THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION AND THE MASSIVE THIRRING MODEL

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The linearization and Bäcklund transformations are obtained for a class of nonlinear partial differential equations, which contains the derivative nonlinear Schrödinger equation and the equations for the massive Thirring model.

1. *Integral equation.* Consider the integral equation

$$\phi_k^{(n)}(x,t) + f(k) \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}[kx-\omega(k)t]}\,\mathrm{e}^{-\mathrm{i}[l'x-\omega^*(l')t]}}{(k-l')(l'-l)} \phi_l^{(n)}(x,t) = (1/k^n)\,\mathrm{e}^{\mathrm{i}[kx-\omega(k)t]} \quad (n \text{ integer}), \tag{1}$$

in which the integrations are performed over an arbitrary contour C and its complex conjugate C* in the complex k plane with an arbitrary measure $\mathrm{d}\lambda(k)$ and its complex conjugate $\mathrm{d}\lambda^*(k)$. The measure $\mathrm{d}\lambda(k)$ and the contour C are to be chosen in such a way that the integral equation has a unique solution $\phi_k^{(n)}(x,t)$, n being an integer [1–5]. The dispersion $\omega(k)$ is a meromorphic function $\omega(k) = \Sigma_r \lambda_r k^r$ (r integer), which can be specified in order to obtain various (integrable) partial differential equations (PDEs). Integral equations of the type (1) with $f(k) = 1$ have been investigated systematically in refs. [2–4], leading e.g. to solutions of the nonlinear Schrödinger equation (NLS), the isotropic Heisenberg spin chain (IHSC) and various other PDEs, see also refs. [1,5] for linear integral equations containing only one integration over a contour.

In this letter we shall investigate the case that $f(k) = k$ and in a more extended publication [6] the case of general meromorphic $f(k)$ will be treated.

We introduce vectors $\boldsymbol{\phi}_k(x,t)$ and $\boldsymbol{\psi}_k(x,t)$ with components $\phi_k^{(n)}(x,t)$ and

$$\psi_k^{(n)}(x,t) = \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}[kx-\omega(k)t]}}{k-l'} \phi_{l'}^{(n)*}(x,t), \tag{2}$$

and matrices $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ with elements

$$\phi_{n,m}(x,t) = \int_C \mathrm{d}\lambda(k)\, k^{-m} \phi_k^{(n)}(x,t), \quad \psi_{n,m} = \int_C \mathrm{d}\lambda(k)\, k^{-m} \psi_k^{(n)}(x,t) \quad (n, m \text{ integers}). \tag{3}$$

From the integral equation (1), taking into account the uniqueness of its solution, and following the line of reasoning given in refs. [2–4], one can derive the constitutive relations

$$-\mathrm{i}\partial_x \boldsymbol{\phi}_k = (\mathbf{1} - \boldsymbol{\Psi}^* \cdot \mathrm{O}) \cdot \mathrm{J}^{\mathrm{T}} \cdot \boldsymbol{\phi}_k, \quad -\mathrm{i}\partial_x \boldsymbol{\psi}_k = k\boldsymbol{\psi}_k - \mathrm{J}^{\mathrm{T}} \cdot \boldsymbol{\psi}_k + \boldsymbol{\psi} \cdot \mathrm{O} \cdot \mathrm{J}^{\mathrm{T}} \cdot \boldsymbol{\psi}_k, \tag{4a, b}$$

$$k^p \boldsymbol{\phi}_k = (\mathrm{J}^{\mathrm{T}})^p \cdot \boldsymbol{\phi}_k - \boldsymbol{\Psi}^* \cdot \mathrm{O}_p \cdot \mathrm{J}^{\mathrm{T}} \cdot \boldsymbol{\phi}_k - \boldsymbol{\Phi} \cdot \mathrm{O}_p \cdot \mathrm{J}^{\mathrm{T}} \cdot \boldsymbol{\psi}_k - \boldsymbol{\Phi} \cdot \mathrm{O}_p \cdot \boldsymbol{\Phi}^{\dagger} \cdot \mathrm{O} \cdot \boldsymbol{\phi}_k, \tag{4c}$$

$$k^p \boldsymbol{\psi}_k = (\mathrm{J}^{\mathrm{T}})^p \cdot \boldsymbol{\psi}_k - \boldsymbol{\Psi} \cdot \mathrm{O}_p \cdot \mathrm{J}^{\mathrm{T}} \cdot \boldsymbol{\psi}_k - \boldsymbol{\Phi}^* \cdot \mathrm{O}_p \cdot \boldsymbol{\Phi}^{\mathrm{T}} \cdot \mathrm{O} \cdot \boldsymbol{\psi}_k + \boldsymbol{\Phi}^* \cdot \mathrm{O}_p \cdot \mathrm{J}^{\mathrm{T}} \cdot k^{-1} \boldsymbol{\phi}_k, \tag{4d}$$

