathbf{U} \cdot \mathbf{O} \cdot \boldsymbol{u}_k, \tag{7.9}$$

$$2i\partial_t \boldsymbol{u}_k = k^r \boldsymbol{u}_k + \mathbf{J}^{\mathrm{T}r} \cdot \boldsymbol{u}_k - i\mathbf{U} \cdot \mathbf{R}_r \cdot \boldsymbol{u}_k, \quad (r \text{ odd}). \tag{7.10}$$

Eqs. (7.3), (7.9) and (7.10) are the constitutive relations associated with the integral equation and may be also derived directly from (7.2). Eq. (7.3) for $p = 2$, and eq. (7.9), can be combined in order to eliminate $\mathbf{J}^{\mathrm{T}}$ and this gives

$$(k + i\partial_x)i\partial_x \boldsymbol{u}_k = -(\partial_x \mathbf{U}) \cdot \mathbf{O} \cdot \boldsymbol{u}_k, \tag{7.11}$$

where we have also used the relation

$$-2i\partial_x \mathbf{U} = \mathbf{U} \cdot \mathbf{J} + \mathbf{J}^{\mathrm{T}} \cdot \mathbf{U} - i\mathbf{U} \cdot \mathbf{O} \cdot \mathbf{U}, \tag{7.12}$$

which can be derived multiplying (7.9) by k^{-m} and integrating over the contour C.

As an example we consider the special value $r = 3$, corresponding to the dispersion relation $\omega(k) = k^3$. In that case, eq. (7.10) can be further evaluated, using eqs. (C.2a) and (C.2b) of appendix C and the identification

$$\mathrm{i}\partial_t u_k = Y_k^{(3)} - \mathrm{i}Z_k^{(3)}. \tag{7.13}$$

The result is

$$(\partial_t - \partial_x^3) u_k = -3(\partial_x \mathbf{U}) \cdot \mathbf{O} \cdot \partial_x u_k, \tag{7.14}$$

and integrating over C we obtain

$$(\partial_t - \partial_x^3)\mathbf{U} = -3(\partial_x \mathbf{U}) \cdot \mathbf{O} \cdot \partial_x \mathbf{U}. \tag{7.15}$$

The $(0, 0)$ element of (7.15) is given by

$$(\partial_t - \partial_x^3) u_{0,0} + 3(\partial_x u_{0,0})^2 = 0, \tag{7.16}$$

which is the potential Korteweg–de Vries equation, i.e. $\partial_x u_{0,0}$ satisfies the KdV.

The Miura transformation, mapping the solutions $v_{0,0}$ of the MKdV on solutions $\partial_x u_{0,0}$ of the KdV, can be inferred from the $(0, 0)$ element of the matrix relation

$$\partial_x \mathbf{U} = \partial_x \mathbf{V} + \mathbf{V} \cdot \mathbf{O} \cdot \mathbf{V}, \tag{7.17}$$

which follows from (7.5) and (3.26d).

A PDE for $u_{1,0}$ can be obtained, taking the $(1, 0)$ element of (7.15). We have

$$(\partial_t - \partial_x^3) u_{1,0} = -3(\partial_x u_{1,0}) \partial_x u_{0,0}. \tag{7.18}$$

The factor $\partial_x u_{0,0}$ can be eliminated from (7.18), using the $(0, 1)$ element of the matrix relation

$$\partial_x^2 \mathbf{U} = \mathrm{i}\partial_x \mathbf{U} \cdot \mathbf{J} + (\partial_x \mathbf{U}) \cdot \mathbf{O} \cdot \mathbf{U}, \tag{7.19}$$

which follows by integrating (7.11) over the contour C. From (7.19) we have

$$\partial_x u_{0,0} = (\mathrm{i} + u_{1,0})^{-1} \partial_x^2 u_{1,0}, \tag{7.20}$$

and (7.18) can be expressed as

$$(\partial_t - \partial_x^3) u_{1,0} = \frac{-3(\partial_x u_{1,0}) \partial_x^2 u_{1,0}}{\mathrm{i} + u_{1,0}}. \tag{7.21}$$

Eq. (7.21) is equivalent to (4.38), which is the MKdV, as a consequence of the relation

$$v_{0,0} = \partial_x \ln(\mathrm{i} + u_{1,0}), \tag{7.22}$$

which can be inferred from the $(1, 0)$ element of (7.5), together with the $(1, 0)$ element of (3.26d). As a result, the solutions of the MKdV can also be inferred from the integral equation of Fokas and Ablowitz, i.e. (7.2) with $n = 0$, using

$$v_{0,0} = \partial_x \ln\left[\mathrm{i} + \int_C \mathrm{d}\lambda(k)\, u_k^{(0)} k^{-1}\right]. \tag{7.23}$$

Also for $u_{1,1}$, one can derive a PDE, which is equivalent to the MKdV.

Remark.

For the case $r = 5$, taking into account the $n = 0$ components of the vectors $Y_k^{(5)}$ and $Z_k^{(5)}$ given in eqs. (C.3a) and (C.3b) of appendix C and performing an integration over the contour C, it can be shown that

$$\begin{aligned}\partial_t u_{0,0} &= -\mathrm{i}(\mathbf{Y}^{(5)} - \mathrm{i}\mathbf{Z}^{(5)})_{0,0} \\ &= -\partial_x^5 u_{0,0} + 10(\partial_x^3 u_{0,0})\partial_x u_{0,0} + 5(\partial_x^2 u_{0,0})^2 - 10(\partial_x u_{0,0})^3.\end{aligned} \tag{7.24}$$

Eq. (7.24) is a (potential) higher-order Korteweg–de Vries equation, and the higher-order KdV for $v \equiv \partial_x u_{0,0}$ has already been given in ref. 32. The $(0, 0)$ element of (7.17) provides again the Miura transformation mapping a solution of (4.39) on a solution of (7.24). Eq. (7.24) may also be derived using the constitutive relations associated with the integral equation (7.2).

