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THE ANISOTROPIC HEISENBERG SPIN CHAIN AND THE NONLINEAR SCHRÖDINGER EQUATION

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It is shown that the solutions of the continuous Anisotropic Heisenberg Spin Chain (AHSC) can be obtained from the linear integral equation which was proposed in a previous paper for the solutions of the Isotropic Heisenberg Spin Chain (IHSC) and the Nonlinear Schrödinger equation (NLS). An explicit expression is obtained for the Miura transformation which maps the solutions of the AHSC on solutions of the NLS. In the second part of the paper we investigate the similarity solutions of these partial differential equations which leads to ordinary differential equations of Painlevé type. As an application we discuss some new solutions of Painlevé IV.

1. Introduction

Consider the linear integral equation[1,2])

$$\phi_k(x,t) + \frac{\alpha}{2}\int_C \mathrm{d}\lambda(l)\int_{C^*} \mathrm{d}\lambda^*(l')\,\frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')(l'-l)}\,\phi_l(x,t) = \mathrm{e}^{\mathrm{i}(kx-k^2t)}, \tag{1.1}$$

in which $\mathrm{d}\lambda(k)$ is an arbitrary measure with the complex conjugated measure $\mathrm{d}\lambda^*(k')$ and in which the integrations are performed over an arbitrary contour C and its complex conjugate C^*.

In refs. 1 and 2 we have shown how, for any choice of the contour C and the measure $\mathrm{d}\lambda(k)$ under rather general conditions, one can obtain from the solution ϕ_k of (1.1) a solution $\phi(x,t)$ of the nonlinear Schrödinger equation (NLS)

$$\mathrm{i}\partial_t\phi + \partial_x^2\phi + \alpha|\phi|^2\phi = 0, \tag{1.2}$$

as well as a solution $S(x,t)$ of the equation of motion of the isotropic Heisenberg spin chain in the continuum limit (IHSC), i.e.

$$\partial_t \boldsymbol{S} = \boldsymbol{S}\times\partial_x^2\boldsymbol{S} + \boldsymbol{S}\times B\boldsymbol{e}^z, \quad \boldsymbol{S}\cdot\boldsymbol{S} = 1, \tag{1.3}$$

where $\boldsymbol{B} = B\boldsymbol{e}^z$ is a magnetic field in the z direction.

In the present paper we will show how, starting from any solution ϕ_k of (1.1), we can construct a second solution $\tilde{\phi}(x,t)$ of (1.2) and also a solution

$S(x, t)$ of the anisotropic Heisenberg spin chain in the continuum limit (AHSC), i.e.

$$\partial_t \boldsymbol{S} = \boldsymbol{S} \times \partial_x^2 \boldsymbol{S} + \boldsymbol{S} \times (AS^z + B)\boldsymbol{e}^z, \quad \boldsymbol{S} \cdot \boldsymbol{S} = 1, \tag{1.4}$$

where A arises from a uniaxial anisotropy.

The construction of $\tilde{\phi}(x, t)$ and $S(x, t)$ is very closely related to the derivation of the Bäcklund transformation (BT) of the NLS by Chen3) and in section 2 some of the relevant features of this derivation will be treated. The connection with the AHSC will be established in section 3 where we also derive an explicit (Miura) transformation which maps an arbitrary solution of the AHSC with $A > 0$ on a solution of the NLS. In a special case we obtain a new partial differential equation (PDE) which will be called the modified nonlinear Schrödinger equation (MNLS).

The similarity solutions, which depend essentially on one variable which is a function of x and t, of the NLS, AHSC, IHSC and MNLS will be studied in section 4. These similarity solutions satisfy ordinary differential equations (ODE's) without movable critical points, in agreement with the paper by Ablowitz et al.4) on the similarity solutions of integrable PDE's. In section 5 we discuss the transformations between these ODE's and as an application some new solutions of Painlevé IV are obtained. A preliminary account of some of the considerations in this paper was given in ref. 5.

2. Bäcklund transformation

2.1. *The linear problem*

As a first step in deriving the BT of the NLS we shall give the linear problem in terms of the functions $\phi_k(x, t)$ satisfying (1.1) and the associated functions

$$\psi_k(x, t) = \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx - k^2 t)}}{k - l'} \phi_{l'}^*(x, t). \tag{2.1}$$

The functions $\psi_k(x, t)$ satisfy the integral equation

$$\psi_k(x, t) + \frac{\alpha}{2} \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx - k^2 t)}\, \mathrm{e}^{-\mathrm{i}(l'x - l'^2 t)}}{(k - l')(l' - l)} \psi_l(x, t)$$
$$= \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx - k^2 t)}\, \mathrm{e}^{-\mathrm{i}(l'x - l'^2 t)}}{k - l'}, \tag{2.2}$$

as follows from (1.1) and (2.1).

Assuming that the homogeneous integral equation, corresponding to (1.1) and (2.2), has only the zero solution and that the differentiations with respect to x and t may be shifted through the integrals, one can derive the constitutive relations of the linear problem,

$$(\mathrm{i}\partial_t + \partial_x^2)\phi_k = -\alpha\,\mathrm{i}(\partial_x\psi^*)\phi_k, \tag{2.3}$$

$$(\mathrm{i}\partial_t + \partial_x^2)\psi_k = 2\mathrm{i}(\partial_x\phi^*)\phi_k, \tag{2.4}$$

$$-\mathrm{i}\partial_x\psi_k = \phi^*\phi_k, \tag{2.5}$$

$$(-k - \mathrm{i}\partial_x)\phi_k = \tfrac{1}{2}\alpha\phi\psi_k, \tag{2.6}$$

in which

$$\phi = \int_C \mathrm{d}\lambda(k)\phi_k, \quad \psi = \int_C \mathrm{d}\lambda(k)\psi_k. \tag{2.7}$$

For the special case $\alpha = 2$ these relations were given in refs. 1 and 2 and the derivation for $\alpha \neq 2$ proceeds in the same way.

From (2.3), (2.5) and (2.7) it is clear that the function $\phi(x, t)$ satisfies the NLS (1.2). (The function $y(x, t) = -\frac{1}{2}\mathrm{i}\psi(x, t)$ is a solution of the potential NLS in terms of one real variable introduced in refs. 6–8.) In refs. 1 and 2 it was also shown that the function

$$\Phi(x, t) = \int_C \mathrm{d}\lambda(k)k^{-1}\phi_k, \tag{2.8}$$

in which ϕ_k is the solution of (1.1) with $\alpha = 2$, satisfies the PDE

$$(\mathrm{i}\partial_t + \partial_x^2)\Phi = \frac{-2|\partial_x\Phi|^2}{1-|\Phi|^2}\,\Phi, \tag{2.9}$$

which is equivalent to the IHSC (1.3). The polar angles of the spin vector, i.e.

$$S = (\sin\theta\cos\chi, \sin\theta\sin\chi, \cos\theta), \tag{2.10}$$

are explicitly given by

$$\theta = 2\,\mathrm{arc}\,\sin|\Phi|, \tag{2.11}$$

and

$$\partial_x\chi = \frac{\partial_x \operatorname{Im}\ln\Phi}{1-|\Phi|^2}, \quad \partial_t\chi = \frac{\partial_t \operatorname{Im}\ln\Phi}{1-|\Phi|^2} - \frac{|\partial_x\Phi|^2}{(1-|\Phi|^2)^2} - B, \tag{2.12}$$

as can be inferred from eq. (5.21) of ref. 2. Furthermore it was shown in refs. 1 and 2 that the linear problems of Zakharov and Shabat[9]) for the NLS and of Takhtadzhyan[10]) for the IHSC, as well as their gauge equivalence[11]), can be derived from the integral equation (1.1). Hence, the integral equation (1.1) yields the linearization of the NLS as well as of the IHSC. (A similar integral equation was proposed by Fokas and Ablowitz[12]) for the linearization of the

Korteweg–de Vries equation (KdV), and the solutions of the modified Korteweg–de Vries equation (MKdV) can be obtained from the same integral equation[1]). Furthermore, integral equations have been found for the real and complex sine Gordon equation[13]) and the Boussinesq equation[14]). The actual form of the integral equation can be inferred from a treatment by Rosales[15]).)

2.2. *The Bäcklund transformation*

We now proceed with the derivation of the BT of the NLS following the treatment of ref. 3. From the linear problem (2.3)–(2.6) one can derive the following Riccati equations:

$$\partial_x u = -\tfrac{1}{2}\mathrm{i}\alpha\phi + \mathrm{i}ku + \mathrm{i}\phi^* u^2, \tag{2.13}$$

$$\partial_t u = \tfrac{1}{2}\alpha\partial_x\phi + \mathrm{i}\frac{\alpha}{2}k\phi + \mathrm{i}(\alpha\phi^*\phi - k^2)u + (-\mathrm{i}k\phi^* + \partial_x\phi^*)u^2, \tag{2.14}$$

for the function

$$u(x, t) = -\phi_k(x, t)/\psi_k(x, t), \tag{2.15}$$

which for $\alpha = 2$ have been given in ref. 3 in a slightly different notation.