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for all integer p, independent of the dispersion $\omega(k)$, and

$$\mathrm{i}\partial_t \boldsymbol{\phi}_k = \sum_r \lambda_r [(\mathbf{J}^{\mathrm{T}})^r \cdot \boldsymbol{\phi}_k - \boldsymbol{\Psi}^* \cdot \mathbf{Q}_r \cdot \mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{\phi}_k], \quad \mathrm{i}\partial_t \boldsymbol{\psi}_k = \sum_r \lambda_r [k^r \boldsymbol{\psi}_k - (\mathbf{J}^{\mathrm{T}})^r \cdot \boldsymbol{\psi}_k + \boldsymbol{\Psi} \cdot \mathbf{Q}_r \cdot \mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{\psi}_k], \tag{4e, f}$$

depending on the dispersion relation $\omega(k) = \Sigma_r \lambda_r k^r$. In (4) $\mathbf{A}^*$, $\mathbf{A}^\dagger$ and $\mathbf{A}^{\mathrm{T}}$ denote the complex conjugate, the hermitean adjoint and the transposed matrix of a matrix $\mathbf{A}$ respectively; $\mathbf{J}$ and $\mathbf{J}^{\mathrm{T}}$ are index-raising and index-lowering matrices, i.e. $(\mathbf{J})_{n,m} = \delta_{m,n+1}$, $(\mathbf{J}^{\mathrm{T}})_{n,m} = \delta_{n,m+1}$; $\mathbf{O}$ is a projection matrix $(\mathbf{O})_{n,m} = \delta_{n,0}\delta_{m,0}$; and the matrix $\mathbf{Q}_p$ is given by

$$\mathbf{Q}_p = (\operatorname{sgn} p) \sum_{j=0}^{|p|-1} \mathbf{J}^{(p-|p|)/2+j} \cdot \mathbf{O} \cdot (\mathbf{J}^{\mathrm{T}})^{(p+|p|)/2-j-1} \tag{5}$$

for all integers p. From eq. (1) one can also derive the symmetry relations

$$\boldsymbol{\Phi}^{\mathrm{T}} \cdot \mathbf{J} = \mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{\Phi}, \quad \boldsymbol{\Psi} + \boldsymbol{\Psi}^\dagger = \boldsymbol{\Psi} \cdot \mathbf{O} \cdot \boldsymbol{\Psi}^\dagger, \quad \mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{\Psi} \cdot \mathbf{J}^{\mathrm{T}} + \boldsymbol{\Psi}^\dagger = -\mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{J}^{\mathrm{T}} \cdot \mathbf{O} \cdot \boldsymbol{\Phi} \cdot \mathbf{J}^{\mathrm{T}}, \tag{6}$$

from which one immediately has $|1 - \psi_{0,0}|^2 = 1$, $\phi_{-1,1} = \phi_{0,0}$ and $\psi^*_{0,0} + \psi_{-1,1} = -\phi^*_{-1,1}\phi_{0,1}$. Using (3) one can integrate eqs. (4) over the contour C, replacing $k^p \boldsymbol{\phi}_k$ and $k^p \boldsymbol{\psi}_k$ by $\boldsymbol{\Phi} \cdot \mathbf{J}^p$ and $\boldsymbol{\Psi} \cdot \mathbf{J}^p$ respectively, in order to obtain matrix equations in terms of $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$.

2. Partial differential equations. One can use the integrated versions of eqs. (4) in order to derive closed PDEs in terms of the elements of the matrix $\boldsymbol{\Phi}$. All these PDEs are completely integrable in the sense that solutions for these PDEs, such as e.g. soliton solutions, can be found immediately from the linear integral equation (1) by appropriate choices of measure and contour.

