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An alternating integrable map whose square is the QRT map

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Abstract

We present a novel alternating integrable map (i.e., a non-autonomous integrable map whose sole time-dependence is a term of the form $(-1)^n$). We show that the square (i.e., second iterate) of this map is the QRT map.

1. Introduction

Integrable dynamical systems have long played a prominent role in science, the two-body problem and soliton equations such as the Korteweg–de Vries equation being important examples [1].

More recently, interest has grown in discrete integrable systems. Here I will concentrate on integrable maps in two dimensions. The reader interested in discrete soliton equations or higher-dimensional integrable maps can consult Refs. [2,3], respectively [4,5].

The first integrable map of the plane was introduced by McMillan [6], and contained 6 arbitrary parameters. This map was generalized to an 18-parameter map, that is now commonly called the QRT map, in [7–9]. De-autonomization of the QRT map led to many of the discrete Painlevé equations [10] cf. also [11].

In a parallel development it has been realized that to obtain the most general instances of many discrete phenomena, one should study inhomogeneous/non-autonomous systems. This has led to inhomogeneous soliton equations [12], to terms of the form $(-1)^n$ in Painlevé equations [10], and to k-symmetries, k-integrals, etc. [13,14].

In this Letter we give very general integrable non-autonomous versions of the McMillan map and of the symmetric QRT map, and discuss their relationship with the autonomous QRT map.

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2. Two examples of alternating integrable maps

Example 1. Consider a discrete modified Korteweg–de Vries equation [7,15]:

$$\frac{d}{dt}x_n = \left(1+x_n^2\right)\left[x_{n-1} - x_{n+1} + \frac{1}{2}(x_{n+2}+x_n)\left(1+x_{n+1}^2\right) - \frac{1}{2}(x_n + x_{n-2})\left(1+x_{n-1}^2\right)\right]. \tag{1}$$

Stationary solutions of this equation are given by

$$x_{n+1} - \frac{1}{2}(x_n + x_{n+2})\left(1+x_{n+1}^2\right) = x_{n-1} - \frac{1}{2}(x_{n-2}+x_n)\left(1+x_{n-1}^2\right). \tag{2}$$

Note that this can be integrated to give

$$x_{n+1} - \frac{1}{2}(x_n + x_{n+2})\left(1+x_{n+1}^2\right) = K_1 + (-1)^n K_2, \tag{3}$$

where K_1, K_2 are integration constants. We rewrite this as the alternating map

$$x_{n+1} + x_{n-1} = \frac{x_n - K_1 + (-1)^n K_2}{\frac{1}{2}x_n^2 + \frac{1}{2}}. \tag{4}$$

This mapping turns out to be integrable.[1] Its alternating integral is

$$I(x_n, x_{n+1}, n) = \frac{1}{2}x_n^2 x_{n+1}^2 + \frac{1}{2}\left(x_n^2 + x_{n+1}^2\right) - x_n x_{n+1} + K_1(x_n + x_{n+1}) - (-1)^n K_2(x_n - x_{n+1}). \tag{5}$$

By this we mean that

$$I(x_n, x_{n+1}, n) = I(x_{n+1}, x_{n+2}, n+1) \tag{6}$$

provided x_n satisfies (4).

Example 2. Consider the alternating map [10]

$$x_{n+1} + x_{n-1} = -\frac{x_n^2 - a x_n - \beta - \gamma(-1)^n}{x_n}. \tag{7}$$

This mapping is also integrable. Its alternating integral is

$$I(x_n, x_{n+1}, n) = x_n^2 x_{n+1} + x_n x_{n+1}^2 - a x_n x_{n+1} - \beta(x_n + x_{n+1}) - (-1)^n \gamma(x_n - x_{n-1}). \tag{8}$$

Both alternating maps (4) and (7) will turn out to be special cases of a more general integrable alternating McMillan map that we present below.

3. Autonomous and alternating McMillan maps

Consider the autonomous McMillan map [6]:

$$x_{n+1} + x_{n-1} = -\frac{\beta x_n^2 + \epsilon x_n + \xi}{\alpha x_n^2 + \beta x_n + \gamma}, \tag{9}$$

where $\alpha, \beta, \gamma, \epsilon$ and ξ are arbitrary constants.

[1] See below.

It has the integral

$$I(x_n, x_{n+1}) = \alpha x_n^2 x_{n+1}^2 + \beta\left(x_n^2 x_{n+1} + x_n x_{n+1}^2\right) + \gamma\left(x_n^2 + x_{n+1}^2\right) + \epsilon x_n x_{n+1} + \xi(x_n + x_{n+1}), \tag{10}$$

i.e.,

$$I(x_n, x_{n+1}) = I(x_{n+1}, x_{n+2}) \tag{11}$$

provided x_n satisfies Eq. (9).