From (2.13) for $\partial_x u$ and the complex conjugate for $\partial_x u^*$ one can solve

$$-\mathrm{i}\phi = \frac{\tfrac{1}{2}\alpha(\partial_x u - \mathrm{i}ku) - (\partial_x u^* + \mathrm{i}k^* u^*)u^2}{\tfrac{1}{4}\alpha^2 - u^2 u^{*2}}. \tag{2.16}$$

Inserting (2.16) in (2.14) one obtains after a lengthy calculation

$$\begin{aligned}\partial_t u = {} & \tfrac{1}{2}\alpha i\partial_x\left(\frac{\tfrac{1}{2}\alpha\partial_x u - u^2\partial_x u^*}{\tfrac{1}{4}\alpha^2 - u^2u^{*2}}\right) - \mathrm{i}u^2\partial_x\left(\frac{\tfrac{1}{2}\alpha\partial_x u^* - u^{*2}\partial_x u}{\tfrac{1}{4}\alpha^2 - u^2u^{*2}}\right) \\ & + \frac{\mathrm{i}\alpha u}{\tfrac{1}{4}\alpha^2 - u^2u^{*2}}(\tfrac{1}{2}\alpha\partial_x u - u^2\partial_x u^*)(\tfrac{1}{2}\alpha\partial_x u^* - u^{*2}\partial_x u) \\ & + \frac{\alpha(k + k^*)u^2\partial_x u^*}{(\tfrac{1}{2}\alpha - uu^*)^2} + \frac{\tfrac{1}{2}\alpha^2\mathrm{i}(k + k^*)^2u^3u^{*2}}{(\tfrac{1}{4}\alpha^2 - u^2u^{*2})^2} + \frac{\alpha\mathrm{i}kk^*u^2u^*}{(\tfrac{1}{2}\alpha + uu^*)^2}.\end{aligned} \tag{2.17}$$

Eq. (2.17) is an integrable PDE in the sense that its solutions can be obtained directly from the linear integral equation (1.1) and in section 3 it will be shown that (2.17) is equivalent to the AHSC.

The PDE for u is invariant under the transformation $k \to k^*$. Therefore a new solution $\bar{\phi}$ of the NLS can be obtained from (2.16) with k replaced by k^*, i.e.

$$-\mathrm{i}\bar{\phi} = \frac{\tfrac{1}{2}\alpha(\partial_x u - \mathrm{i}k^* u) - (\partial_x u^* + \mathrm{i}ku^* u)^2}{\tfrac{1}{4}\alpha^2 - u^2u^{*2}}. \tag{2.18}$$

Applying the same transformation to (2.13) and (2.14) we get

$$\partial_x u = -\tfrac{1}{2}\mathrm{i}\alpha\bar{\phi} + \mathrm{i}k^* u + \mathrm{i}\bar{\phi}^* u^2, \tag{2.19}$$

$$\partial_t u = \tfrac{1}{2}\alpha\partial_x\tilde{\phi} + \frac{i}{2}\alpha k^*\tilde{\phi} + i(\alpha\tilde{\phi}^*\tilde{\phi} - k^{*2})u + (-ik^*\tilde{\phi} + \partial_x\tilde{\phi}^*)u^2, \tag{2.20}$$

given in ref. 3 for $\alpha = 2$.

Subtracting (2.19) from (2.13) we have

$$0 = -\tfrac{1}{2}i\alpha(\phi - \tilde{\phi}) + 4\sigma u + i(\phi^* - \tilde{\phi}^*)u^2, \quad \sigma = -\tfrac{1}{2}\,\mathrm{Im}\,k, \tag{2.21}$$

leading to

$$u = \frac{-4\sigma + \{16\sigma^2 - 2\alpha|\phi - \tilde{\phi}|^2\}^{1/2}}{2i(\phi^* - \tilde{\phi}^*)}. \tag{2.22}$$

(The other solution with the minus sign in front of the square root leads essentially to the same results and will be ignored.) From (2.13) and (2.19) we have the relation

$$2\partial_x u = -\tfrac{1}{2}i\alpha(\phi + \tilde{\phi}) - 4i\xi u + i(\phi^* + \tilde{\phi}^*)u^2, \quad \xi = -\tfrac{1}{2}\,\mathrm{Re}\,k. \tag{2.23}$$

Taking the derivative of (2.21) with respect to x and using (2.22) and (2.23) we get

$$\partial_x(\phi - \tilde{\phi}) = -2i\xi(\phi - \tilde{\phi}) - \tfrac{1}{2}(\phi + \tilde{\phi})\{16\sigma^2 - 2\alpha|\phi - \tilde{\phi}|^2\}^{1/2}. \tag{2.24}$$

From the fact that both ϕ and $\tilde{\phi}$ satisfy the NLS we have

$$\partial_t(\phi - \tilde{\phi}) = i\partial_x^2(\phi - \tilde{\phi}) + \alpha i\{|\phi|^2\phi - |\tilde{\phi}|^2\tilde{\phi}\}. \tag{2.25}$$

Differentiating (2.24) with respect to x, the right-hand side of (2.25) can be worked out to give

$$\begin{aligned}\partial_t(\phi - \tilde{\phi}) = {}& 2\xi\partial_x(\phi - \tilde{\phi}) - \tfrac{1}{2}(\partial_x\phi + \partial_x\tilde{\phi})\{16\sigma^2 - 2\alpha|\phi - \tilde{\phi}|^2\}^{1/2} \\ & + \frac{\alpha}{2}i(\phi - \tilde{\phi})(|\phi|^2 + |\tilde{\phi}|^2).\end{aligned} \tag{2.26}$$

Eqs. (2.24) and (2.26) together form the BT of the NLS, given in ref. 3 for $\alpha = 2$ and in ref. 16 for general α.

3. *Relation with the AHSC*

3.1. *The anisotropic Heisenberg spin chain*

In order to derive the relation between eq. (2.17) and the AHSC (1.4) it is convenient to define a new function $b(x, t)$ by the relation

$$\phi - \tilde{\phi} = \left(\frac{2A}{|\alpha|}\right)^{1/2} b(x, t), \tag{3.1}$$

where for the moment A is a positive constant. Solving $\phi - \bar{\phi}$ from (2.21) we immediately have

$$\left(\frac{2A}{|\alpha|}\right)^{1/2} b(x, t) = -\frac{4\sigma u \mathrm{i}}{\frac{1}{2}\alpha + |u|^2}, \tag{3.2}$$

so that eq. (2.17) for $u(x, t)$ can be equivalently expressed as a PDE in terms of $b(x, t)$ for $\sigma \neq 0$. From (3.1) and (2.24) we can express ϕ and $\bar{\phi}$ in terms of b, i.e.

$$\phi = |\alpha|^{-1/2}\left\{(\tfrac{1}{2}A)^{1/2} b - \frac{\partial_x b + 2\mathrm{i}\xi b}{(8\sigma^2 A^{-1} - 2\nu|b|^2)^{1/2}}\right\}, \tag{3.3}$$

$$\bar{\phi} = |\alpha|^{-1/2}\left\{-(\tfrac{1}{2}A)^{1/2} b - \frac{\partial_x b + 2\mathrm{i}\xi b}{(8\sigma^2 A^{-1} - 2\nu|b|^2)^{1/2}}\right\}, \quad \nu = \operatorname{sgn} \alpha. \tag{3.4}$$

Eq. (3.2) implies that for $\alpha > 0$, $(\nu = 1)$, $|b| \leq 2|\sigma A^{-1/2}|$, so that $8\sigma^2 A^{-1} - 2\nu|b|^2 > 0$.

Inserting (3.3) and (3.4) in (2.25) we obtain the PDE in terms of b, i.e.

$$\begin{aligned} &\mathrm{i}\partial_t b + \partial_x^2 b + \tfrac{1}{2}\nu A|b|^2 b + \nu\{8\sigma^2 A^{-1} - 2\nu|b|^2\}^{-1} \\ &\quad \times \{b^*(\partial_x b)^2 + 2b|\partial_x b|^2 + 4\mathrm{i}\xi b^2 \partial_x b^* + 4\xi^2|b|^2 b\} = 0. \end{aligned} \tag{3.5}$$

Eq. (3.5) is a completely integrable PDE in the sense that solutions can be obtained from the linear integral equation (1.1), cf. (2.1), via $u = -\phi_k/\psi_k$ and (3.2). Furthermore, eq. (3.3) can be considered as the Miura transformation mapping a solution of (3.5) on a solution of the NLS, similar to the well-known transformation[17]) between the MKdV and the KdV which can be derived from the BT of the KdV[3,18]).

As a further simplification we note that (3.5) for general $\xi \neq 0$ is equivalent to (3.5) for $\xi = 0$. In fact, introducing a new variable $a(x, t)$ by

$$b(x, t) = a(x + 4\xi t, t) \exp[-4\mathrm{i}\xi^2 t - 2\mathrm{i}\xi x], \tag{3.6}$$

we obtain

$$\mathrm{i}\dot{a} + a'' + \tfrac{1}{2}\nu A|a|^2 a + \frac{a^* a'^2 + 2a|a'|^2}{2s^2\nu - 2|a|^2} = 0, \quad s^2 = 4\sigma^2 A^{-1}, \tag{3.7}$$

in which primes and dots have been used to denote the differentiations with respect to $x + 4\xi t$ and t, respectively.

The solutions of (3.7) can be derived from a potential equation in terms of one real variable $q(x, t)$. In fact, substituting

$$a = \kappa\, \mathrm{e}^{\mathrm{i}\gamma} \quad (\kappa \geq 0, \gamma \text{ real}), \tag{3.8}$$

in (3.7) and taking the real and imaginary part, we obtain

$$-\partial_t(s^2 - \nu\kappa^2)^{1/2} + \partial_x\left(\frac{\nu\kappa^2\gamma'}{(s^2 - \nu\kappa^2)^{1/2}}\right) = 0, \tag{3.9}$$

and

$$\dot{\gamma} = \frac{\kappa''}{\kappa} - \gamma'^2 + \frac{3\kappa'^2 + \kappa^2\gamma'^2}{2\nu s^2 - 2\kappa^2} + \tfrac{1}{2}A\nu\kappa^2. \tag{3.10}$$

Eq. (3.9) can be formally solved introducing a new variable $q(x, t)$ by

$$\kappa = (\nu s^2 - \nu q'^2)^{1/2}, \tag{3.11}$$

so that (3.9) is the compatibility relation for q and

$$\gamma' = \frac{\dot{q}q'}{s^2 - q'^2}. \tag{3.12}$$

From (3.2) we have $s^2 - \nu\kappa^2 \geq 0$, so that $q'^2 \geq 0$; from (3.10) and $\kappa \geq 0$ it follows that $q'^2 \leq s^2$ for $\nu = 1$ ($\alpha > 0$), and $q'^2 \geq s^2$ for $\nu = -1$ ($\alpha < 0$).

