LINEARIZATION OF THE BOUSSINESQ EQUATION AND THE MODIFIED BOUSSINESQ EQUATION

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Received 1 July 1982

A description in terms of one and the same inhomogeneous linear integral equation is proposed for the solutions of the Boussinesq equation and the modified Boussinesq equation. New similarity solutions of these equations are obtained, as well as two-parameter families of solutions of Painlevé II and Painlevé IV.

1. Introduction. In the first part of this letter we propose a new linear inhomogeneous integral equation from which one can evaluate in a direct way the solutions of the Boussinesq equation (BSQ), as well as of the modified Boussinesq equation (MBSQ) [1], thereby proving that the MBSQ is completely integrable. In the second part we consider the similarity solutions of the BSQ and the MBSQ and the Miura transformation between MBSQ and BSQ is used to obtain two-parameter families of solutions of Painlevé II and Painlevé IV [2].

2. Integral equation. Consider the linear integral equation

$$\phi_k^{(n)}(x,t) + \mathrm{i}\, \mathrm{e}^{\mathrm{i}(kx+k^2t)} \int_C \mathrm{d}\lambda(l)\, \frac{\phi_l^{(n)}(x,t)}{\alpha k + \alpha^* l} = k^{-n}\, \mathrm{e}^{\mathrm{i}(kx+k^2t)}\,, \tag{1}$$

where n is an integer and $\mathrm{d}\lambda(k)$ an arbitrary measure over the complex variable k and the integration is performed over a contour C. The measure and the contour are to be chosen such that the homogeneous integral equation has only the zero solution and we assume that the differentiations with respect to x and t may be shifted through the integral. Integral equations of this type have been proposed for the linearization of the Korteweg–de Vries equation [3], the nonlinear Schrödinger equation (NLS) and the isotropic Heisenberg spin chain (IHSC) [4,5], the modified Korteweg–de Vries equation and the sine-Gordon equation etc. [6]. The form of the integral equation may be inferred from a treatment by Rosales [7].

From eq. (1) one can derive the relations

$$(\alpha^* + \alpha)(-\mathrm{i}\partial_x \boldsymbol{\phi}_k) = \alpha \mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{\phi}_k + \alpha^* k \boldsymbol{\phi}_k - \mathrm{i}\boldsymbol{\Phi}\cdot \mathbf{O} \cdot \boldsymbol{\phi}_k\,, \tag{2}$$

$$(\alpha^{*2} - \alpha^2)(-\partial_t \boldsymbol{\phi}_k) = -\alpha^2 \mathbf{J}^{\mathrm{T}^2} \cdot \boldsymbol{\phi}_k + \alpha^{*2} k^2 \boldsymbol{\phi}_k - \mathrm{i}\alpha^* \boldsymbol{\Phi}\cdot \mathbf{J} \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k + \mathrm{i}\alpha \boldsymbol{\Phi}\cdot \mathbf{O} \cdot \mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{\phi}_k\,. \tag{3}$$

Here $\boldsymbol{\phi}_k$ denotes a vector with components $\phi_k^{(n)}$, $\boldsymbol{\Phi}$ is a matrix with elements $\phi_{n,m} = \int_C \mathrm{d}\lambda(l)\, \phi_l^{(n)} l^{-m}$ and the matrices $\mathbf{J}$, $\mathbf{J}^{\mathrm{T}}$ and $\mathbf{O}$ are defined by $J_{n,m} = J^{\mathrm{T}}_{m,n} = \delta_{m,n+1}$, $O_{m,n} = \delta_{n,0}\delta_{m,n}$. If furthermore $(-\alpha/\alpha^*)^p = 1$, for some natural number p, we have the algebraic relation

$$\alpha^*(k^p \boldsymbol{\phi}_k - \mathbf{J}^{\mathrm{T}^p} \cdot \boldsymbol{\phi}_k) = \mathrm{i} \sum_{j=1}^{p-1} (-\alpha/\alpha^*)^{p-j-1} \boldsymbol{\Phi}\cdot \mathbf{J}^j \cdot \mathbf{O} \cdot \mathbf{J}^{\mathrm{T}^{p-j-1}} \cdot \boldsymbol{\phi}_k\,. \tag{4}$$