7.2. *The modified nonlinear Schrödinger equation*

In this subsection we consider the integral equation of type I, (1.3) in the case that $\eta = -1$. In that case it is possible to express the constitutive relations in terms of the vector

$$p_k \equiv \phi_k + \mathrm{i}\psi_k \tag{7.25a}$$

and the corresponding matrix

$$\mathbf{P} = \mathbf{\Phi} + \mathrm{i}\mathbf{\Psi}. \tag{7.25b}$$

From the algebraic relations (2.21a), (2.21b) and eqs. (2.34a)–(2.34d) with $\eta = -1$, we have

$$-\mathrm{i}\partial_x p_k = \mathbf{J}^{\mathrm{T}} \cdot \phi_k + \mathrm{i}\mathbf{P}^* \cdot \mathbf{O} \cdot \phi_k, \tag{7.26a}$$

$$(-k - \mathrm{i}\partial_x) p_k = -\mathrm{i}\mathbf{J}^{\mathrm{T}} \cdot \psi_k - \mathbf{P} \cdot \mathbf{O} \cdot \psi_k, \tag{7.26b}$$

and

$$\mathrm{i}\partial_t p_k = \mathbf{J}^{\mathrm{T}r} \cdot \phi_k + \mathrm{i}\mathbf{P}^* \cdot \mathbf{Q}_r \cdot \phi_k. \tag{7.27}$$

Considering, as an example, the dispersion relation $\omega(k) = \lambda_2 k^2 + \lambda_3 k^3$, leading to

$$\mathrm{i}\partial_t \rho_k = (\lambda_2 k^2 + \lambda_3 k^3)\rho_k, \quad D \equiv -2\lambda_2 + 3\mathrm{i}\lambda_3\partial_x, \tag{7.28}$$

eq. (7.27) can be expressed as

$$(\mathrm{i}\partial_t + \lambda_2\partial_x^2 - \mathrm{i}\lambda_3\partial_x^3)\boldsymbol{p}_k = (\partial_x \mathbf{P}^*)\cdot\mathbf{O}\cdot D\boldsymbol{\phi}_k + 3\lambda_3(\partial_x\mathbf{P})\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\psi}_k, \tag{7.29}$$

which can be derived, using eqs. (B.2a), (B.2b), (B.3a) and (B.3b) of appendix B with $\eta = -1$.

From the $n = 1$ component of (2.26a), (2.26b) with $\eta = -1$, one can solve $\phi_k^{(0)}$ and $\psi_k^{(0)}$, in terms of $p_k^{(1)}$ and $p_{1,0}$. The result is

$$\phi_k^{(0)} = \frac{-\mathrm{i}\partial_x p_k^{(1)}}{1+\mathrm{i}p_{1,0}^*}, \tag{7.30a}$$

and

$$\psi_k^{(0)} = \frac{\mathrm{i}(-k-\mathrm{i}\partial_x)p_k^{(1)}}{1-\mathrm{i}p_{1,0}}. \tag{7.30b}$$

Inserting (7.30a) in (7.29) we obtain

$$\begin{aligned}(\mathrm{i}\partial_t + \lambda_2\partial_x^2 - \mathrm{i}\lambda_3\partial_x^3)p_k^{(1)} &= 2\mathrm{i}\lambda_2\frac{(\partial_x p_{1,0}^*)\partial_x p_k^{(1)}}{1+\mathrm{i}p_{1,0}^*} \\ &+ 3\mathrm{i}\lambda_3(\partial_x p_{1,0}^*)\left[-\mathrm{i}\partial_x\left(\frac{\partial_x p_k^{(1)}}{1+\mathrm{i}p_{1,0}^*}\right) - \frac{\partial_x p_{1,0}}{1-\mathrm{i}p_{1,0}}\frac{\partial_x p_k^{(1)}}{1+\mathrm{i}p_{1,0}^*}\right],\end{aligned} \tag{7.31a}$$

and from (7.30a), (7.30b) and the $n = 0$ component of (2.34d) we have

$$\partial_x\left(\frac{(-k-\mathrm{i}\partial_x)p_k^{(1)}}{1-\mathrm{i}p_{1,0}^*}\right) = \frac{(\partial_x p_{1,0}^*)\partial_x p_k^{(1)}}{|1-\mathrm{i}p_{1,0}|^2}. \tag{7.31b}$$

The relations (7.31a) and (7.31b) form the linear problem associated with the PDE for $p_{1,0}$, which after substituting

$$q \equiv 1 - \mathrm{i}p_{1,0} \tag{7.32}$$

can be expressed as

$$(\mathrm{i}\partial_t + \lambda_2\partial_x^2 - \mathrm{i}\lambda_3\partial_x^3)q = \frac{2\lambda_2|\partial_x q|^2}{q^*} - 3\mathrm{i}\lambda_3(\partial_x q^*)\left(\partial_x\left(\frac{\partial_x q}{q^*}\right) + \frac{(\partial_x q)^2}{|q|^2}\right). \tag{7.33}$$

The Miura transformation mapping solutions $p_{1,0}$ of (7.33), cf. (7.32), on solutions of equation (4.24) with $\eta = -1$, can be found integrating (7.30a) over the contour C.

For $\lambda_3 = 0$, $\lambda_2 = 1$, eq. (7.33) reduces to

$$(\mathrm{i}\partial_t + \partial_x^2)q = \frac{2|\partial_x q|^2}{q^*}, \tag{7.34}$$

which is the modified nonlinear Schrödinger equation (MNLS). Note in this connection that the substitution $q = \sqrt{a}$ yields eq. (3.7) of ref. 15 in the limiting case $s \to 0$, $A \to 0$. In ref. 15 we have also shown that the corresponding potential equation, in terms of one real variable f, i.e.

$$\partial_t\left(\frac{\dot f}{f'}\right) + \partial_x\left(\frac{f'''}{f'} - \frac{3}{2}\frac{f''^2 + \dot f^2}{f'^2}\right) = 0, \tag{7.35}$$

has similarity solutions of the type $f = \delta t + F(x/\sqrt{t})$, satisfying Painlevé IV. For some details on the derivation of (7.35) from (7.34) we refer to appendix F.

The considerations given above imply in particular that the MNLS and the potential MNLS are completely integrable, since solutions can be obtained from the linear integral equations (2.1a), (2.1b). Furthermore, on substituting $s = \partial_x \ln q$, eq. (7.34) for the MNLS can be expressed as

$$\mathrm{i}\partial_t s + \partial_x^2 s = \{2\partial_x|s|^2 - \partial_x s^2\}, \tag{7.36}$$

which may be regarded as a complex extension of the Burgers equation.

In the special case $\lambda_2 = 0$, $\lambda_3 = 1$, the substitution $s = \partial_x \ln q$ in (7.33) leads to

$$\partial_t s - \partial_x^3 s = \partial_x[3(\partial_x s)(s - s^*) + s^3 - 3s^*|s|^2 + 6|s|^2(s^* - s)], \tag{7.37}$$

which is an integrable complex version of the modified Korteweg–de Vries equation, which differs from the complex MKdV given in (4.18).