Using (3.11) and (3.12) the right-hand side of (3.10) can be worked out to give

$$\dot{\gamma} = -\frac{q'q''}{s^2 - q'^2} + \frac{1}{2}\frac{(\dot{q}^2 + q''^2)}{s^2 - q'^2} - \frac{q'^2(\dot{q}^2 + q''^2)}{(s^2 - q'^2)^2} + \frac{A}{2}(s^2 - q'^2), \tag{3.13}$$

and the compatibility relation $\partial_t\gamma' = \partial_x\dot{\gamma}$ leads to, taking $q' \neq 0$,

$$\frac{\ddot{q} + q''''}{s^2 - q'^2} + \frac{4q'(q''q''' + \dot{q}\dot{q}')}{(s^2 - q'^2)^2} + \frac{q''(s^2 + 3q'^2)(\dot{q}^2 + q''^2)}{(s^2 - q'^2)^3} + Aq'' = 0. \tag{3.14}$$

Eq. (3.14) can be rewritten in the form

$$\partial_t\left(\frac{\dot{q}}{s^2 - q'^2}\right) + \partial_x\left(\frac{q'''}{s^2 - q'^2} + \frac{q'(\dot{q}^2 + q''^2)}{(s^2 - q'^2)^2} + Aq'\right) = 0, \tag{3.15}$$

which for $s = 1$ is the potential AHSC which is an equivalent description in terms of one real variable $q(x, t)$ for the AHSC and which has been given in refs. 6 and 7. Eq. (3.15) is integrable in the sense that solutions of (3.15) can be obtained from the linear integral equation (1.1) using (2.1), (2.15), (3.2), (3.6) and

$$q' = (s^2 - \nu|a|^2)^{1/2},$$

$$\dot{q} = \frac{\nu|a|^2 \operatorname{Im} \partial_x \ln a}{(s^2 - \nu|a|^2)^{1/2}}. \tag{3.16}$$

Physical solutions of the AHSC (1.4) have $q'^2 \leq 1$ and therefore $\nu = 1$. The relation between $a(x, t)$ and the polar angles $\theta(x, t)$ and $\chi(x, t)$ of the spin vector, cf. (2.10), can be inferred from (3.16) and from the relations

$$\theta = \arccos q', \tag{3.17}$$

$$\chi' = \frac{\dot{q}}{1 - q'^2}, \quad \dot{\chi} = \frac{-q'''}{1 - q'^2} - \frac{q'(\dot{q}^2 + q''^2)}{(1 - q'^2)^2} - Aq' - B, \tag{3.18}$$

which have been given in refs. 6 and 7. The result can be expressed as

$$\theta = \arcsin|a|, \tag{3.19}$$

$$\chi' = \frac{\operatorname{Im} \partial_x \ln a}{(1-|a|^2)^{1/2}}, \tag{3.20}$$

$$\dot{\chi} = \frac{\operatorname{Im} \partial_t \ln a}{(1-|a|^2)^{1/2}} - \frac{1}{2}\frac{|\partial_x a|^2}{(1-|a|^2)^{3/2}} - \frac{1}{2}A\frac{(2-|a|^2)}{(1-|a|^2)^{1/2}} - B.$$

From (3.19) and (3.20) it is clear that solutions of the AHSC in the case of Ising anisotropy ($A > 0$) can be obtained from the linear integral equation (1.1), cf. (2.1), (2.15), (3.2) and (3.6). (The complete integrability of (1.4) was first proved by Borovik[19]), and Sklyanin[20]) showed the integrability of the Landau–Lifshitz equation, i.e. (1.14) for $B = 0$ in the more general case of orthorhombic anisotropy.)

3.2. *The Miura transformation*

We now consider the Miura transformation between the AHSC and the NLS. From (3.2) and (3.6) we have

$$\phi = |\alpha|^{-1/2}\left\{(\tfrac{1}{2}A)^{1/2}a - \frac{\partial_x a}{(2s^2 - 2\nu a^2)^{1/2}}\right\}, \quad s^2 = 4\sigma^2 A^{-1}. \tag{3.21}$$

From (3.21) it is also straightforward to derive the Miura transformation between the potential AHSC, as given by (3.14), (3.15), and the potential NLS which is given by[6–8])

$$\partial_t\left(\frac{-\dot{y}}{2y'}\right) = \partial_x\left(\frac{y'''}{2y'} - \frac{y''^2 + \dot{y}^2}{4y'^2} + 2\alpha y'\right), \tag{3.22}$$

in which dots and primes have been used to denote the differentiations with respect to t and x. Eq. (3.22) is equivalent to the NLS (1.2). The transformation between the two is given by

$$y' = \tfrac{1}{2}\phi^*\phi, \qquad \dot{y} = \operatorname{Im} \phi\phi'^*, \tag{3.23}$$

and its inverse

$$\phi(x,t) = (2y')^{1/2}\exp\left[\mathrm{i}\int_\Gamma \left\{\mathrm{d}l_x\left(\frac{-\dot{y}}{2y'}\right) + \mathrm{d}l_t\left(\frac{y'''}{2y'} - \frac{y''^2 + \dot{y}^2}{4y'^2} + 2\alpha y'\right)\right\}\right], \tag{3.24}$$

in which Γ is an arbitrary curve connecting $(0,0)$ and (x,t) and $\mathrm{d}l = (\mathrm{d}l_x, \mathrm{d}l_t)$ is an infinitesimal two-dimensional vector tangent to Γ. Solutions of (3.22) can be obtained directly from (2.2), (2.7) and $y = -\frac{1}{2}\mathrm{i}\psi$.

Using (3.10)–(3.12) to evaluate ϕ in (3.21) in terms of q and inserting the result in (3.23), we obtain

$$2\alpha y' = \frac{1}{2}\frac{\dot{q}^2 + q''^2}{s^2 - q'^2} + \tfrac{1}{2}A(s^2 - q'^2) + A^{1/2}q'', \tag{3.25}$$

$$2\alpha\dot{y} = \frac{\dot{q}'q'' - \dot{q}q''}{s^2 - q'^2} - \frac{\dot{q}q'(\dot{q}^2 + q''^2)}{(s^2 - q'^2)^2} - A\dot{q}q' - A^{1/2}\dot{q}'. \tag{3.26}$$

Eqs. (3.25) and (3.26) provide the Miura transformation which maps the solutions of the potential AHSC (3.15) on solutions of the potential NLS (3.22). Eqs. (3.25) and (3.26) were also given in ref. 5 and in the special case $A = 0$ also in ref. 7. For the anisotropic Heisenberg spin chain (1.4) we have, taking $\nu = 1$ ($\alpha > 0$) and $s = 1$, from (3.17), (3.18) and (2.10)

$$2\alpha y' = \tfrac{1}{2}S' \cdot S' + \tfrac{1}{2}A(1 - S^{z^2}) + A^{1/2}S^{z'}, \tag{3.27}$$

$$2\alpha\dot{y} = S' \cdot \dot{S} + B(S \times S') \cdot e^z + A^{1/2}\dot{S}^z. \tag{3.28}$$

In the special case of the IHSC with $A = B = 0$, the expressions for $2\alpha y'$ and $-2\alpha\dot{y}$ can be interpreted as the energy density and the current density[21] and eqs. (3.27) and (3.28) are equivalent to the result derived by Lakshmanan[22]. The explicit relation which maps the spin components of the AHSC on a solution $\phi(x, t)$ of the NLS can be inferred from (3.27), (3.28) and (3.24).

The IHSC may be obtained as a special case of the AHSC, if we take $\nu = 1$, $4\sigma^2A^{-1} = 1$ and consider the limit $A \to 0$, $\sigma \to 0$. This limiting case can also be described by (2.17) with $k + k^* = 0$, or by the substitution $b(x, t) = -|2\alpha|^{1/2}u\mathrm{i}(\tfrac{1}{2}\alpha + |u|^2)^{-1}$, cf. (3.2), and eq. (3.7) with $A = 0$. Therefore, one can also obtain solutions of the IHSC from the integral equation (1.1) in the limit $k + k^* \to 0$ without performing an integration as in (2.7).

3.3. *The modified nonlinear Schrödinger equation*

Another special case of interest, especially in connection with the investigation of similarity solutions, occurs for $\nu = -1$, $q'^2 \geq s^2$ in the limit $s^2 = 4\sigma^2A^{-1} \to 0$.

Eq. (3.14) is identical to

$$\partial_t\left(\frac{\dot{q}}{s - q'}\right) + \partial_x\left(\frac{q'''}{s - q'} + \frac{1}{2}\frac{\dot{q}^2 + q''^2}{(s - q')^2} - \frac{\dot{q}^2 + q''^2}{s^2 - q'^2} + \tfrac{1}{2}Aq'^2 + Asq'\right) = 0, \tag{3.29}$$

$(q' + s) \neq 0$, and, if we take the limit $s \to 0$, together with $q(x, t) \to f(x, t)$, we obtain

$$\partial_t\left(\frac{\dot{f}}{f'}\right) + \partial_x\left(\frac{f'''}{f'} - \frac{3}{2}\frac{\dot{f}^2 + f''^2}{f'^2} - \tfrac{1}{2}Af'^2\right) = 0. \tag{3.30}$$

Eq. (3.30) will be called the modified (potential) nonlinear Schrödinger equation (MNLS). Information about solutions of the MNLS may be obtained also from the linear integral equations (1.1) and (2.2), but further details will be presented in ref. 13.

The Miura transformation which maps the solutions of the MNLS on solutions of the potential NLS is given by

$$2\alpha y' = -\frac{1}{2}\frac{\dot{f}^2 + f''^2}{f'^2} - \frac{1}{2}Af'^2 + A^{1/2}f'', \tag{3.31}$$

$$2\alpha \dot{y} = -\frac{(\dot{f}'f'' - \dot{f}f''')}{f'^2} - \frac{\dot{f}(\dot{f}^2 + f''^2)}{f'^3} - A\dot{f}f' + A^{1/2}\dot{f}'. \tag{3.32}$$

Finally, the constant A in (3.30) may be omitted without loss of generality. In fact, inserting $f \to g = e^{\alpha f}$ in (3.30) with $A = 0$, we obtain

$$\partial_t\left(\frac{\dot{f}}{f'}\right) + \partial_x\left(\frac{f'''}{f'} - \frac{3}{2}\frac{\dot{f}^2 + f''^2}{f'^2} - \frac{1}{2}\alpha^2 f'^2\right) = 0. \tag{3.33}$$

This implies that with every solution q of (3.30) with $A = 0$, there corresponds a solution $f = A^{-1/2}\ln q$ of (3.30) with $A \neq 0$.