In particular it is straightforward to derive closed PDEs for the matrix elements $\phi_{0,0}$ and $\phi_{0,1}$, making use of (6) and simple relations like $-\mathrm{i}\partial_x \phi_{0,1} = (1 - \psi^*_{0,0})\phi_{0,0}$ and $\mathrm{i}\partial_x \ln(1 - \psi_{0,0}) = |\phi_{0,0}|^2$. More specifically, taking $\omega(k) = \lambda_2 k^2 + \lambda_3 k^3$ we have

$$(\mathrm{i}\partial_t + \lambda_2 \partial_x^2 - \mathrm{i}\lambda_3 \partial_x^3)\phi_{0,1} = 2\mathrm{i}\lambda_2 |\partial_x \phi_{0,1}|^2 \partial_x \phi_{0,1} + 6\lambda_3 |\partial_x \phi_{0,1}|^2 (\partial_x^2 \phi_{0,1} - \mathrm{i}|\partial_x \phi_{0,1}|^2 \partial_x \phi_{0,1}), \tag{7}$$

$$\begin{aligned}(\mathrm{i}\partial_t + \lambda_2 \partial_x^2 - \mathrm{i}\lambda_3 \partial_x^3)\phi_{0,0} &= 2\mathrm{i}\lambda_2 |\phi_{0,0}|^2 \partial_x \phi_{0,0} + 3\lambda_3 \phi^*_{0,0} (\partial_x \phi_{0,0})^2 \\ &\quad + 3\lambda_3 |\phi_{0,0}|^2 (\partial_x^2 \phi_{0,0} - \mathrm{i}|\phi_{0,0}|^2 \partial_x \phi_{0,0}).\end{aligned} \tag{8}$$

Eq. (7) for $\lambda_3 = 0$ is a "potential" form of the derivative nonlinear Schrödinger equation (DNLS), i.e. $\mathrm{i}\partial_t q + \partial_x^2 q = 2\mathrm{i}\partial_x(|q|^2 q)$ with $q \equiv \partial_x \phi_{0,1}$, which has been treated within the inverse scattering formalism [7]; eq. (7) for $\lambda_2 = 0$ may be regarded as a derivative version of the complex modified Korteweg–de Vries equation (CMKdV), cf. ref. [4], and in the general case with $\lambda_2, \lambda_3 \neq 0$ as a derivative version of Hirota's equation [4,8]. Eq. (8) for $\lambda_3 = 0$ is a new derivative nonlinear Schrödinger equation, which has been introduced in ref. [9]. For this equation multisoliton solutions were found in ref. [10] and the prolongation algebra was treated in ref. [11].

Eq. (8), however, is completely equivalent to eq. (7), cf. also ref. [12]. In fact, one can express $\phi_{0,0}$ in terms of $q \equiv \partial_x \phi_{0,1}$, using

$$|\phi_{0,0}| = |q|, \quad \partial_x \ln \phi_{0,0} = \partial_x \ln q - \mathrm{i}|q|^2, \tag{9}$$

and the relation for $\ln \phi_{0,0}$ in terms of q, which can be derived from (8) and (9). The inverse transformation can be found from (9), i.e. $|q| = |\phi_{0,0}|$, $\partial_x \ln q = \partial_x(\ln \phi_{0,0}) + \mathrm{i}|\phi_{0,0}|^2$, and the relation for $\mathrm{i}\partial_t \ln q$ in terms of $\phi_{0,0}$, which follows from (7).

For $\omega(k) = k^{-1}$ we find from the integrated version of (4), taking into account (6) and the simple relations given above [13],

$$\partial_x \partial_t \phi_{0,1} = \phi_{0,1} + 2\mathrm{i} |\phi_{0,1}|^2 \partial_x \phi_{0,1} , \tag{10}$$

and the quantities $\psi_1 \equiv \phi_{0,1}(1 - \psi_{0,0})$ and $\psi_2 \equiv \phi_{0,0}$ satisfy

$$-\mathrm{i}\partial_x \psi_1 = \psi_2 - |\psi_2|^2 \psi_1 , \quad \mathrm{i}\partial_t \psi_2 = \psi_1 - |\psi_1|^2 \psi_2 ; \tag{11}$$

which are the equations for the massive Thirring model (MTM) [14–16].