Remark.

The vector $\boldsymbol{p}_k$ can be solved directly from the integral equation

$$p_k^{(n)} - \mathrm{i}\rho_k \int_{C^*} \mathrm{d}\lambda^*(l')\frac{p_{l'}^{(n)*}}{k-l'} = \frac{1}{k^n}\rho_k, \tag{7.38a}$$

which can be inferred from (2.1a) and (2.1b), or equivalently from the integral equation

$$p_k^{(n)} - \int_C \mathrm{d}\lambda(l)\int_{C^*}\mathrm{d}\lambda^*(l')\frac{\rho_k\rho_{l'}^*}{(k-l')(l'-l)}p_l^{(n)} = \frac{1}{k^n}\rho_k + \mathrm{i}\int_{C^*}\mathrm{d}\lambda^*(k')\frac{\rho_k\rho_{k'}^*}{k'^n(k-k')}. \tag{7.38b}$$

Note that only eq. (7.38b) can be used for a direct derivation of the constitutive relations, since eq. (7.38a) is not of the right type for that purpose.

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Appendix A

In order to show that the integration constants drop out in eq. (4.10), we consider the relation

$$G_k^{(p)} = -(\mathbf{J}^{\mathrm{T}} - \eta \boldsymbol{\Psi} \cdot \mathbf{O}) \cdot \mathbf{J}^{\mathrm{T}p-1} \cdot \psi_k + k^p \psi_k + \eta \boldsymbol{\Psi} \cdot \mathbf{J} \cdot \mathbf{Q}_{p-1} \cdot \psi_k, \tag{A.1}$$

which can be found from (4.5), inserting (4.2). From the algebraic relations (2.21b) for $k\psi_k$ and (2.35b) for $\boldsymbol{\Psi} \cdot \mathbf{J}$, we find

$$\begin{aligned} G_k^{(p)} &= (\mathbf{J}^{\mathrm{T}} - \eta \boldsymbol{\Psi} \cdot \mathbf{O}) \cdot (k^{p-1}\psi_k - \mathbf{J}^{\mathrm{T}p-1} \cdot \psi_k + \eta \boldsymbol{\Psi} \cdot \mathbf{Q}_{p-1} \cdot \psi_k) \\ &\quad + \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot (k^{p-1}\boldsymbol{\phi}_k + \eta \boldsymbol{\Phi} \cdot \mathbf{Q}_{p-1} \cdot \boldsymbol{\Psi}_k) \\ &= (\mathbf{J}^{\mathrm{T}} - \eta \boldsymbol{\Psi} \cdot \mathbf{O}) \cdot G_k^{(p-1)} + \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot F_k^{(p-1)}. \end{aligned} \tag{A.2}$$

Assuming, by induction, that $G_k^{(p-1)}$ can be written in the form

$$G_k^{(p-1)} = \sum_s (\partial_x^s \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot G_k^{(p-1,s)}, \tag{A.3}$$

we have from (A.2) and the complex conjugate of (2.35c)

$$\begin{aligned} G_k^{(p)} &= \mathrm{i} \sum_s (\partial_x^{s+1} \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot G_k^{(p-1,s)} + \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot F_k^{(p-1)} \\ &\quad + \eta \sum_s \sum_{t=1}^{s} \binom{s}{t} (\partial_x^t \boldsymbol{\Psi}) \cdot \mathbf{O} \cdot (\partial_x^{s-t} \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot G_k^{(p-1,s)}, \end{aligned} \tag{A.4}$$

implying with (2.35d) that indeed

$$G_k^{(p)} = \sum_s (\partial_x^s \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot G_k^{(p,s)}, \tag{A.5}$$

with the recursion relation

$$\begin{aligned} G_k^{(p,s)} &= \mathrm{i} G_k^{(p-1,s-1)} + F_k^{(p-1)} \delta_{s,0} \\ &\quad + \mathrm{i}\eta \sum_{t \geq s+1} \sum_{u \geq t} \binom{u}{t} \binom{t-1}{s} (\partial_x^{t-1-s} \boldsymbol{\Phi}) \cdot \mathbf{O} \cdot (\partial_x^{u-t} \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot G_k^{(p-1,u)}. \end{aligned} \tag{A.6}$$

Eqs. (A.5) and (A.6), in combination with (4.4), show that $F_k^{(p)}$ and $G_k^{(p)}$, for positive p, can be evaluated in a recursive way starting from $F_k^{(0)} = \boldsymbol{\phi}_k$ and $G_k^{(0)} = 0$. This implies in particular that no undetermined integration constants will appear in the right-hand side of (4.10).

In a similar way, in the case of the integral equation of type II, we have the relation

$$Z_k^{(p)} = -(\mathbf{J}^{\mathrm{T}} - \mathbf{W} \cdot \mathbf{O}) \cdot Z_k^{(p-1)} + \mathbf{V} \cdot \mathbf{O} \cdot Y_k^{(p-1)}, \tag{A.7}$$

from which it can be shown by induction that

$$Z_k^{(p)} = \sum_s (\partial_x^s \mathbf{V}) \cdot \mathbf{O} \cdot Z_k^{(p,s)}, \tag{A.8}$$

where $Z_k^{(p,s)}$ satisfies the recursion relation

$$\begin{aligned} Z_k^{(p,s)} &= \mathrm{i} Z_k^{(p-1,s-1)} + \mathbf{V} \cdot \mathbf{O} \cdot Y_k^{(p-1)} \\ &\quad - \mathrm{i} \sum_{t \geq s+1} \sum_{u \geq t} \binom{u}{t} \binom{t-1}{s} (\partial_x^{t-1-s} \mathbf{V}) \cdot \mathbf{O} \cdot (\partial_x^{u-t} \mathbf{V}) \cdot \mathbf{O} \cdot Z_k^{(p-1,u)}. \end{aligned} \tag{A.9}$$

Eq. (A.9), in combination with (A.8) and (4.28) shows that the vectors $Y_k^{(p)}$ and $Z_k^{(p)}$, for positive p, can be evaluated in a recursive way, so that the integration arising from ∂_x^{-1} will not produce an undetermined constant in (4.33).