4. Similarity solutions

4.1. *Infinitesimal transformations*

In this section we investigate the similarity solutions of the potential NLS (1.10), of the potential AHSC (3.10), with the potential IHSC for $A = 0$ as a special case, and of the MNLS (3.27). In the derivation use will be made of the property that (1.10), (3.10) and (3.27) can be derived from a Lagrangian density $\mathscr{L}(\dot{\chi}, \chi', \chi'')$ using the Lagrange equations

$$-\partial_t \frac{\partial \mathscr{L}}{\partial \dot{\chi}} - \partial_x \frac{\partial \mathscr{L}}{\partial \chi'} + \partial_x^2 \frac{\partial \mathscr{L}}{\partial \chi''} = 0. \tag{4.1}$$

The Lagrangian densities in the three cases under consideration are given by

$$\mathscr{L}(\dot{y}, y', y'') = \frac{\dot{y}^2 - y''^2}{4y'} + \alpha y'^2, \tag{4.2}$$

$$\mathscr{L}(\dot{q}, q', q'') = \frac{1}{2}\left(\frac{\dot{q}^2 - q''^2}{s^2 - q'^2}\right) + \frac{1}{2}Aq'^2, \tag{4.3}$$

$$\mathscr{L}(\dot{f}, f', f'') = -\frac{1}{2}\left(\frac{\dot{f}^2 - f''^2}{f'^2}\right) + \frac{1}{2}Af'^2, \tag{4.4}$$

and, using (4.1) with $\chi = y, q, f$, it is straightforward to derive (1.10), (3.10) and (3.27), respectively.

We now consider an infinitesimal transformation of the form

$$\begin{aligned} \bar\chi &= \chi + \epsilon(c_0\chi + c_1), \\ \bar x &= x + \epsilon(c_2 x + c_3 t + c_4), \\ \bar t &= t + \epsilon(c_5 x + c_6 t + c_7), \end{aligned} \tag{4.5}$$

in which ϵ is infinitesimal and the c_α are constants independent of x and t, and we require that

$$\bar{\mathcal{L}} \equiv \mathcal{L}[\partial_{\bar t}\bar\chi, \partial_{\bar x}\bar\chi, \partial^2_{\bar x}\chi] = \lambda\mathcal{L} + D, \tag{4.6}$$

in which λ is constant and D a total derivative. Then $\bar\chi(\bar x, \bar t)$, as a function of $\bar x$ and $\bar t$, satisfies the same PDE as $\chi(x, t)$. Similarity solutions which depend essentially on one variable η, where η is a well-defined function of x and t, can then be obtained requiring that the function χ is invariant under the transformation (4.5), i.e. $\chi \equiv \bar\chi$.

For infinitesimal transformations with $\epsilon \to 0$ we can introduce $\mathrm{d}\chi = \bar\chi(\bar x, \bar t) - \chi(x, t) = \chi(\bar x, \bar t) - \chi(x, t)$, $\mathrm{d}x = \bar x - x$, $\mathrm{d}t = \bar t - t$ and we have the obvious relations

$$\frac{\mathrm{d}\chi}{c_0\chi + c_1} = \frac{\mathrm{d}x}{c_2 x + c_3 t + c_4} = \frac{\mathrm{d}t}{c_5 x + c_6 t + c_7}. \tag{4.7}$$

For the three Lagrangians (4.2)–(4.4) we shall investigate first under which conditions for c_α, ($\alpha = 0, \ldots, 7$), eq. (4.6) is satisfied, and next eqs. (4.7) will be solved to obtain the appropriate expression for the similarity solution χ as a function of essentially one variable η. It should be noted that this procedure of finding similarity solutions is not a systematic one. Firstly one may consider more general transformations than the one in (4.5) and secondly one may have similarity solutions which by a specific cancellation of terms do not arise from a symmetry property of the Lagrangian density. For the three PDE's under consideration it is not clear that interesting similarity solutions will be obtained by a more general approach, apart from a specific similarity solution for the MNLS which will be given at the end of this section.

Using the transformation (4.5) we have the explicit expressions up to order ϵ

$$\begin{aligned} \partial_x\chi &= [1 + \epsilon(c_2 - c_0)]\partial_{\bar x}\bar\chi + \epsilon c_5\partial_{\bar t}\bar\chi, \\ \partial_t\chi &= [1 + \epsilon(c_6 - c_0)]\partial_{\bar t}\bar\chi + \epsilon c_3\partial_{\bar t}\bar\chi, \\ \partial^2_x\chi &= [1 + \epsilon(2c_2 - c_0)]\partial^2_{\bar x}\bar\chi + 2\epsilon c_5\partial_{\bar x}\partial_{\bar t}\bar\chi, \end{aligned} \tag{4.8}$$

and inserting (4.8) in (4.2)–(4.4) it is easy to verify under which conditions (4.6) is satisfied.

As a result we have

$$c_5 = 0, \quad c_1, c_4 \text{ and } c_7 \text{ arbitrary}, \tag{4.9}$$

and the conditions for c_0, c_2, c_3 and c_6 depend on the specific PDE. We now give these conditions for the different PDE's, as well as the corresponding ODE's.

4.2. *The potential nonlinear Schrödinger equation*

Potential NLS:

$$\partial_t\left(\frac{-\dot{y}}{2y'}\right) = \partial_x\left(\frac{y'''}{2y'} - \frac{y''^2 + \dot{y}^2}{4y'^2} + 2\alpha y'\right). \tag{3.22}$$

Inserting (4.8) in (4.2) and taking into account (4.6) we obtain the following conditions:

$$2c_0 = -2c_2 = -c_6, \quad c_3 \text{ arbitrary}, \tag{4.10}$$

and from (4.9), (4.10) and (4.7) we have

$$\frac{\mathrm{d}y}{c_0 y + c_1} = \frac{\mathrm{d}x}{-c_0 x + c_3 t + c_4} = \frac{\mathrm{d}t}{-2c_0 t + c_7}. \tag{4.11}$$

We consider the cases $c_0 \neq 0$ and $c_0 = 0$ separately.

4.2.1 *The case $c_0 \neq 0$*

If $c_0 \neq 0$, we get from (4.11)

$$\ln\left[x + \frac{c_3}{c_0} t - \frac{c_4}{c_0} - \frac{c_3 c_7}{c_0^2}\right] = \tfrac{1}{2} \ln\left[t - \frac{c_7}{2c_0}\right] + \text{constant}, \tag{4.12}$$

$$\tfrac{1}{2} \ln\left[t - \frac{c_7}{2c_0}\right] = -\ln\left[y + \frac{c_1}{c_0}\right] + \text{constant}. \tag{4.13}$$

The integration constant in (4.12) can be identified with $\ln \eta$, where η is the new similarity variable; the constant in (4.13) is then a function of η. The constants c_1, c_4 and c_7 can be taken to be zero by an appropriate choice of origin. From (4.12), (4.13) we obtain

$$y(x, t) = \frac{Y(\eta)}{\sqrt{(t)}}, \quad \text{with } \eta = \frac{x - at}{\sqrt{(t)}}, \quad a = \frac{-c_3}{c_0}. \tag{4.14}$$

Inserting (4.14) in (3.22) we obtain

$$\tfrac{1}{8}\eta\left(\frac{Y + \eta Y'}{Y'}\right) + \frac{Y'''}{2Y'} - \frac{Y''^2}{4Y'^2} + 2\alpha Y' - \tfrac{1}{16}\frac{(Y + \eta Y')^2}{Y'^2} = \gamma, \tag{4.15}$$

where γ is an integration constant, and the primes denote differentiations with respect to the similarity variable η.

Equation (4.15) can be rewritten as

$$\partial_\eta\left[\frac{Y''^2+\frac{1}{4}(Y-\eta Y')^2}{Y'}\right]+8\alpha Y'Y''-4\gamma Y''=0, \tag{4.16}$$

and after a second integration we obtain, see also ref. 16,

$$Y''^2+4\alpha Y'^3+\tfrac{1}{4}(\eta Y'-Y)^2-4\gamma Y'^2-\gamma' Y'=0, \tag{4.17}$$

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in which γ' is another integration constant.

Introducing a new variable

$$Z=Y+\Gamma\eta, \tag{4.18}$$

we obtain

$$Z''^2+\tfrac{1}{4}(\eta Z'-Z)^2-4(\lambda Z'-\beta)(Z'-\Gamma)(Z'-\Delta)=0, \tag{4.19}$$

with

$$\begin{aligned}
&\lambda=-\alpha,\\
&\lambda\Delta+\beta=2\lambda\Gamma-\gamma,\\
&\Delta\beta=\lambda\Gamma^2-\gamma\Gamma+\tfrac{1}{4}\gamma'.
\end{aligned} \tag{4.20}$$

Eq.(4.19) is an ODE without movable critical points as has been shown by Chazy[23]). It will be referred to as Chazy IV in view of the relation with Painlevé IV[24]) which will be shown in section 5.