In this letter we shall restrict ourselves to the value $p = 3$, taking $\alpha = 1 \mp \frac{1}{3}\mathrm{i}\sqrt{3}$. From eqs. (2)–(4) one can derive after a lengthy calculation

$$(\pm 1/\sqrt{3})\partial_t \boldsymbol{\phi}_k = \partial_x^2 \boldsymbol{\phi}_k - 2\kappa \partial_x \boldsymbol{\phi}_k - \partial_x \boldsymbol{\Phi}\cdot \mathbf{O} \cdot \boldsymbol{\phi}_k\,, \tag{5}$$

$$(\partial_x^3 - 3\kappa\partial_x^2 + 3\kappa^2 \partial_x)\boldsymbol{\phi}_k = \tfrac{3}{4}\partial_x^2 \boldsymbol{\Phi}\cdot \mathbf{O} \cdot \boldsymbol{\phi}_k \pm \tfrac{1}{4}\sqrt{3}\,\partial_t \boldsymbol{\Phi}\cdot \mathbf{O} \cdot \boldsymbol{\phi}_k + \tfrac{3}{2}\partial_x \boldsymbol{\Phi}\cdot \mathbf{O} \cdot \partial_x \boldsymbol{\phi}_k - \tfrac{3}{2}\kappa \partial_x \boldsymbol{\Phi}\cdot \mathbf{O} \cdot \boldsymbol{\phi}_k\,, \tag{6}$$

where $\kappa \equiv \mathrm{i}(\frac{1}{2} \pm \frac{1}{6}\mathrm{i}\sqrt{3})k$. From (5) and (6), using an integration over the contour C, one obtains the matrix partial differential equation (PDE):

$$-(\partial_t^2 + \partial_x^4)\mathbf{\Phi} \pm \sqrt{3}(\partial_t \mathbf{\Phi} \cdot \mathbf{O} \cdot \partial_x \mathbf{\Phi} - \partial_x \mathbf{\Phi} \cdot \mathbf{O} \cdot \partial_t \mathbf{\Phi})$$
$$+ 3\partial_x(\partial_x \mathbf{\Phi} \cdot \mathbf{O} \cdot \partial_x \mathbf{\Phi}) = 0 . \tag{7}$$

Introducing the function $w \equiv -\frac{1}{2}\phi_{0,0}$ and the new time scale $t \to t/\sqrt{3}$, as will be used throughout the rest of this paper, we obtain from (7)

$$\tfrac{1}{3} w_{tt} + w_{xxxx} + 12 w_x w_{xx} = 0 , \tag{8}$$

which is the (potential) Boussinesq equation. From the integral equation (1) one can evaluate solutions of the usual BSQ, i.e.

$$\tfrac{1}{3}\psi_{tt} + \psi_{xxxx} + 6(\psi^2)_{xx} + \psi_{xx} = 0 ,$$

using

$$\psi = -\tfrac{1}{12} + \partial_x w = -\tfrac{1}{12} - \tfrac{1}{2}\partial_x \phi_{0,0} .$$