Appendix B

In this appendix we give the explicit expressions for the vectors $F_k^{(p)}$ and $G_k^{(p)}$, for $p = 1, 2, 3, 4, 5$, defined by (4.1) and (4.5), in the case of the integral equation of type I. The results are

$$p = 1, \quad F_k^{(1)} = -\mathrm{i}\partial_x \boldsymbol{\phi}_k, \tag{B.1a}$$

$$G_k^{(1)} = \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k, \tag{B.1b}$$

$$p = 2, \quad F_k^{(2)} = -\partial_x^2 \boldsymbol{\phi}_k - 2\eta \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k, \tag{B.2a}$$

$$G_k^{(2)} = \mathrm{i}(\partial_x \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k - \mathrm{i}\boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \partial_x \boldsymbol{\phi}_k, \tag{B.2b}$$

$$\begin{aligned} p = 3, \quad F_k^{(3)} &= \mathrm{i}\partial_x^3 \boldsymbol{\phi}_k + 3\mathrm{i}\eta(\partial_x \boldsymbol{\Phi}) \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k \\ &\quad + 3\mathrm{i}\eta \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \partial_x \boldsymbol{\phi}_k, \end{aligned} \tag{B.3a}$$

$$\begin{aligned} G_k^{(3)} &= -(\partial_x^2 \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k - \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \partial_x^2 \boldsymbol{\phi}_k + (\partial_x \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot \partial_x \boldsymbol{\phi}_k \\ &\quad - 3\eta \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k, \end{aligned} \tag{B.3b}$$

$$\begin{aligned} p = 4, \quad F_k^{(4)} &= \partial_x^4 \boldsymbol{\phi}_k + \eta[4(\partial_x^2 \boldsymbol{\Phi}) \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k + 4\boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \partial_x^2 \boldsymbol{\phi}_k \\ &\quad + 2\boldsymbol{\Phi} \cdot \mathbf{O} \cdot (\partial_x^2 \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k + 2(\partial_x \boldsymbol{\Phi}) \cdot \mathbf{O} \cdot (\partial_x \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k \\ &\quad + 2\boldsymbol{\Phi} \cdot \mathbf{O} \cdot (\partial_x \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot \partial_x \boldsymbol{\phi}_k + 6(\partial_x \boldsymbol{\Phi}) \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \partial_x \boldsymbol{\phi}_k] \\ &\quad + 6\boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k, \end{aligned} \tag{B.4a}$$

$$G_k^{(4)} = -\mathrm{i}(\partial_x^3\Phi^*)\cdot\mathbf{O}\cdot\phi_k + \mathrm{i}(\partial_x^2\Phi^*)\cdot\mathbf{O}\cdot\partial_x\phi_k - \mathrm{i}(\partial_x\Phi^*)\cdot\mathbf{O}\cdot\partial_x^2\phi_k + \mathrm{i}\Phi^*\cdot\mathbf{O}\cdot\partial_x^3\phi_k + \mathrm{i}\eta[-4(\partial_x\Phi^*)\cdot\mathbf{O}\cdot\Phi\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\phi_k + 2\Phi^*\cdot\mathbf{O}\cdot(\partial_x\Phi)\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\phi_k - 2\Phi^*\cdot\mathbf{O}\cdot\Phi\cdot\mathbf{O}\cdot(\partial_x\Phi^*)\cdot\mathbf{O}\cdot\phi_k + 4\Phi^*\cdot\mathbf{O}\cdot\Phi\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\partial_x\phi_k], \tag{B.4b}$$

$p=5,$

$$F_k^{(5)} = -\mathrm{i}\partial_x^5\phi_k - 5\mathrm{i}\eta[(\partial_x^3\Phi)\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\phi_k + \Phi\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\partial_x^3\phi_k + (\partial_x^2\Phi)\cdot\mathbf{O}\cdot(\partial_x\Phi^*)\cdot\mathbf{O}\cdot\phi_k + \Phi\cdot\mathbf{O}\cdot(\partial_x\Phi^*)\cdot\mathbf{O}\cdot\partial_x^2\phi_k + (\partial_x\Phi)\cdot\mathbf{O}\cdot(\partial_x^2\Phi^*)\cdot\mathbf{O}\cdot\phi_k + \Phi\cdot\mathbf{O}\cdot(\partial_x^2\Phi^*)\cdot\mathbf{O}\cdot\partial_x\phi_k + 2(\partial_x^2\Phi)\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\partial_x\phi_k + 2(\partial_x\Phi)\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\partial_x^2\phi_k + 2(\partial_x\Phi)\cdot\mathbf{O}\cdot(\partial_x\Phi^*)\cdot\mathbf{O}\cdot\partial_x\phi_k] - 10\mathrm{i}[(\partial_x\Phi)\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\Phi\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\phi_k + \Phi\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\Phi\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\partial_x\phi_k + \Phi\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot(\partial_x\Phi)\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\phi_k], \tag{B.5a}$$

$$G_k^{(5)} = (\partial_x^4\Phi^*)\cdot\mathbf{O}\cdot\phi_k + \Phi^*\cdot\mathbf{O}\cdot\partial_x^4\phi_k - (\partial_x^3\Phi^*)\cdot\mathbf{O}\cdot\partial_x\phi_k - (\partial_x\Phi^*)\cdot\mathbf{O}\cdot\partial_x^3\phi_k + (\partial_x^2\Phi^*)\cdot\mathbf{O}\cdot\partial_x^2\phi_k + 5\eta[(\partial_x^2\Phi^*)\cdot\mathbf{O}\cdot\Phi\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\phi_k + \Phi^*\cdot\mathbf{O}\cdot\Phi\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\partial_x^2\phi_k + \Phi^*\cdot\mathbf{O}\cdot(\partial_x^2\Phi)\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\phi_k + \Phi^*\cdot\mathbf{O}\cdot\Phi\cdot\mathbf{O}\cdot(\partial_x^2\Phi^*)\cdot\mathbf{O}\cdot\phi_k + (\partial_x\Phi^*)\cdot\mathbf{O}\cdot\Phi\cdot\mathbf{O}\cdot(\partial_x\Phi^*)\cdot\mathbf{O}\cdot\phi_k + \Phi^*\cdot\mathbf{O}\cdot(\partial_x\Phi)\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\partial_x\phi_k - (\partial_x\Phi^*)\cdot\mathbf{O}\cdot\Phi\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\partial_x\phi_k + \Phi^*\cdot\mathbf{O}\cdot(\partial_x\Phi)\cdot\mathbf{O}\cdot(\partial_x\Phi^*)\cdot\mathbf{O}\cdot\phi_k] + 10[\Phi^*\cdot\mathbf{O}\cdot\Phi\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\Phi\cdot\mathbf{O}\cdot\Phi^*\cdot\mathbf{O}\cdot\phi_k]. \tag{B.5b}$$