Eq. (10), however, is completely equivalent to (11), (13). In fact, using $\mathrm{i}\partial_x \ln(1 - \psi_{0,0}) = |\phi_{0,0}|^2$, $\mathrm{i}\partial_t \ln(1 - \psi_{0,0}) = |\phi_{0,1}|^2$, we find

$$\psi_1 = \phi_{0,1}\, \mathrm{e}^{-\mathrm{i}\chi} , \quad \psi_2 = -\mathrm{i}(\partial_x \phi_{0,1})\, \mathrm{e}^{-\mathrm{i}\chi} , \quad \chi = \int_\Gamma (\mathrm{d}l_x |\partial_x \phi_{0,1}|^2 + \mathrm{d}l_t |\phi_{0,1}|^2), \tag{12}$$

in which Γ is an arbitrary curve connecting $(0, 0)$ and (x, t) in the two-dimensional space–time plane and $\mathrm{d}l = (\mathrm{d}l_x, \mathrm{d}l_t)$ is an infinitesimal two-dimensional vector tangent to Γ. The inverse transformation reads

$$\phi_{0,1} = \psi_1 \exp\left(\mathrm{i} \int_\Gamma (\mathrm{d}l_x |\psi_2|^2 + \mathrm{d}l_t |\psi_1|^2)\right) . \tag{13}$$

Eq. (10) may also be rewritten as $-\mathrm{i}\partial_x \phi_{0,1} = \psi_3$, $\mathrm{i}\partial_t \psi_3 = \phi_{0,1} - 2|\phi_{0,1}|^2 \psi_3$, which is a semilinear version of the MTM. Furthermore, due to a space–time symmetry of eq. (1) for $\omega(k) = k^{-1}$, we have another manifestation of the MTM, namely $\partial_x \partial_t \psi_4 = \psi_4 - 2\mathrm{i}|\psi_4|^2 \partial_t \psi_4$, $\psi_4 = (1 - \psi_{0,0})\phi_{0,0}$.

3. Bäcklund transformations. Bäcklund transformations are induced by singular transformations of measures in the integral equation [17]. Applying the transformation $\mathrm{d}\lambda(l) \to \mathrm{d}\tilde{\lambda}(l) = (p - l)(p^* - l)^{-1}\, \mathrm{d}\lambda(l)$ in (1), one obtains the following relations between the solutions $\tilde{\boldsymbol{\phi}}_k(x, t)$ and $\tilde{\boldsymbol{\psi}}_k(x, t)$ of eqs. $(\tilde{1})$ and $(\tilde{2})$ with the measure $\mathrm{d}\tilde{\lambda}(l)$ instead of $\mathrm{d}\lambda(l)$ on the one hand, and the solutions $\boldsymbol{\phi}_k(x, t)$ and $\boldsymbol{\psi}_k(x, t)$ of eqs. (1) and (2) on the other hand:

$$(p - k)\tilde{\boldsymbol{\phi}}_k = p\boldsymbol{\phi}_k - \mathsf{J}^{\mathrm{T}}\cdot\boldsymbol{\phi}_k + \tilde{\boldsymbol{\Psi}}^*\cdot\mathsf{O}\cdot\mathsf{J}^{\mathrm{T}}\cdot\boldsymbol{\phi}_k + \tilde{\boldsymbol{\Phi}}\cdot\mathsf{O}\cdot\mathsf{J}^{\mathrm{T}}\cdot\boldsymbol{\psi}_k + \tilde{\boldsymbol{\Phi}}\cdot\mathsf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathsf{O}\cdot\boldsymbol{\phi}_k , \tag{14a}$$

$$(p - k)\tilde{\boldsymbol{\psi}}_k = p^*\boldsymbol{\psi}_k - \mathsf{J}^{\mathrm{T}}\cdot\boldsymbol{\psi}_k + \tilde{\boldsymbol{\Psi}}\cdot\mathsf{O}\cdot\mathsf{J}^{\mathrm{T}}\cdot\boldsymbol{\psi}_k - (p\tilde{\boldsymbol{\Phi}}^* - p^*\boldsymbol{\Phi}^*)\cdot\mathsf{J}^{\mathrm{T}}\cdot\mathsf{O}\cdot\boldsymbol{\phi}_k - (1 - \tilde{\boldsymbol{\Psi}}\cdot\mathsf{O})\cdot\boldsymbol{\Phi}^\dagger\cdot\mathsf{O}\cdot\boldsymbol{\phi}_k , \tag{14b}$$

in which $\tilde{\boldsymbol{\Phi}}$ and $\tilde{\boldsymbol{\Psi}}$ are defined in terms of $\tilde{\boldsymbol{\phi}}_k$, $\tilde{\boldsymbol{\psi}}_k$ and $\mathrm{d}\tilde{\lambda}(k)$, cf. (3).