We now generalize these results as follows, by introducing a very simple alternating time-dependence. This leads to the alternating McMillan map:[2]

$$x_{n+1} + x_{n-1} = -\frac{(\beta - (-1)^n \tilde{\beta})x_n^2 + \epsilon x_n + \xi + (-1)^n \tilde{\xi}}{\alpha x_n^2 + (\beta + (-1)^n \tilde{\beta})x_n + \gamma + (-1)^n \tilde{\gamma}}, \tag{12}$$

where $\alpha, \beta, \tilde{\beta}, \gamma, \tilde{\gamma}, \epsilon, \xi$ and $\tilde{\xi}$ are arbitrary constants. It has the alternating integral

$$\begin{aligned} I(x_n, x_{n+1}, n) = {} & \alpha x_n^2 x_{n+1}^2 + \beta\left(x_n^2 x_{n+1} + x_n x_{n+1}^2\right) + \gamma\left(x_n^2 + x_{n+1}^2\right) + \epsilon x_n x_{n+1} + \xi(x_n + x_{n+1}) \\ & + (-1)^n\left[\tilde{\beta}\left(x_n^2 x_{n+1} - x_n x_{n+1}^2\right) + \tilde{\gamma}\left(x_n^2 - x_{n+1}^2\right) + \tilde{\xi}(x_n - x_{n+1})\right], \end{aligned} \tag{13}$$

i.e., $I(x_n, x_{n+1}, n)$ satisfies Eq. (6), provided x_n satisfies Eq. (12).

Autonomous cases of the map (12) are obtained by taking either $\tilde{\beta} = \tilde{\gamma} = \tilde{\xi} = 0$, leading to the map (9), or $\alpha = \beta = \gamma = \epsilon = \xi = 0$, leading to

$$x_{n+1} + x_{n-1} = \frac{\tilde{\beta} x_n^2 - \tilde{\xi}}{\tilde{\beta} x_n + \tilde{\gamma}}. \tag{14}$$

Note that (14) is *not* of the form (9) and has an alternating integral.

There are several ways to "autonomize" the remaining alternating maps. One method is given in Ref. [10]. Here we describe an alternative method. Both methods relate the alternating McMillan map to special cases of the QRT map. To exhibit this connection, we first rewrite Eq. (12) as 2 coupled first-order equations:

$$x_{n+1} = -y_n - \frac{(\beta - (-1)^n \tilde{\beta})x_n^2 + \epsilon x_n + \xi + (-1)^n \tilde{\xi}}{\alpha x_n^2 + (\beta + (-1)^n \tilde{\beta})x_n + \gamma + (-1)^n \tilde{\gamma}}, \qquad y_{n+1} = x_n. \tag{15}$$

In turn this can be split into 2 autonomous cases, depending on whether n is even

$$x_{2m+1} = -y_{2m} - \frac{(\beta - \tilde{\beta})x_{2m}^2 + \epsilon x_{2m} + \xi + \tilde{\xi}}{\alpha x_{2m}^2 + (\beta + \tilde{\beta})x_{2m} + \gamma + \tilde{\gamma}}, \qquad y_{2m+1} = x_{2m} \tag{16}$$

or odd

$$x_{2m} = -y_{2m-1} - \frac{(\beta + \tilde{\beta})x_{2m-1}^2 + \epsilon x_{2m-1} + \xi - \tilde{\xi}}{\alpha x_{2m-1}^2 + (\beta - \tilde{\beta})x_{2m-1} + \gamma - \tilde{\gamma}}, \qquad y_{2m} = x_{2m-1}. \tag{17}$$

[2] Note that Eq. (12) can be written in a unique standard form, using

$$\frac{a + b(-1)^n}{c + d(-1)^n} = \frac{ac - bd + (bc - ad)(-1)^n}{c^2 - d^2}.$$

Substituting (17) into (16) we find that the odd x and y satisfy a non-symmetric QRT map:

$$x_{2m+1} = -x_{2m-1} - \frac{(\beta - \tilde{\beta})y_{2m+1}^2 + \epsilon y_{2m+1} + \xi + \tilde{\xi}}{\alpha y_{2m+1}^2 + (\beta + \tilde{\beta})y_{2m+1} + \gamma + \tilde{\gamma}},$$
$$y_{2m+1} = -y_{2m-1} - \frac{(\beta + \tilde{\beta})x_{2m-1}^2 + \epsilon x_{2m-1} + \xi - \tilde{\xi}}{\alpha x_{2m-1}^2 + (\beta - \tilde{\beta})x_{2m-1} + \gamma - \tilde{\gamma}}. \tag{18}$$

This special case of the QRT map was given in [9].