Appendix C

In this appendix we give the explicit expressions for the vectors $Y_k^{(p)}$ and $Z_k^{(p)}$, for $p=1,3,5$, defined by (4.25) and (4.26) in the case of the integral equation of type II. The results are

$$p=1,\quad Y_k^{(1)} = -\mathrm{i}\partial_x v_k, \tag{C.1a}$$

$$Z_k^{(1)} = \mathbf{V}\cdot\mathbf{O}\cdot v_k, \tag{C.1b}$$

$$p=3,\quad Y_k^{(3)} = \mathrm{i}\partial_x^3 v_k - 3\mathrm{i}(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot v_k - 3\mathrm{i}\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x v_k, \tag{C.2a}$$

$$Z_k^{(3)} = -\mathbf{V}\cdot\mathbf{O}\cdot\partial_x^2 v_k - (\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot v_k + (\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\partial_x v_k + 3\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot v_k, \tag{C.2b}$$

$$p=5,\quad Y_k^{(5)} = -\mathrm{i}\partial_x^5 v_k + 5\mathrm{i}[(\partial_x^3\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot v_k + \mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x^3 v_k + (\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot v_k + \mathbf{V}\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\partial_x^2 v_k + (\partial_x\mathbf{V})\cdot\mathbf{O}\cdot(\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot v_k + \mathbf{V}\cdot\mathbf{O}\cdot(\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot\partial_x v_k + 2(\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x v_k + 2(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x^2 v_k + 2(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\partial_x v_k] - 10\mathrm{i}[(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot v_k + \mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot v_k + \mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x v_k], \tag{C.3a}$$

$$Z_k^{(5)} = (\partial_x^4\mathbf{V})\cdot\mathbf{O}\cdot v_k + \mathbf{V}\cdot\mathbf{O}\cdot\partial_x^4 v_k - (\partial_x^3\mathbf{V})\cdot\mathbf{O}\cdot\partial_x v_k - (\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\partial_x^3 v_k + (\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot\partial_x^2 v_k - 5[(\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot v_k + \mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x^2 v_k + \mathbf{V}\cdot\mathbf{O}\cdot(\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot v_k + \mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot(\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot v_k + (\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot v_k + \mathbf{V}\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x v_k - (\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x v_k + \mathbf{V}\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot v_k] + 10\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot v_k. \tag{C.3b}$$

The expressions for the vectors $Y_k^{(2)}$, $Z_k^{(2)}$, $Y_k^{(4)}$, $Z_k^{(4)}$, which have been used in the derivation of (C.2) and (C.3), respectively, have no direct meaning, in connection with a PDE, and are not given here.

Appendix D

In this appendix we shall show how three coupled equations for $\phi_{0,0}$, $\phi_{1,0}$ and $\phi_{1,1}$ can be derived for negative values of r, $(\eta=1)$.

In fact, using (4.2) for $p=-1$, we have

$$\mathbf{J}\cdot\tilde{\mathbf{O}} = \mathbf{O}\cdot\mathbf{J},\quad \mathbf{J}^{\mathrm{T}}\cdot\mathbf{O} = \tilde{\mathbf{O}}\cdot\mathbf{J}^{\mathrm{T}}, \tag{D.1}$$

where

$$\tilde{\mathbf{O}} = -\mathbf{Q}_{-1} = \mathbf{J}^{\mathrm{T}}\cdot\mathbf{O}\cdot\mathbf{J},\quad (\tilde{\mathbf{O}})_{m,n} = \delta_{m,1}\delta_{n,1}. \tag{D.2}$$

Inserting (D.2) in (4.2) we obtain a relation connecting p and $p+1$, viz.

$$\mathbf{Q}_p = \mathbf{Q}_{p+1}\cdot\mathbf{J} - \mathbf{J}^{p+1}\cdot\tilde{\mathbf{O}} = \mathbf{J}^{\mathrm{T}}\cdot\mathbf{Q}_{p+1} - \tilde{\mathbf{O}}\cdot\mathbf{J}^{\mathrm{T}p+1}. \tag{D.3}$$

Hence eq. (4.4), which has been derived using (4.2) and which therefore is valid for negative values of p as well, together with (4.5) and (4.6), leads to

$$\begin{aligned} \boldsymbol{F}_k^{(p)} &= -\mathrm{i}\partial_x \boldsymbol{F}_k^{(p-1)} - \boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot(\mathbf{J}^{\mathrm{T}}\cdot\mathbf{Q}_p - \tilde{\mathbf{O}}\cdot\mathbf{J}^{\mathrm{T}p})\cdot\boldsymbol{\phi}_k \\ &\quad - \boldsymbol{\Phi}\cdot(\mathbf{Q}_p\cdot\mathbf{J} - \mathbf{J}^p\cdot\tilde{\mathbf{O}})\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k. \end{aligned} \tag{D.4}$$

Next we use (2.35a) for $p=-1$, i.e.

$$\boldsymbol{\Phi}\cdot\mathbf{J}^{\mathrm{T}} = \mathbf{J}\cdot\boldsymbol{\Phi} + \boldsymbol{\Psi}^*\cdot\tilde{\mathbf{O}}\cdot\boldsymbol{\Phi} + \boldsymbol{\Phi}\cdot\tilde{\mathbf{O}}\cdot\boldsymbol{\Psi}, \tag{D.5}$$