4.2.2. *The case* $c_0=0$

If $c_0=0$ we obtain from (4.11)

$$x-\frac{c_4}{c_7}t-\frac{1}{2}\frac{c_3}{c_7}t^2=\text{constant}, \tag{4.21}$$

$$\frac{y}{c_1}-\frac{t}{c_7}=\text{constant}. \tag{4.22}$$

The right-hand side of (4.21) is identified with the new similarity variable η, the right-hand side of (4.22) is then a function of η. We have

$$y(x,t)=\delta t+Y(\eta),\ \text{with}\ \eta=x-vt-\tfrac{1}{2}at^2,\quad \delta=\frac{c_1}{c_7},\quad v=\frac{c_4}{c_7},\quad a=\frac{c_3}{c_7}. \tag{4.23}$$

Inserting (4.23) in (3.22) we obtain

$$\frac{Y'''}{2Y'}-\frac{Y''^2+(\delta-vY')^2}{4Y'^2}+2\alpha Y'-\frac{v\delta}{2Y'}-\tfrac{1}{2}a\eta=\gamma, \tag{4.24}$$

in which γ is an integration constant. Eq. (4.24) can be rewritten as

$$\frac{1}{Y''}\partial_\eta\left[\frac{Y''^2+(\delta-vY')^2}{4Y'}\right]-\tfrac{1}{2}a\eta+2\alpha Y'-\tfrac{1}{2}v^2=\gamma, \tag{4.25}$$

and after a second integration we obtain

$$Y''^2+(\delta-vY')^2+2a(Y-\eta Y')Y'+4\alpha Y'^3-(2v^2+4\gamma)Y'^2-4\gamma'Y'=0, \tag{4.26}$$

in which γ' is another integration constant.

For $a=0$, eq. (4.26) is solvable in terms of elliptic functions. For $a\neq 0$ we can apply a shift

$$\eta\to\eta-\tfrac{1}{2}(v^2+4\gamma)a^{-1}, \tag{4.27}$$

and introduce a new function Z by

$$Y=Z+(v\delta+2\gamma')a^{-1}, \tag{4.28}$$

leading to

$$Z''^2+4\alpha Z'^3+2a(Z-\eta Z')Z'+\delta^2=0. \tag{4.29}$$

Eq. (4.29) is another ODE without movable critical points introduced by Chazy23). It will be referred to as Chazy II, since it can be related to Painlevé II, cf. ref. 25.

4.2.3. *Remark*

From the results given above it is straightforward to derive the similarity solutions of the NLS in terms of the function ϕ, as have been given in ref. 26. In fact, from (4.14), (3.23) and (3.24) we can obtain the similarity solution

$$\phi(x,t)=t^{i\gamma-1/2}\,\Phi(\eta)\,e^{i(ax/4-a^2t/16)},\quad \eta=\frac{x-at}{\sqrt{t}}, \tag{4.30}$$

in which the integration constant γ is given by (4.15). Inserting (4.30) in the NLS (1.2) one obtains the ODE for Φ:

$$(\gamma-\tfrac{1}{2}i)\Phi-\tfrac{1}{2}i\eta\Phi'+\Phi''+\alpha|\Phi|^2\Phi=0. \tag{4.31}$$

In the same way from (4.23), (3.23) and (3.24) one can obtain the similarity solution

$$\phi(x,t)=\Phi(\eta)\exp[i(\tfrac{1}{2}axt+\tfrac{1}{2}vx+\gamma t-\tfrac{1}{2}vat^2-\tfrac{1}{6}a^2t^3)],\quad \eta=x-vt-\tfrac{1}{2}at^2, \tag{4.32}$$

in which γ is given by (4.24). Inserting (4.32) in (1.2) one derives the ODE for

Φ, which is given by

$$\Phi'' - \tfrac{1}{2}a\eta\Phi - (\gamma + \tfrac{1}{4}v^2)\Phi + \alpha|\Phi|^2\Phi = 0. \tag{4.33}$$

4.3. *The potential anisotropic Heisenberg spin chain*

Potential AHSC:

$$\partial_t\left(\frac{\dot{q}}{s^2 - q'^2}\right) + \partial_x\left(\frac{q'''}{s^2 - q'^2} + \frac{q'(\dot{q}^2 + q''^2)}{(s^2 - q'^2)^2} + Aq'\right) = 0. \tag{3.15}$$

From (4.6) we have the condition

$$2c_0 = 2c_2 = c_6, \qquad c_3 = 0, \tag{4.34}$$

and together with (4.7) this gives us

$$\frac{dq}{c_0 q + c_1} = \frac{dx}{c_0 x + c_4} = \frac{dt}{2c_0 t + c_7}, \tag{4.35}$$

with the additional requirement

$$c_0 = 0, \quad \text{if } A \neq 0. \tag{4.36}$$

4.3.1. *The case* $c_0 \neq 0$

In this case, which only applies to the potential IHSC with $A = 0$, we obtain from (4.35)

$$\ln\left[x + \frac{c_4}{c_0}\right] = \tfrac{1}{2}\ln\left[t + \frac{c_7}{2c_0}\right] + \text{constant}, \tag{4.37}$$

$$\tfrac{1}{2}\ln\left[t + \frac{c_7}{2c_0}\right] = \ln\left[q + \frac{c_1}{c_0}\right] + \text{constant}. \tag{4.38}$$

Again, choosing $c_1 = c_4 = c_7 = 0$ without loss of generality, we obtain

$$q(x,t) = \sqrt{t}\,Q(\eta), \quad \eta = \frac{x}{\sqrt{t}}. \tag{4.39}$$

Inserting (4.39) in (3.15), with $A = 0$, we find

$$-\frac{\frac{1}{4}\eta(Q - \eta Q')}{s^2 - Q'^2} + \frac{Q'''}{s^2 - Q'^2} + \frac{[\frac{1}{4}(Q - \eta Q')^2 + Q''^2]Q'}{(s^2 - Q'^2)^2} = \hat{\gamma}, \tag{4.40}$$

in which $\hat{\gamma}$ is an integration constant. Eq. (4.40) can be rewritten as

$$\frac{1}{2Q''}\,\partial_\eta\left[\frac{Q''^2 + \frac{1}{4}(Q - \eta Q')^2}{s^2 - Q'^2}\right] = \hat{\gamma}, \tag{4.41}$$

leading to

$$Q''^2 + \tfrac{1}{4}(Q - \eta Q')^2 = (2\hat{\gamma}Q' + \hat{\gamma}')(s^2 - Q'^2), \tag{4.42}$$

where $\hat{\gamma}'$ is a second integration constant.

Eq. (4.42) is a special case of the Chazy equation (4.19), cf. the substitutions $Z \to Q$, $\lambda \to -\frac{1}{2}\hat{\gamma}$, $B \to -\frac{1}{4}\hat{\gamma}'$, $\Gamma \to s$, $\Delta \to -s$. Eq. (4.42) may also be derived from (4.17) and the Miura transformation. From (3.25) and (3.26) and the similarity solutions (4.39) and (4.14) for $a = 0$, it can be shown that

$$2\alpha Y = \hat{\gamma}Q + \tfrac{1}{2}\hat{\gamma}'\eta, \tag{4.43}$$

and inserting (4.43) in (4.17) one obtains (4.42) with $\hat{\gamma}' = 2\gamma$, $s^2\hat{\gamma}^2 = \gamma'\alpha + \frac{1}{4}\gamma^2$.

The similarity solutions for the polar angles θ and χ in the IHSC, with $s = 1$, can be inferred from (4.39), (3.17) and (3.18). We obtain

$$\theta = \Theta\left(\frac{x}{\sqrt{t}}\right), \quad \chi = X\left(\frac{x}{\sqrt{t}}\right) - \hat{\gamma} \ln t, \tag{4.44}$$

in which $\hat{\gamma}$ is given by (4.40) and

$$\cos \Theta = Q', \quad X' = \tfrac{1}{2}\frac{Q - \eta Q'}{s^2 - Q'^2}. \tag{4.45}$$

4.3.2. *The case* $c_0 = 0$

In this case, which applies to the AHSC with arbitrary A, we have from (4.7)

$$x - vt = \text{constant}, \quad v = \frac{c_4}{c_7}, \tag{4.46}$$

$$q = \delta t + \text{constant}, \quad \delta = \frac{c_1}{c_7}, \tag{4.47}$$

and from this we obtain

$$q(x, t) = \delta t + Q(\eta), \quad \eta = x - vt, \tag{4.48}$$

cf. (4.23) with $a = 0$. Inserting (4.48) in (3.15) we obtain

$$\frac{-v(\delta - vQ')}{s^2 - Q'^2} + \frac{Q'''}{s^2 - Q'^2} + \frac{[(\delta - vQ')^2 + Q''^2]Q'}{(s^2 - Q'^2)^2} + AQ' = \hat{\gamma}, \tag{4.49}$$

where $\hat{\gamma}$ is an integration constant. Eq. (4.49) can be rewritten as

$$\frac{1}{2Q''}\,\partial_\eta\left[\frac{(\delta - vQ')^2 + Q''^2}{s^2 - Q'^2}\right] + AQ' = \hat{\gamma}, \tag{4.50}$$

and after a second integration we find

$$Q''^2 + (\delta - vQ')^2 = (\hat{\gamma}' + 2\hat{\gamma}Q' + AQ'^2)(s^2 - Q'^2), \tag{4.51}$$

which may be solved in terms of elliptic functions. For special values of the integration constants, eq. (4.51) has one-soliton solutions. These solutions are given in appendix A, cf. also refs. 32 and 33 for special cases.

The similarity solutions for the polar angles θ and χ, in the AHSC with $s = 1$, can be derived directly from (4.48), (3.17) and (3.18). The result is

$$\theta(x, t) = \Theta(x - vt), \quad \chi(x, t) = X(x - vt) - \hat{\gamma} t, \tag{4.52}$$

in which $\hat{\gamma}$ is given by (4.49) and

$$\cos \Theta = Q', \quad X' = \frac{\delta - vQ'}{s^2 - Q'^2}. \tag{4.53}$$

4.4. *The modified nonlinear Schrödinger equation*

4.4.1. *Similarity solutions and Painlevé IV*

MNLS:

$$(A \neq 0) \qquad \partial_t\left(\frac{\dot{f}}{f'}\right) + \partial_x\left(\frac{f'''}{f'} - \frac{3}{2}\frac{\dot{f}^2 + f''^2}{f'^2} - \tfrac{1}{2}Af'^2\right) = 0. \tag{3.30}$$

From (4.6) we have the conditions

$$c_6 = 2c_2, \quad c_3 = 0, \quad c_0 = 0. \tag{4.54}$$

Taking $c_2 \neq 0$ we obtain from (4.11)

$$\tfrac{1}{2}\ln\left[t + \frac{c_7}{2c_2}\right] = \ln\left[x + \frac{c_4}{c_2}\right] + \text{constant}, \tag{4.55}$$

$$f = \frac{c_1}{2c_2}\ln\left[t + \frac{c_7}{2c_2}\right] + \text{constant}, \tag{4.56}$$

and choosing $c_4 = c_7 = 0$ we obtain

$$f = \delta \ln t + F(\eta), \quad \eta = \frac{x}{\sqrt{t}}, \quad \delta = \frac{c_1}{2c_2}. \tag{4.57}$$

Eq. (4.11) for $c_2 = 0$ gives (4.48) with q and Q replaced by f and F and leads to (4.51) with $s = 0$.