The linear problem for the BSQ, as given in ref. [8], can be inferred from the $n = 0$ component of eqs. (5) and (6) after some obvious transformations. Substituting

$$q = -2 \ln \phi_k^{(0)} + 2\kappa x \mp 6\kappa^2 t \tag{9}$$

in the $n = 0$ component of (5) and (6) and eliminating $\phi_{0,0}$ one obtains, [taking into account the upper signs in (5), (6) and (9)],

$$\tfrac{1}{3} q_{tt} - q_t q_{xx} - \tfrac{3}{2} q_x^2 q_{xx} + q_{xxxx} = 0 , \tag{10}$$

which is the MBSQ [1]. The MBSQ is completely integrable, since its solutions can be obtained directly from the linear integral equation (1) with $n = 0$. Furthermore from (7) it can be shown that the functions

$$q = -2\ln[1 - (\mathrm{i}/\alpha^*)\phi_{0,1}] ,$$

$$q = 2\ln[1 - (\mathrm{i}/\alpha)\phi_{1,0}]$$

also satisfy the MBSQ. We have also derived the linear problem for $\phi_k^{(1)}$, which corresponds to a Lax representation for the MBSQ, see ref. [9] for further details.

From the linear problem for $\phi_k^{(0)}$ one can derive the Bäcklund transformation of the BSQ, cf. (8), which is given by [10,11]

$$\partial_t(w - \widetilde{w}) = -3\partial_x^2(w + \widetilde{w}) - 6(w - \widetilde{w})\partial_x(w - \widetilde{w}) ,$$
$$\partial_t(w + \widetilde{w}) = \partial_x^2(w - \widetilde{w}) + 6(w - \widetilde{w})\partial_x(w + \widetilde{w})$$
$$+ 4(w - \widetilde{w})^3 + 2\delta , \tag{11}$$

($\delta = -2\kappa^3$), and from the relation $w - \widetilde{w} = \frac{1}{2}\partial_x q$, one obtains the Miura transformation, cf. [1], which maps the solutions of the MBSQ on solutions of the BSQ, i.e.

$$w_x = -\tfrac{1}{12} q_t + \tfrac{1}{4} q_{xx} - \tfrac{1}{8} q_x^2 ,$$
$$w_t = \tfrac{1}{4} q_{xt} + \tfrac{1}{4} q_{xxx} - \tfrac{1}{4} q_x q_t - \tfrac{1}{8} q_x^3 + \delta . \tag{12}$$

3. Similarity solutions. We first treat the solutions with similarity variable $\eta = x - vt - \frac{1}{2}at^2$. Inserting

$$w(x,t) = -\tfrac{1}{216} a^3 t^4 - \tfrac{1}{54} v a^2 t^3 - 3\gamma t^2 - \alpha t$$
$$- \tfrac{1}{36} a x^2 + \tfrac{1}{18} a \eta^2 - \tfrac{1}{36} v^2 \eta + S(\eta) , \tag{13}$$

where v, a, γ and α are arbitrary constants, in (8), and

$$q(x,t) = \tfrac{1}{3} x(v + at) + Q(\eta), \quad V(\eta) \equiv -\tfrac{1}{2} Q'(\eta) , \tag{14}$$

in (10), we obtain, after an obvious shift,

$$S''^2 + 4S'^3 - \tfrac{2}{3} a(S - \eta S')S' - 2\beta(S - \eta S') - 2\hat{\gamma} = 0 \tag{15}$$

and

$$V'' = 2V^3 + \tfrac{1}{3} a\eta V - \tfrac{1}{6} v^2 V + \mu - \tfrac{1}{2} , \tag{16}$$

where $\beta = \frac{1}{36} a v^2 - 2\gamma$ and $\hat{\gamma}$ and μ are constants of the integration. The Miura transformation (12) yields the relation

$$S' = -\tfrac{1}{12} a\eta + \tfrac{1}{72} v^2 - \tfrac{1}{2} V' - \tfrac{1}{2} V^2 , \tag{17}$$

under the conditions

$$6\gamma - \tfrac{1}{18} a v^2 = 0, \quad \tfrac{1}{4} - \tfrac{1}{2}\mu + \tfrac{1}{12} a + \alpha - \tfrac{1}{54} v^3 = -\delta ,$$

together with the identification

$$\hat{\gamma} = \tfrac{1}{2}(\tfrac{1}{12} a + \tfrac{1}{2}\mu - \tfrac{1}{4})^2 - v^6/23328$$

of the integration constants.