giving

$$\begin{aligned} \boldsymbol{F}_k^{(p)} &= -\mathrm{i}\partial_x \boldsymbol{F}_k^{(p-1)} - \boldsymbol{\Phi}\cdot\mathbf{O}\cdot(\mathbf{J}\cdot\boldsymbol{\Phi}^* + \boldsymbol{\Psi}\cdot\tilde{\mathbf{O}}\cdot\boldsymbol{\Phi}^* + \boldsymbol{\Phi}^*\cdot\tilde{\mathbf{O}}\cdot\boldsymbol{\Psi}^*)\cdot\mathbf{Q}_p\cdot\boldsymbol{\phi}_k \\ &\quad - \boldsymbol{\Phi}\cdot\mathbf{Q}_p\cdot(\boldsymbol{\Phi}^*\cdot\mathbf{J}^{\mathrm{T}} - \boldsymbol{\Psi}\cdot\tilde{\mathbf{O}}\cdot\boldsymbol{\Phi}^* - \boldsymbol{\Phi}^*\cdot\tilde{\mathbf{O}}\cdot\boldsymbol{\Psi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k \\ &\quad + \boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\tilde{\mathbf{O}}\cdot\mathbf{J}^{\mathrm{T}p}\cdot\boldsymbol{\phi}_k + \boldsymbol{\Phi}\cdot\mathbf{J}^p\cdot\tilde{\mathbf{O}}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k \\ &= -\mathrm{i}\partial_x \boldsymbol{F}_k^{(p-1)} - (\boldsymbol{\Phi}\cdot\mathbf{J} + \boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Psi})\cdot\tilde{\mathbf{O}}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{Q}_p\cdot\boldsymbol{\phi}_k \\ &\quad + \boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\tilde{\mathbf{O}}\cdot(\mathbf{J}^{\mathrm{T}p}\cdot\boldsymbol{\phi}_k - \boldsymbol{\Psi}^*\cdot\mathbf{Q}_p\cdot\boldsymbol{\phi}_k) \\ &\quad - \boldsymbol{\Phi}\cdot\mathbf{Q}_p\cdot\boldsymbol{\Phi}^*\cdot\tilde{\mathbf{O}}\cdot(\mathbf{J}^{\mathrm{T}}\cdot\boldsymbol{\phi}_k - \boldsymbol{\Psi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k) \\ &\quad + (\boldsymbol{\Phi}\cdot\mathbf{J}^p + \boldsymbol{\Phi}\cdot\mathbf{Q}_p\cdot\boldsymbol{\Psi})\cdot\tilde{\mathbf{O}}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k. \end{aligned} \tag{D.6}$$

Inserting (2.34c), (2.35c) with the algebraic relation (2.35a), (4.1), (4.8), (4.5) and (4.6), we have

$$\begin{aligned} \boldsymbol{F}_k^{(p)} &= -\mathrm{i}\partial_x \boldsymbol{F}_k^{(p-1)} - (-\mathrm{i}\partial_x\boldsymbol{\Phi})\cdot\tilde{\mathbf{O}}\cdot\boldsymbol{G}_k^{(p)} - \mathbf{G}^{(p)*}\cdot\tilde{\mathbf{O}}\cdot(-\mathrm{i}\partial_x\boldsymbol{\phi}_k) \\ &\quad + \boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\tilde{\mathbf{O}}\cdot\boldsymbol{F}_k^{(p)} + \mathbf{F}^{(p)}\cdot\tilde{\mathbf{O}}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k. \end{aligned} \tag{D.7}$$

Hence, using also (4.9), which again is valid for negative values of p as well, we finally get

$$\boldsymbol{F}_k^{(p-1)} = \tilde{\Omega}[\boldsymbol{F}_k^{(p)}], \tag{D.8}$$

in which the action of the operator $\tilde{\Omega}$ on an arbitrary vector $\boldsymbol{a}_k$ is defined by, cf. (4.12),

$$\begin{aligned} \tilde{\Omega}[\boldsymbol{a}_k] &= \mathrm{i}\partial_x^{-1}[\boldsymbol{a}_k + (-\mathrm{i}\partial_x\boldsymbol{\Phi})\cdot\tilde{\mathbf{O}}\cdot[\mathrm{i}\partial_x^{-1}(\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{a}_k - \mathbf{A}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k)] \\ &\quad + [-\mathrm{i}\partial_x^{-1}(\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\mathbf{A}^* - \mathbf{A}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*)]\cdot\tilde{\mathbf{O}}\cdot(-\mathrm{i}\partial_x\boldsymbol{\phi}_k) \\ &\quad - \boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\tilde{\mathbf{O}}\cdot\boldsymbol{a}_k - \mathbf{A}\cdot\tilde{\mathbf{O}}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k]. \end{aligned} \tag{D.9}$$

The equation for $\boldsymbol{\Phi}$, cf. (4.16), in the case of negative r, can be expressed in the form

$$\mathrm{i}\partial_t\boldsymbol{\Phi} = \tilde{\Omega}^{|r|}[\boldsymbol{\Phi}], \tag{D.10}$$

in which the action of the operator $\tilde{\Omega}$ on an arbitrary matrix $\mathbf{A}$ is defined by

$$\begin{aligned} \tilde{\Omega}[\mathbf{A}] &= \mathrm{i}\partial_x^{-1}[\mathbf{A} + (-\mathrm{i}\partial_x\boldsymbol{\Phi})\cdot\tilde{\mathbf{O}}\cdot[\mathrm{i}\partial_x^{-1}(\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\mathbf{A} - \mathbf{A}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi})] \\ &\quad + [-\mathrm{i}\partial_x^{-1}(\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\mathbf{A}^* - \mathbf{A}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*)]\cdot\tilde{\mathbf{O}}\cdot(-\mathrm{i}\partial_x\boldsymbol{\Phi}) \\ &\quad - \boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\tilde{\mathbf{O}}\cdot\mathbf{A} - \mathbf{A}\cdot\tilde{\mathbf{O}}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi}]. \end{aligned} \tag{D.11}$$

Hence, the right-hand side of (D.10) can be, in principle, evaluated starting with $\mathbf{A}=\boldsymbol{\Phi}$, and from (D.2) in combination with (2.24), it is clear that (D.10) will lead to 3 coupled equations containing only $\phi_{0,0}$, $\phi_{1,0}$ and $\phi_{1,1}$.