Similarly, the even x and y satisfy the transposed QRT map

$$x_{2m+2} = -x_{2m} - \frac{(\beta + \tilde{\beta})y_{2m+2}^2 + \epsilon y_{2m+2} + \xi - \tilde{\xi}}{\alpha y_{2m+2}^2 + (\beta - \tilde{\beta})y_{2m+2} + \gamma - \tilde{\gamma}},$$
$$y_{2m+2} = -y_{2m} - \frac{(\beta - \tilde{\beta})x_{2m}^2 + \epsilon x_{2m} + \xi + \tilde{\xi}}{\alpha x_{2m}^2 + (\beta + \tilde{\beta})x_{2m} + \gamma + \tilde{\gamma}}. \tag{19}$$

4. Autonomous and alternating symmetric QRT maps

More generally, consider the autonomous symmetric QRT map

$$x_{n+1}x_{n-1}f_3(x_n) + (x_{n+1} + x_{n-1})f_2(x_n) - f_1(x_n) = 0, \tag{20}$$

where

$$\big(f_1(x_n), f_2(x_n), f_3(x_n)\big) = (A_0 X_n) \times (A_1 X_n), \tag{21}$$

where

$$X_n := \begin{pmatrix} x_n^2 \\ x_n \\ 1 \end{pmatrix} \tag{22}$$

and the symmetric matrices A_i are given by

$$A_i := \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \beta_i & \epsilon_i & \xi_i \\ \gamma_i & \xi_i & \mu_i \end{pmatrix}, \quad i = 0, 1, \tag{23}$$

where $\alpha_i, \beta_i, \gamma_i, \epsilon_i$ and μ_i $(i = 0, 1)$ are arbitrary constants.

It has the integral[3]

$$I(x_n, x_{n+1}) = \frac{X_n^t A_0 X_{n+1}}{X_n^t A_1 X_{n+1}}, \tag{24}$$

i.e., $I(x_n, x_{n+1})$ satisfies Eq. (11) provided x_n satisfies Eq. (20).

Once again we generalize this map by introducing an alternating time-dependence.

This leads to the alternating map

$$x_{n+1}x_{n-1}f_3(x_n, n) + (x_{n+1} + x_{n-1})f_2(x_n, n) - f_1(x_n, n) = 0, \tag{25}$$

[3] Here and below, the superscript t denotes the transposed.

where

$$\big(f_1(x_n,n), f_2(x_n,n), f_3(x_n,n)\big) = \big(A_0(n)X_n\big) \times \big(A_1(n)X_n\big), \tag{26}$$

where X_n is again given by (22), but the matrices A_i are now time-dependent (alternating) and non-symmetric:

$$A_i(n) := \begin{pmatrix} \alpha_i & \beta_i + (-1)^n\tilde{\beta}_i & \gamma_i + (-1)^n\tilde{\gamma}_i \\ \beta_i - (-1)^n\tilde{\beta}_i & \epsilon_i & \xi_i + (-1)^n\tilde{\xi}_i \\ \gamma_i - (-1)^n\tilde{\gamma}_i & \xi_i - (-1)^n\tilde{\xi}_i & \mu_i \end{pmatrix}, \quad i = 0, 1, \tag{27}$$

where $\alpha_i, \beta_i, \tilde{\beta}_i, \gamma_i, \tilde{\gamma}_i, \epsilon_i, \xi_i, \tilde{\xi}_i$, and μ_i $(i = 0, 1)$ are arbitrary parameters.

It has the alternating integral

$$I(x_n, x_{n+1}, n) = \frac{X_n^t A_0(n) X_{n+1}}{X_n^t A_1(n) X_{n+1}}, \tag{28}$$

i.e., $I(x_n, x_{n+1}, n)$ satisfies Eq. (6), provided x_n satisfies Eq. (25). Rewriting (25) as 2 coupled first-order equations in x_n and y_n, it can be shown (in a similar way to above, and using that $A_i(n+1) = A_i^t(n)$) that the odd x and y satisfy the general QRT map, and the even x and y satisfy the transposed QRT map.

In this sense, one may say that the alternating map given by Eqs. (25)–(27) is a square root of the QRT map. Together with the existence of the integral (28), we take this to be sufficient to warrant calling the alternating map (25)–(27) integrable.

5. Concluding remarks

Other special cases of the alternating map (25) have appeared in the literature [10], or occur as reductions of integrable partial difference equations. Moreover, higher-dimensional cases also abound. We intend to present and expand upon the derivation of the results presented here elsewhere.

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