Inserting (4.57) in (3.30) we obtain

$$-\tfrac{1}{2}\eta\left(\frac{\delta}{F'} - \tfrac{1}{2}\eta\right) + \frac{F'''}{F'} - \frac{3}{2}\frac{F''^2}{F'^2} - \frac{3}{2}\left(\frac{\delta}{F'} - \tfrac{1}{2}\eta\right)^2 - \tfrac{1}{2}AF'^2 = c, \tag{4.58}$$

where c is an integration constant. Substituting $W = 1/F'$ we get

$$W''W = \tfrac{1}{2}W'^2 - \tfrac{3}{2}\delta^2 W^4 + \eta\delta W^3 - (\tfrac{1}{8}\eta^2 + c)W' - \tfrac{1}{2}E^2, \tag{4.59}$$

with

$$E = A^{1/2}. \tag{4.60}$$

Eq. (4.59) is Painlevé IV apart from a trivial transformation.

The Miura transformation (3.31), (3.32), applied to the similarity solutions (4.14) for $a = 0$ and (4.57) leads also to a direct relation between Chazy IV and Painlevé IV, as will be shown in section 5.

4.4.2. *Special similarity solution*

Finally, the MNLS has a similarity solution

$$f = F(\eta), \quad \eta = x - vt - \tfrac{1}{2}at^2, \tag{4.61}$$

which does not follow from (4.6). In fact, substituting (4.61) in (3.30) we obtain

$$-a + \partial_\eta\left(\frac{F'''}{F'} - \frac{3}{2}\frac{F''^2}{F'^2} - \tfrac{1}{2}AF'^2\right) = 0, \tag{4.62}$$

which depends only on η. (In (4.62) F', F'', and F''' denote the derivatives with respect to η.)

Substituting $W = 1/F'$ in (4.62) we obtain, c being a constant,

$$aW + 2(a\eta + c)W' + W''' = 0, \tag{4.63}$$

which can be solved in terms of hypergeometric functions. Using example 10 on p. 298 of ref. 27 the solution of (4.63) can be expressed as

$$W = c_1u^2 + c_2uv + c_3v^2, \tag{4.64}$$

where u and v are independent solutions of Airy's equation

$$U'' + \tfrac{1}{2}(a\eta + c)U = 0, \tag{4.65}$$

with $U = (u, v)$, cf. ref. 28, ch. 10.

In this section we have derived the ordinary differential equations satisfied by the similarity solutions of the NLS, IHSC, AHSC and MNLS. Some interesting results on similarity solutions can also be found in ref. 32.

5. The Chazy equation and Painlevé IV

5.1. *Transformations between Chazy IV and Painlevé IV*

In this section we derive the transformation between the Chazy equation (4.19) which is satisfied by the similarity solutions (4.14) of the NLS and (4.39) of the IHSC, cf. (4.42), and Painlevé IV (4.59) which is satisfied by the similarity solutions (4.57) of the MNLS. As an application of this transformation one may derive Bäcklund transformations for Painlevé IV[30-32]), as well as for Chazy IV, and one can obtain explicit solutions of Painlevé IV for special values of the integration constants.

The Chazy equation (4.19) may be expressed in the form

$$R^2 + S^2 = 16(Z' - \Gamma)(Z' - \Delta), \tag{5.1}$$

where

$$R = 2Z'(\lambda Z' - \beta)^{-1/2}, \tag{5.2}$$

$$S = (Z - \eta Z')(\lambda Z' - \beta)^{-1/2}, \tag{5.3}$$

Introducing

$$\begin{aligned} \sin\omega &= \{2(2Z' - \Gamma - \Delta)S - 2\mathrm{i}(\Gamma - \Delta)R\}\{16(Z' - \Gamma)(Z' - \Delta)\}^{-1}, \\ \cos\omega &= \{-2(2Z' - \Gamma - \Delta)R - 2\mathrm{i}(\Gamma - \Delta)S\}\{16(Z' - \Gamma)(Z' - \Delta)\}^{-1}, \end{aligned} \tag{5.4}$$

we have the relations

$$R\sin\omega + S\cos\omega = -2\mathrm{i}(\Gamma - \Delta), \tag{5.5}$$

$$-R\cos\omega + S\sin\omega = 4Z' - 2\Gamma - 2\Delta, \tag{5.6}$$

from which one can solve

$$S = -2\mathrm{i}(\Gamma - \Delta)\cos\omega + (4Z' - 2\Gamma - 2\Delta)\sin\omega. \tag{5.7}$$

An explicit relation for ω' can be derived evaluating S' in two different ways. In fact, from (5.3) one has

$$S' = -\eta Z''(\lambda Z' - \beta)^{-1/2} - \tfrac{1}{2}\lambda(Z - \eta Z')(\lambda Z' - \beta)^{-3/2}Z'', \tag{5.8}$$

and (5.7) together with (5.5), (5.6) and (5.2) leads to

$$S' = 4Z''\sin\omega - 2Z''\omega'(\lambda Z' - \beta)^{-1/2}. \tag{5.9}$$

Comparing (5.8) and (5.9) we have

$$\omega' = \tfrac{1}{4}\lambda S(\lambda Z' - \beta)^{-1/2} + \delta W, \tag{5.10}$$

in which the quantity δW is defined by

$$\delta W - \tfrac{1}{2}\eta = 2(\lambda Z' - \beta)^{1/2}\sin\omega. \tag{5.11}$$

We now proceed to show that δW satisfies (4.59). Differentiating (5.11) we have

$$\delta W' = \tfrac{1}{2} + \lambda Z''(\lambda Z' - \beta)^{-1/2}\sin\omega + 2(\lambda Z' - \beta)^{1/2}\omega'\cos\omega, \tag{5.12}$$

and the right-hand side can be evaluated using (5.10), (5.2) and (5.5) to give

$$\delta W' = \tfrac{1}{2} - \mathrm{i}\lambda(\Gamma - \Delta) + 2\delta W(\lambda Z' - \beta)^{1/2}\cos\omega. \tag{5.13}$$

Differentiating (5.12) and using (5.10), (5.2) and (5.6), it can be shown that

$$\begin{aligned}\delta W'' = {} & -(2Z' - \Gamma - \Delta)\lambda\delta W - 2\delta^2 W^2(\lambda Z' - \beta)^{1/2}\sin\omega \\ & + 2\delta W'(\lambda Z' - \beta)^{1/2}\cos\omega.\end{aligned} \tag{5.14}$$

Using (5.11) to eliminate $2(\lambda Z' - \beta)^{1/2}\sin\omega$, (5.13) for $2(\lambda Z' - \beta)^{1/2}\cos\omega$ and the relation

$$4(\lambda Z' - \beta) = (\delta W - \tfrac{1}{2}\eta)^2 + (\delta W' - \tfrac{1}{2} + \mathrm{i}(\Gamma - \Delta)\lambda)^2(\delta W)^{-2}, \tag{5.15}$$

for $2Z' - \Gamma - \Delta$, eq. (5.14) can be expressed as

$$\begin{aligned}\delta W'' = {} & \{(\Gamma + \Delta)\lambda - 2\beta\}\delta W - \tfrac{1}{2}\delta W(\delta W - \tfrac{1}{2}\eta)^2 - \delta^2 W^2(\delta W - \tfrac{1}{2}\eta) \\ & - \tfrac{1}{2}\{\delta W' - \tfrac{1}{2} + \mathrm{i}(\Gamma - \Delta)\lambda\}^2(\delta W)^{-2} \\ & + \delta W'(\delta W)^{-1}\{\delta W' - \tfrac{1}{2} + \mathrm{i}(\Gamma - \Delta)\lambda\}.\end{aligned} \tag{5.16}$$

With the identifications

$$-\tfrac{1}{2} + \mathrm{i}(\Gamma - \Delta)\lambda = E\delta, \tag{5.17}$$

$$2\beta - \Gamma\lambda - \Delta\lambda = c, \tag{5.18}$$

eq. (5.16) can be expressed as

$$WW'' = \tfrac{1}{2}W'^2 - \tfrac{3}{2}\delta^2 W^4 + \eta\delta W^3 - (\tfrac{1}{8}\eta^2 + c)W^2 - \tfrac{1}{4}E^2, \tag{4.59}$$

which is Painlevé IV.

So far we have derived the transformation mapping a solution Z of Chazy IV (4.19) on a solution W of Painlevé IV. The explicit transformation formula can be inferred from (5.11) inserting (5.4), (5.2) and (5.3). The result is

$$\boxed{\delta W = \tfrac{1}{2}\eta + \frac{(2Z' - \Gamma - \Delta)(Z - \eta Z') - 2\mathrm{i}Z''(\Gamma - \Delta)}{4(Z' - \Gamma)(Z' - \Delta)}.} \tag{5.19}$$

It is also straightforward to derive the inverse transformation of (5.9). In fact, from (5.3) and (5.7) we have

$$\begin{aligned}\lambda Z = {} & -2\mathrm{i}(\Gamma - \Delta)\lambda(\lambda Z' - \beta)^{1/2}\cos\omega \\ & + \lambda(4Z' - 2\Gamma - 2\Delta)\lambda(\lambda Z' - \beta)^{1/2}\sin\omega + \eta Z'\lambda.\end{aligned} \tag{5.20}$$

Using (5.11) for $2(\lambda Z' - \beta)^{1/2}\sin\omega$, (5.13) for $2(\lambda Z' - \beta)^{1/2}\cos\omega$, (5.15) for Z' and $(4Z' - 2\Gamma - 2\Delta)$, and (5.17), we obtain

$$\boxed{\begin{aligned}\lambda Z - \beta\eta = {} & -\tfrac{1}{2}c\eta - (E\delta + \tfrac{1}{2})(W' + E)W^{-1} \\ & + \delta W[c + \tfrac{1}{2}(\delta W - \tfrac{1}{2}\eta)^2 + \tfrac{1}{2}(W' + E)^2 W^{-2}],\end{aligned}} \tag{5.21}$$

which is the inverse of (5.19). For $A > 0$ (E real), the transformation (5.21) may also be derived from the Miura transformation (3.31) with (3.32), as will be shown in appendix B. The derivation given in this section, however, applies for arbitrary (complex) values of the integration constants in (4.19) and (4.59).