Appendix E

The Lax representations for the complex sine-Gordon equation can be expressed as

$$-\mathrm{i}\partial_x\boldsymbol{\chi}_k^{(n)} = \mathbf{U}^{(n)}\cdot\boldsymbol{\chi}_k^{(n)}, \tag{E.1a}$$

$$\mathrm{i}\partial_t\boldsymbol{\chi}_k^{(n)} = \mathbf{V}^{(n)}\cdot\boldsymbol{\chi}_k^{(n)}, \quad (n=0,1), \tag{E.1b}$$

where $\boldsymbol{\chi}_k^{(n)}$, for $n=0,1$, is a two-dimensional vector with components

$$\boldsymbol{\chi}_k^{(n)} = \begin{pmatrix} \chi_{k1}^{(n)} \\ \chi_{k2}^{(n)} \end{pmatrix} = \mathrm{e}^{-\frac{1}{2}\mathrm{i}(kx-k^{-1}t)}\begin{pmatrix} \phi_k^{(n)} \\ \psi_k^{(n)} \end{pmatrix}. \tag{E.2}$$

We shall now evaluate the 2×2-matrices $\mathbf{U}^{(0)}$, $\mathbf{V}^{(0)}$, $\mathbf{U}^{(1)}$ and $\mathbf{V}^{(1)}$. From (2.34c) and (2.34d), in combination with the $n=0$ component of (2.21a), we have

$$\begin{aligned} -\mathrm{i}\partial_x\phi_k^{(0)} &= k\phi_k^{(0)} + \phi_{0,0}\psi_k^{(0)}, \\ -\mathrm{i}\partial_x\psi_k^{(0)} &= \phi_{0,0}^*\phi_k^{(0)}, \end{aligned} \tag{E.3}$$

leading with (E.2) to eq. (E.1a) with

$$\mathbf{U}^{(0)} = \begin{pmatrix} \frac{1}{2}k & \phi_{(0,0)} \\ \phi_{0,0}^* & -\frac{1}{2}k \end{pmatrix}. \tag{E.4}$$

In order to find $\mathbf{V}^{(0)}$, we consider the $n=1$ component of the algebraic relations (2.21a), (2.21b), which can be expressed as

$$k\phi_k^{(1)} = (1-\psi_{1,0}^*)\phi_k^{(0)} - \phi_{1,0}\psi_k^{(0)}, \tag{E.5a}$$

$$k\psi_k^{(1)} = \phi_{1,0}^*\phi_k^{(0)} + (1-\psi_{1,0})\psi_k^{(0)}. \tag{E.5b}$$

Eqs. (E.5a) and (E.5b) provide a direct relation between the vectors $\boldsymbol{\chi}_k^{(1)}$ and $\boldsymbol{\chi}_k^{(0)}$, viz.

$$k\boldsymbol{\chi}_k^{(1)} = \mathbf{g}\cdot\boldsymbol{\chi}_k^{(0)}, \tag{E.6}$$

where

$$\mathbf{g}=\begin{pmatrix}1-\psi^*_{1,0} & -\phi_{1,0}\\ \phi^*_{1,0} & 1-\psi_{1,0}\end{pmatrix} \tag{E.7}$$

is a unitary matrix. Taking the $n=0$ component of (2.34a) and (2.34b), we have

$$\mathrm{i}\partial_t\phi^{(0)}_k=(1-\psi_{1,0})\phi^{(1)}_k, \tag{E.8a}$$

$$\mathrm{i}\partial_t\psi^{(0)}_k=-\phi^*_{1,0}\phi^{(1)}_k, \tag{E.8b}$$

which, in combination with (E.2) and (E.7), lead to

$$\left(\frac{1}{2k}+\mathrm{i}\partial_t\right)\chi^{(0)}_k=\begin{pmatrix}1-\psi_{1,0} & 0\\ -\phi^*_{1,0} & 0\end{pmatrix}\chi^{(1)}_k, \tag{E.9}$$

or to (E.1b) with

$$\mathbf{V}^{(0)}=\frac{1}{2k}\left\{2\begin{pmatrix}1-\psi_{1,0} & 0\\ -\phi^*_{1,0} & 0\end{pmatrix}\cdot\mathbf{g}-\mathbf{1}\right\}=\frac{1}{2k}\mathbf{g}^{-1}\cdot\boldsymbol{\sigma}_3\cdot\mathbf{g}, \tag{E.10}$$

where $\boldsymbol{\sigma}_3$ is the Pauli matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$.

Using (6.4) and (6.5), the right-hand side of (E.10) can be evaluated to be

$$\mathbf{V}^{(0)}=\frac{1}{2k}\begin{pmatrix}(1+4|\partial_t\phi_{0,0}|^2)^{1/2} & 2\mathrm{i}\partial_t\phi_{0,0}\\ -2\mathrm{i}\partial_t\phi^*_{0,0} & -(1-4|\partial_t\phi_{0,0}|^2)^{1/2}\end{pmatrix}. \tag{E.11}$$

From (2.34c) and (2.34d), taking the $n=1$ component we have

$$-\mathrm{i}\partial_x\phi^{(1)}_k=(1-\psi^*_{1,0})\phi^{(0)}_k, \tag{E.12a}$$

$$-\mathrm{i}\partial_x\psi^{(1)}_k=\phi^*_{1,0}\phi^{(0)}_k, \tag{E.12b}$$

or

$$(\tfrac{1}{2}k-\mathrm{i}\partial_x)\chi^{(1)}_k=\begin{pmatrix}1-\psi^*_{1,0} & 0\\ \phi^*_{1,0} & 0\end{pmatrix}\chi^{(0)}_k. \tag{E.13}$$

Eq. (E.13) leads to the first equation of (E.1) with

$$\mathbf{U}^{(1)}=\tfrac{1}{2}k\left\{\begin{pmatrix}2(1-\psi^*_{1,0}) & 0\\ 2\phi^*_{1,0} & 0\end{pmatrix}\cdot\mathbf{g}^{-1}-\mathbf{1}\right\}=\tfrac{1}{2}k\mathbf{g}\cdot\boldsymbol{\sigma}_3\cdot\mathbf{g}^{-1}, \tag{E.14}$$

which with (5.12a) and (5.13) can be rewritten as

$$\mathbf{U}^{(1)}=\tfrac{1}{2}k\begin{pmatrix}(1-4|\partial_x\phi_{1,1}|^2)^{1/2} & -2\mathrm{i}\partial_x\phi_{1,1}\\ 2\mathrm{i}\partial_x\phi^*_{1,1} & -(1-4|\partial_x\phi_{1,1}|^2)^{1/2}\end{pmatrix}. \tag{E.15}$$

Finally, from (2.34a), (2.34b) and (2.21a) for $p=-1$ we have

$$\mathrm{i}\partial_t\phi^{(1)}_k=\frac{1}{k}\phi^{(1)}_k-\phi_{1,1}\psi^{(1)}_k, \tag{E.16a}$$

$$\mathrm{i}\partial_t\psi^{(1)}_k=-\phi^*_{1,1}\phi^{(1)}_k, \tag{E.16b}$$

or

$$\left(\frac{1}{2k}+\mathrm{i}\partial_t\right)\chi^{(1)}_k=\begin{pmatrix}k^{-1} & -\phi_{1,1}\\ -\phi^*_{1,1} & 0\end{pmatrix}\chi^{(1)}_k, \tag{E.17}$$

leading to (E.1b) with

$$\mathbf{V}^{(1)}=\begin{pmatrix}\tfrac{1}{2}k^{-1} & -\phi_{1,1}\\ -\phi^*_{1,1} & -\tfrac{1}{2}k^{-1}\end{pmatrix}. \tag{E.18}$$