Integrating (14a), (14b) over $\mathrm{d}\lambda(k)$, and using $(p^* - k)\, \mathrm{d}\tilde{\lambda}(k) = (p - k)\, \mathrm{d}\lambda(k)$ to evaluate the left-hand sides, we find

$$p^*\tilde{\boldsymbol{\Phi}} - p\boldsymbol{\Phi} = \tilde{\boldsymbol{\Phi}}\cdot\mathsf{J} - \mathsf{J}^{\mathrm{T}}\cdot\boldsymbol{\Phi} + \tilde{\boldsymbol{\Psi}}^*\cdot\mathsf{O}\cdot\mathsf{J}^{\mathrm{T}}\cdot\boldsymbol{\Phi} + \tilde{\boldsymbol{\Phi}}\cdot\mathsf{O}\cdot\mathsf{J}^{\mathrm{T}}\cdot\boldsymbol{\Psi} + \tilde{\boldsymbol{\Phi}}\cdot\mathsf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathsf{O}\cdot\boldsymbol{\Phi} , \tag{15a}$$

$$p^*(\tilde{\boldsymbol{\Psi}} - \boldsymbol{\Psi}) = \tilde{\boldsymbol{\Psi}}\cdot\mathsf{J} - \mathsf{J}^{\mathrm{T}}\cdot\boldsymbol{\Psi} + \tilde{\boldsymbol{\Psi}}\cdot\mathsf{O}\cdot\mathsf{J}^{\mathrm{T}}\cdot\boldsymbol{\Psi} - (p\tilde{\boldsymbol{\Phi}}^* - p^*\boldsymbol{\Phi}^*)\cdot\mathsf{J}^{\mathrm{T}}\cdot\mathsf{O}\cdot\boldsymbol{\Phi} - (1 - \tilde{\boldsymbol{\Psi}}\cdot\mathsf{O})\cdot\boldsymbol{\Phi}^\dagger\cdot\mathsf{O}\cdot\boldsymbol{\Phi} . \tag{15b}$$

From eqs. (15a), (15b), in combination with the integrated version of eqs. (4a)–(4d), one can derive the spatial part of the Bäcklund transformation (BT) for e.g. the PDE for $\phi_{0,1}$. One obtains after a lengthy calculation

$$\begin{aligned} a = &-\tfrac{1}{2}\mathrm{i}p^{-1}(\partial_x \tilde{\phi}_{0,1})\{p + p^* + |a|^2 \pm \mathrm{i}[4|p|^2 - (p + p^* + |a|^2)^2]^{1/2}\} \\ &+ \tfrac{1}{2}\mathrm{i}p^{*-1}(\partial_x \phi_{0,1})\{p + p^* + |a|^2 \mp \mathrm{i}[4|p|^2 - (p + p^* + |a|^2)^2]^{1/2}\} , \end{aligned} \tag{16}$$

$a = p^*\tilde{\phi}_{0,1} - p\phi_{0,1}$ which is independent of the dispersion $\omega(k)$. The time-dependent part of the BT for $\omega(k) = \lambda_2 k^2 + \lambda_3 k^3$ and $\omega(k) = k^{-1}$ can be inferred from eqs. (7) and (10) respectively, see also ref. [18] for a different treatment of the BT of the MTM. Furthermore, one can derive the corresponding (completely integrable) PDEs for a, which may be called the modified equations of the PDEs (7) and (10). As an example we give the modified derivative nonlinear Schrödinger equation (MDNLS) $(\lambda_3 = 0)$,

$$(\mathrm{i}\partial_t + \partial_x^2)a = \frac{2\mathrm{i}}{|p|^2}\left(\tfrac{1}{4}|\partial_x a|^2 \partial_x a + \frac{2(\partial_x a)\big||p|^2 a + \tfrac{1}{2}\mathrm{i}|a|^2 \partial_x a\big|^2 + (\partial_x a^*)(|p|^2 a + \tfrac{1}{2}\mathrm{i}|a|^2 \partial_x a)^2}{4|p|^2 - |a|^4}\right) , \tag{17}$$

where we have taken $p^* = -p$ for convenience. Note that eq. (16), with $\partial_x \phi_{0,1} = q$, $\partial_x \tilde{\phi}_{0,1} = (p^*)^{-1}(\partial_x a + pq)$, provides the Miura-transformation mapping solutions of the modified PDE for a on solutions of the PDE for $q = \partial_x \phi_{0,1}$. For eq. (17), as well as for other modified PDEs, it is straightforward to derive the corresponding linear integral equations, in the same way as a linear integral equation of type (1) with $f(k) = |p|^2 + k^2$ was obtained in ref. [17] for the anisotropic Heisenberg spin chain (AHSC) from the one for the NLS, but further details will be given in ref. [6].

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