5.2. *Bäcklund transformation*

i) The Painlevé equation (4.59) is invariant under the substitution $E \to -E$. Using (5.21) for $\lambda \neq 0$ we can define $\lambda Z - \beta$ as the function corresponding to E and $\lambda \tilde{Z} - \beta$ as the function corresponding to $-E$ in the right-hand side. Noting that $\lambda(Z - \tilde{Z}) = -E/W$ and that $\beta \to \beta$, $\Gamma \to \Delta - \frac{1}{2}\mathrm{i}\lambda^{-1}$, $\Delta \to \Gamma + \frac{1}{2}\mathrm{i}\lambda^{-1}$, we obtain the BT

$$\begin{aligned}&\tilde{Z}(\beta, \Delta - \tfrac{1}{2}\mathrm{i}\lambda^{-1}, \Gamma + \tfrac{1}{2}\mathrm{i}\lambda^{-1})\\ &\quad = Z - \frac{4(Z' - \Gamma)(Z' - \Delta)[\frac{1}{2}\lambda^{-1} - \mathrm{i}(\Gamma - \Delta)]}{-2\mathrm{i}Z''(\Gamma - \Delta) + 2ZZ' - (\Gamma + \Delta)(Z + Z'\eta) + 2\eta\Gamma\Delta},\end{aligned} \tag{5.22}$$

with $Z = Z(\beta, \Gamma, \Delta)$. The same result can be derived for $A > 0$ from the Miura transformation (3.31), (3.32), leading to $2\alpha(Y - \tilde{Y}) = 2A^{1/2}F'$, and thus to $\lambda(Z - \tilde{Z}) = -\alpha(Y - \tilde{Y}) = -E/W$.

ii) The Chazy equation (4.19) for $\lambda = 1$ is invariant under permutations of β, Γ and Δ. Using (5.19) one can derive various transformations between solutions $W(\eta, c, E)$ and $\tilde{W}(\eta, \tilde{c}, \tilde{E})$ corresponding to different values of c and E in eq. (4.59) with $\delta = 1$. As an example we give the transformation connecting solutions $\tilde{W}(\eta, c, E + 1)$ and $W(\eta, c, E)$ of (4.59) with $\delta = 1$, i.e.

$$\tilde{W}(\eta, c, E + 1) = \tfrac{1}{2}\eta - \frac{2R'(R - \eta R') + 2(E + \frac{1}{2})R''}{4R'^2 + (E + \frac{1}{2})^2}, \tag{5.23}$$

where

$$R = -(E + \tfrac{1}{2})(W' + E)W^{-1} + W[c + \tfrac{1}{2}(W - \tfrac{1}{2}\eta)^2 + \tfrac{1}{2}(W' + E)^2 W^{-2}], \tag{5.24}$$

with $W \equiv W(\eta, c, E)$. A few other transformations are given in appendix C. All these transformations can be inferred from the ones given in ref. 30.

5.3. *Special solutions of Painlevé IV*

As an application of the transformations used in subsections (5.1) and (5.2) we discuss the construction of solutions of Painlevé IV, i.e. (4.59) with $\delta = 1$, for special values of E.

i) Consider eq. (4.59) for $\delta = 0$. Differentiating this equation gives

$$W''' = -\tfrac{1}{4}\eta W - (\tfrac{1}{8}\eta^2 + c)2W' = 0. \tag{5.25}$$

Using example 10 on p. 298 of ref. 27, the solution of (5.25) can be expressed as

$$W = c_1 u^2 + c_2 uv + c_3 v^2, \tag{5.26}$$

where

$$u = \eta^{-1/2} U[-\tfrac{1}{2}c, \tfrac{1}{4}, \tfrac{1}{4}\eta^2], \quad v = \eta^{-1/2} V[-\tfrac{1}{2}c, \tfrac{1}{4}, \tfrac{1}{4}\eta^2], \tag{5.27}$$

and U and V are two independent solutions of the differential equation of Whittaker, i.e.

$$4\xi^2 \partial_\xi^2 U = (\xi^2 - 4k\xi + 4m^2 - 1)U, \tag{5.28}$$

with

$$\xi = \tfrac{1}{4}\eta^2, \quad m = \tfrac{1}{4}, \quad k = -\tfrac{1}{2}c, \tag{5.29}$$

cf. eq. (2.273) on p. 473 and eq. (10) on p. 475 of ref. 33. The requirement that W satisfies (5.25) imposes a relation between c and the constants c_1, c_2 and c_3 in (5.26), but we shall not go into further details.

Using eq. (5.21) for $\lambda = 1$, $\delta = 0$, i.e.

$$Z = \tfrac{1}{2}\eta(\Gamma + \Delta) - \tfrac{1}{2}(W' + E)W^{-1}, \tag{5.30}$$

one obtains from the solution W of (5.25), a solution $Z(\beta, \Gamma, \Delta)$ of the Chazy equation[†] (4.19) under the condition

$$\Gamma - \Delta = -\tfrac{1}{2}\mathrm{i}. \tag{5.31}$$

Applying (5.19) with $\delta = 1$ and taking into account (5.31) one obtains from the solution $Z(\beta, \Gamma, \Delta)$ of (4.19) a solution $\bar{W}(\eta, c, 0)$ of P. IV, i.e. (4.59) with $\delta = 1$, with $E = 0$, for arbitrary values of c.

ii) Consider eq. (4.31) for $\alpha = 0$. In this case the solutions can be expressed as

$$\Phi = \eta^{-1/2} \mathrm{e}^{\mathrm{i}\eta^2/8} U(\mathrm{i}\gamma + \tfrac{1}{4}, \tfrac{1}{4}, -\tfrac{1}{4}\mathrm{i}\eta^2), \tag{5.32}$$

in which U satisfies the equation of Whittaker, i.e. (5.28) with

$$\xi = -\tfrac{1}{4}\mathrm{i}\eta^2, \quad m = \tfrac{1}{4}, \quad k = \mathrm{i}\gamma + \tfrac{1}{4}, \tag{5.33}$$

cf. eq. (2.273) on p. 473 and eq. (10) on p. 475 of ref. 33. The solution of the Chazy equation $Z(\beta, \Gamma, \Delta)$ of (4.19) with $\lambda = -\alpha = 0$ can be found from (5.32) noting that

$$|\Phi|^2 = 2Y', \quad \partial_\eta \operatorname{Im} \ln \Phi = \tfrac{1}{4}\left\{\frac{Y + \eta Y'}{Y'}\right\}, \tag{5.34}$$

as follows from (3.23) applied to the similarity solutions (4.14) and (4.30).

[†] This solution also satisfies a Riccati equation which can be derived directly from (4.59).

From (5.34) and (4.18) we obtain the solution

$$Z(\beta, \Gamma, \Delta) = |\Phi|^2\{2\partial_\eta \operatorname{Im} \ln \Phi - \tfrac{1}{2}\eta\} + \Gamma\eta, \tag{5.35}$$

of eq. (4.19) with $\lambda = 0$. Using (5.19) and (5.17) one obtains from $Z(\beta, \Gamma, \Delta)$ a solution $W(\eta, c, -\frac{1}{2})$ of Painlevé IV[†] with $E = -\frac{1}{2}$, for arbitrary values of c.

iii) In i) and ii) we have shown how to derive solutions $W(\eta, c, E)$ of Painlevé IV for $E = 0$ and $E = -\frac{1}{2}$. The transformation (5.23) can then be used to obtain the solutions $W(\eta, c, E)$ for half-integer values $E = \frac{1}{2}n$ of E. With the transformations given in appendix C and in refs. 30–32 more special solutions of Painlevé IV may be obtained.

Remark. In a similar way special solutions can be obtained for Painlevé II. This will be dealt with in a following paper[14]), where we consider the similarity solutions of the Boussinesq equation and their relation to Painlevé II and Painlevé IV.

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Appendix A

One-soliton solutions of the AHSC

In this appendix we discuss the one-soliton solutions $z = z(x - vt)$ for the z-component of the spin in the AHSC ($z = S^z = Q'$). From (4.51) we have

$$z'^2 = P(z), \tag{A.1}$$

with

$$P(z) = (\hat{\gamma}' + 2\hat{\gamma}z - Az^2)(s^2 - z^2) - (\delta - vz)^2. \tag{A.2}$$

We can have a one-soliton solution with $z \to z_0$ for $\eta = x - vt \to \infty$, or $\eta \to -\infty$, when $P(z)$ has a minimum 0 for $z = z_0$ or a triple root for $z = z_0$. We now consider the cases $A < 0$ (planar anisotropy) and $A > 0$ (Ising anisotropy) separately.