The gauge equivalence[13]) between the Lax representations (E.1), for $n=0$ and $n=1$, can be formulated in a straightforward way, using (E.6). We have, e.g.

$$\mathrm{i}\partial_t\chi^{(1)}_k=k^{-1}\mathrm{i}\partial_t(\mathbf{g}\cdot\chi^{(0)}_k)=\mathrm{i}(\partial_t\mathbf{g})\cdot\mathbf{g}^{-1}\cdot\chi^{(1)}_k+\mathbf{g}\cdot\mathbf{V}^{(0)}\cdot\mathbf{g}^{-1}\cdot\chi^{(1)}_k, \tag{E.19}$$

leading to

$$\mathbf{V}^{(1)}=\mathrm{i}(\partial_t\mathbf{g})\cdot\mathbf{g}^{-1}+\mathbf{g}\cdot\mathbf{V}^{(0)}\cdot\mathbf{g}^{-1}, \tag{E.20}$$

and in a similar way it can be shown that

$$\mathbf{U}^{(1)}=-\mathrm{i}(\partial_x\mathbf{g})\cdot\mathbf{g}^{-1}+\mathbf{g}\cdot\mathbf{U}^{(0)}\cdot\mathbf{g}^{-1}. \tag{E.21}$$

Appendix F

In order to derive (7.35), one can insert $q=\kappa\,e^{\mathrm{i}\gamma}$, ($\kappa>0$, γ real) in (7.34). After taking the imaginary and real part one has

$$\tfrac{1}{2}\mathrm{i}\partial_t\kappa^2=-\mathrm{i}\partial_x(\kappa^2\gamma'), \tag{F.1}$$

and

$$\dot\gamma=\frac{\kappa''}{\kappa}-2\frac{\kappa'^2}{\kappa^2}-3\gamma'^2, \tag{F.2}$$

where also dots and primes have been used to denote the differentiations with respect to x and t, respectively.

Introducing a real function f, so that

$$\kappa^2=2f', \tag{F.3}$$

we have from (F.1)

$$\gamma'=\frac{-\dot f}{2f'}, \tag{F.4}$$

and from (F.2)

$$\dot{\gamma} = \frac{f'''}{2f'} - \frac{3}{4}\left(\frac{f''^2 + \dot{f}^2}{f'^2}\right). \tag{F.5}$$

The potential MNLS (7.35) follows immediately from the compatibility relation $\partial_x\dot{\gamma} = \partial_t\gamma'$.

References

1) R.K. Bullough and P.J. Caudrey, Selected Topics in Current Physics **17** (Springer, Berlin, Heidelberg, New York, 1980) and references cited therein.
2) M.J. Ablowitz, A. Ramani and H. Segur, J. Math. Phys. **21** (1980) 1006.
3) V.E. Zakharov and A.B. Shabat, Funct. Anal. Appl. **8** (1974) 226.
4) A.S. Fokas and M.J. Ablowitz, Phys. Rev. Lett. **47** (1981) 1096.
5) F.W. Nijhoff, J. van der Linden, G.R.W. Quispel and H.W. Capel, Phys. Lett. **89A** (1982) 106.
6) F.W. Nijhoff, J. van der Linden, G.R.W. Quispel, H.W. Capel and J. Velthuizen, Physica **116A** (1982) 1.
7) G.R.W. Quispel, F.W. Nijhoff and H.W. Capel, Phys. Lett. **91A** (1982) 143.
8) R.M. Miura, J. Math. Phys. **9** (1968) 1202.
9) M. Lakshmanan, Phys. Lett. **61A** (1977) 53.
10) G.R.W. Quispel and H.W. Capel, Phys. Lett. **85A** (1981) 248 and Physica **110A** (1982) 41.
11) R. Hirota and J. Satsuma, Progr. Theor. Phys. **57** (1977) 797.
12) V.E. Zakharov and L.A. Takhtadzhyan, Theor. Math. Phys. **38** (1979) 17.
13) J. Honerkamp, J. Math. Phys. **22** (1981) 277; H. Eichenherr and J. Honerkamp, J. Math. Phys. **22** (1981) 374.
14) H.-H. Chen, Phys. Rev. Lett. **33** (1974) 925.
15) G.R.W. Quispel and H.W. Capel, Phys. Lett. **88A** (1982) 371 and Physica **117A** (1983) 76.
16) A. Chodos, Phys. Rev. **D21** (1980) 2818.
17) K.M. Case and A.M. Roos, J. Math. Phys. **23** (1982) 392.
18) P. Kaliappan and M. Lakshmanan, J. Math. Phys. **23** (1982) 456.
19) B. Fuchssteiner and E.P. Hefter, Phys. Rev. **D24** (1981) 2769.
20) F.W. Nijhoff and H.W. Capel, Phys. Lett. **91A** (1982) 431.
21) R. Hirota, J. Phys. Soc. Japan **51** (1982) 323.
22) P.J. Caudrey, in: Nonlinear Equations in Physics and Mathematics, A.O. Barut, ed. (Nato Advanced Study Institutes Series **40C** (1977)) p. 177.
23) M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, Stud. Appl. Math. **53** (1974) 249.
24) R.N. Aiyer, J. Phys. **A15** (1982) 397.
25) R. Hirota, J. Math. Phys. **14** (1973) 805.
26) N. Papanicolaou, Phys. Lett. **73A** (1979) 134.
27) K. Pohlmeyer and K. H. Rehren, J. Math. Phys. **20** (1979) 2628.
28) K. Pohlmeyer, Comm. Math. Phys. **46** (1976) 207.
29) F. Lund and T. Regge, Phys. Rev. **D14** (1976) 1524.
30) F. Lund, Phys. Rev. Lett. **38** (1977) 1175.
31) B.S. Getmanov, JETP Lett. **25** (1977) 119 and Teor. Mat. Fiz. **38** (1979) 186.
32) P.D. Lax, Comm. Pure Appl. Math. **21** (1968) 467.