[†]) This solution satisfies a first-order differential equation which can be found from (5.21) taking $\lambda = 0$, $\delta = 1$, and therefore $E = -\frac{1}{2}$, $2\beta = c$.

i) *Planar anisotropy* $A < 0$

a) $P(z)$ has a minimum $P(z_0) = 0$ for $z = z_0$, $z_0^2 \leq s^2$ under the conditions

$$\begin{aligned}&(\hat{\gamma}' + 2\hat{\gamma} z_0 - A z_0^2)(s^2 - z_0^2) - (\delta - \upsilon z_0)^2 = 0,\\&(2\hat{\gamma} - 2Az_0)(s^2 - z_0^2) - 2(\hat{\gamma}' + 2\hat{\gamma} z_0 - A z_0^2) z_0 + 2\upsilon(\delta - \upsilon z_0) = 0,\end{aligned} \tag{A.3}$$

and

$$C \equiv -As^2 + 6Az_0^2 - 6\gamma z_0 - \hat{\gamma}' - \upsilon^2 > 0. \tag{A.4}$$

The one-soliton solutions can be expressed as

$$z = z_0 + \frac{2C}{\{2\hat{\gamma} - 4Az_0\} \pm \{(2\hat{\gamma} - 4Az_0)^2 - 4AC\}^{1/2} \cosh \eta C^{1/2}}. \tag{A.5}$$

b) $P(z)$ has a triple root $P(z_0)$ under the conditions (A.3) and

$$-As^2 + 6Az_0^2 - 6\hat{\gamma} z_0 - \hat{\gamma}' - \upsilon^2 = 0. \tag{A.6}$$

The one-soliton solution is given by

$$z = z_0 - \frac{2\hat{\gamma} - 4Az_0}{\frac{1}{4}\eta^2 (2\hat{\gamma} - 4Az_0)^2 - A}. \tag{A.7}$$

ii) *Ising anisotropy* $A > 0$

a) $P(z)$ can have a minimum $P(z_0) = 0$ for $z_0 = \pm s$ under the conditions

$$\delta = \pm \upsilon s, \quad \hat{\gamma}' \pm 2\hat{\gamma} s - As^2 = 0, \tag{A.8}$$

and

$$C = 2(\hat{\gamma}' + As^2) \geq 0. \tag{A.9}$$

The one-soliton solution is given by

$$z = \pm s \mp \frac{\frac{1}{2} Cs}{(\hat{\gamma}' + 3As\delta) - \{(\hat{\gamma}' 3As^2)^2 - 4ACs^2\}^{1/2} \cosh \eta C^{1/2}}, \tag{A.10}$$

b) $P(z)$ can have two minima $P(z_0) = 0$ for $z_0 = s$ and $z_0 = -s$ under the conditions

$$\upsilon = 0, \quad \delta = 0, \quad \hat{\gamma} = 0, \quad \hat{\gamma}' = As^2. \tag{A.11}$$

We then obtain a stationary solution which is given by

$$z = -s \tanh(\pm 2xAs^2). \tag{A.12}$$

Appendix B

The Miura transformation

In this appendix we discuss the derivation of (5.21) from the Miura transformation (3.31), (3.32). Substituting (4.57) for f and (4.14) with $a = 0$ and

$Y = Z - \hat{\Gamma}$ for y, we obtain

$$2\alpha(Z' - \hat{\Gamma}) = -\tfrac{1}{2}(\delta W - \tfrac{1}{2}\eta)^2 - \tfrac{1}{2}\frac{W'^2}{W^2} - \tfrac{1}{2}\frac{A}{W^2} - \frac{A^{1/2}W'}{W^2}, \tag{B.1}$$

with $W = 1/F'$, and

$$2\alpha[-\tfrac{1}{2}(Z + \eta Z') + \hat{\Gamma}\eta] = \frac{1}{W^2}[-\tfrac{1}{2}(W - \eta W')W' + (\delta W - \tfrac{1}{2}\eta)(-W''W + 2W'^2)$$

$$- (\delta W - \tfrac{1}{2}\eta)^3 W^2 - (\delta W - \tfrac{1}{2}\eta)W'^2 - A(\delta W - \tfrac{1}{2}\eta) - \tfrac{1}{2}A^{1/2}(W - \eta W')]. \tag{B.2}$$

From eqs. (B.1) and (B.2), using also (4.59) to eliminate the terms with W'', one obtains

$$-\alpha Z = -\hat{\Gamma}\alpha\eta - \tfrac{1}{2}c\eta - (A^{1/2}\delta + \tfrac{1}{2})\left(\frac{W' + A^{1/2}}{W}\right)$$

$$+ \delta W\left[c + \tfrac{1}{2}(\delta W - \tfrac{1}{2}\eta)^2 + \tfrac{1}{2}\left(\frac{W' + A^{1/2}}{W}\right)^2\right], \tag{B.3}$$

and eq. (B.3) is identical to (5.21), taking into account that $\lambda = -\alpha$, cf. (4.20), and $E = A^{1/2}$, cf. (4.60), when we identify Γ with $\beta\lambda^{-1}$.

Appendix C

Bäcklund transformations for Painlevé IV

In this appendix we derive the Bäcklund transformations for Painlevé IV arising from the invariance of the Chazy equation (4.19) for $\lambda = 1$ under permutations of β, Γ and Δ. Using (5.21) for $\lambda = 1$ and also (5.18) we have

$$Z = \tfrac{1}{2}\eta(\Gamma + \Delta) + R, \tag{C.1}$$

where

$$R = -(E + \tfrac{1}{2})(W' + E)W^{-1} + W(c + \tfrac{1}{2}(W - \tfrac{1}{2}\eta)^2 + \tfrac{1}{2}(W' + E)^2 W^{-2}), \tag{C.2}$$

and $W \equiv W(\eta, c, E)$ is a solution of (4.59), with $\delta = 1$,

When we insert (C.1) in the inverse relation (5.19), with $\delta = 1$, we obtain an identity. The Chazy equation (4.19) with $\lambda = 1$ is invariant under permutations β_P, Γ_P, Δ_P of β, Γ, Δ i.e. $Z(\beta_P, \Gamma_P, \Delta_P) = Z(\beta, \Gamma, \Delta)$. For any permutation P of β, Γ, Δ we obtain new values $E_P = -\tfrac{1}{2} + \mathrm{i}(\Gamma_P - \Delta_P)$ and $c_P = 2\beta - \Gamma_P - \Delta_P$, which can be expressed as linear combinations of $E = -\tfrac{1}{2} + \mathrm{i}(\Gamma - \Delta)$ and $c = 2\beta - \Gamma - \Delta$. Furthermore a new solution $\tilde{W}_P(\eta, c_P, E_P)$ of (4.59) can be obtained, using (5.19) for $\delta = 1$ with Γ_P and Δ_P instead of Γ and Δ, i.e.

$$\tilde{W}_P(\eta, c_P, E_P) = \tfrac{1}{2}\eta + \frac{(2Z' - \Gamma_P - \Delta_P)(Z - \eta Z') - 2\mathrm{i}Z''(\Gamma_P - \Delta_P)}{4(Z' - \Gamma_P)(Z' - \Delta_P)}. \tag{C.3}$$

Using (C.1), the right-hand side of (C.3) can be expressed in the form

$$\tilde{W}_P(\eta, c_P, E_P) = \tfrac{1}{2}\eta + \frac{(2R' - \lambda_{1P})(R - \eta R') - \lambda_{2P}R''}{4(R' - \lambda_{3P})(R' - \lambda_{4P})} \tag{C.4}$$

in which c_P, E_P, λ_{1P}, λ_{2P}, λ_{3P} and λ_{4P} can be easily evaluated for each of the permutations of β, Γ and Δ. The explicit results, taking $\hat{E} \equiv E + \frac{1}{2}$, are as follows:

β_P	Γ_P	Δ_P	c_P	$\hat{E}_P$	λ_{1P}	λ_{2P}	λ_{3P}	λ_{4P}
β	Γ	Δ	c	$\hat{E}$	0	$2\hat{E}$	$\frac{1}{2}\mathrm{i}\hat{E}$	$-\frac{1}{2}\mathrm{i}\hat{E}$
β	Δ	Γ	c	$-\hat{E}$	0	$-2\hat{E}$	$\frac{1}{2}\mathrm{i}\hat{E}$	$-\frac{1}{2}\mathrm{i}\hat{E}$
Γ	β	Δ	$-\frac{1}{2}c - \frac{3}{2}\mathrm{i}\hat{E}$	$\frac{1}{2}\mathrm{i}c + \frac{1}{2}\mathrm{i}\hat{E}$	$\frac{1}{2}c + \frac{1}{2}\mathrm{i}\hat{E}$	$\mathrm{i}c + \hat{E}$	$\frac{1}{2}c$	$\frac{1}{2}\mathrm{i}\hat{E}$
Δ	Γ	β	$-\frac{1}{2}c + \frac{3}{2}\mathrm{i}\hat{E}$	$-\frac{1}{2}\mathrm{i}c + \frac{1}{2}\mathrm{i}\hat{E}$	$\frac{1}{2}c - \frac{1}{2}\mathrm{i}\hat{E}$	$-\mathrm{i}\hat{c} + \hat{E}$	$\frac{1}{2}c$	$-\frac{1}{2}\mathrm{i}\hat{E}$
Γ	Δ	β	$-\frac{1}{2}c - \frac{3}{2}\mathrm{i}\hat{E}$	$-\frac{1}{2}\mathrm{i}c - \frac{1}{2}\hat{E}$	$\frac{1}{2}c + \frac{1}{2}\mathrm{i}\hat{E}$	$-\mathrm{i}c - \hat{E}$	$\frac{1}{2}c$	$\frac{1}{2}\mathrm{i}\hat{E}$
Δ	β	Γ	$-\frac{1}{2}c + \frac{3}{2}\mathrm{i}\hat{E}$	$\frac{1}{2}\mathrm{i}c - \frac{1}{2}\hat{E}$	$\frac{1}{2}c - \frac{1}{2}\mathrm{i}\hat{E}$	$\mathrm{i}c - \hat{E}$	$\frac{1}{2}c$	$-\frac{1}{2}\mathrm{i}\hat{E}$

(C.5)

The transformation given in (5.23) is a combination of the permutation $(\beta_P, \Gamma_P, \Delta_P) = (\beta, \Delta, \Gamma)$ with $c_P = c$, $E_P = -E - 1$ and the trivial symmetry $W(\eta, c, E) = W(\eta, c, -E)$. All transformations in (C.5) can be inferred from the ones given in ref. 30.

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