For $a = 0$, the function $U(\eta) \equiv S'(\eta)$ satisfies $U'' + 6U^2 + \beta\eta = 0$, which is Painlevé I, cf. [11,12]. For $a \neq 0$, we can choose $v = \beta = 0$, $a = 3$, without loss of generality, and eq. (15) is an ordinary differential equation without movable critical points [13], which may be referred to as Chazy II; eq. (16), for $a = 3$, $v = 0$, is Painlevé II [‡1].

Eq. (17) can be used to construct solutions of Painlevé II, for $\mu = \frac{1}{2}n$, n integer. (For solutions of Painlevé II, see also ref. [3]). In fact, introducing

‡1 In [12] a connection with Painlevé II for the special value $\mu = 1/2$ was found. However, substituting $F = -\xi/2 - y'/2 + y^2/2$ in eq. (14) of [12] one obtains $y'' = 2y^3 - 2\xi y - 1 + (-8k_2)^{1/2}$.

$P(\eta,\mu,\lambda) \equiv \lambda S'(\eta,\mu)/\mu$, we have, $(\upsilon = \beta = 0, a = 3)$,

$$4PP'' + 16\mu\lambda^{-1}P^3 + 4\eta P^2 - 2P'^2 + \tfrac{1}{2}\lambda^2 = 0\,, \qquad (18)$$

and

$$V(\eta,\mu) = \tfrac{1}{2}\,[P'(\eta,\mu) + \tfrac{1}{2}\lambda]/P(\eta,\mu)\,. \qquad (19)$$

For $\mu = 0$, eq. (18) can be solved explicitly. By differentiation we obtain $P''' + 2\eta P' + P = 0$ with the general solution [14] $P = c_1u^2 + c_2uv + c_3v^2$, in which u and v are two independent solutions of Airy's equation $U'' + \frac{1}{2}\eta U = 0$. From (19) we obtain the solutions $V(\eta, 0)$ of Painlevé II for $\mu = 0$, consisting of a one-parameter family corresponding to $\lambda = 0$, given in refs. [15,16], and a new two-parameter family for $\lambda \neq 0$. The solutions for $\mu = \frac{1}{2}n$, n integer, can be obtained from $V(\eta, 0)$, using eqs. (5.6), (4.1) and (4.2) of ref. [15].

Finally we consider the solutions with similarity variable $\eta = x/\sqrt{t}$, cf. [12]. Inserting

$$w(x,t) = t^{-1/2}\,[Z(\eta) - \tfrac{1}{108}\eta^3] + t\delta\,, \qquad (20)$$

and

$$q(x,t) = \Omega \ln t + Q(\eta), \qquad W(\eta) \equiv \tfrac{1}{2}Q'(\eta) + \tfrac{1}{6}\eta \qquad (21)$$

in eqs. (8) and (10) respectively, we obtain

$$Z''^2 - \tfrac{1}{4}(Z - \eta Z')^2 + 4Z'^3 + 2\gamma Z' + 2\gamma' = 0\,, \qquad (22)$$

and

$$W''W = \tfrac{1}{2}W'^2 - \tfrac{3}{2}W^4 + \eta W^3 - \tfrac{1}{8}(\eta^2 + 4\Omega)W^2 - \widetilde{\gamma} = 0\,, \qquad (23)$$

in which γ, γ' and $\widetilde{\gamma}$ are integration constants.

Eq. (22), which has been introduced in ref. [13], also describes the similarity solutions of the NLS and the IHSC [17,18], and eq. (23) is Painlevé IV. The Miura transformation (12) yields

$$Z = \tfrac{1}{2}(1+\Omega)(W - \tfrac{1}{6}\eta) - \tfrac{1}{2}W'^2W^{-1} + \tfrac{1}{2}W^3$$
$$+ \tfrac{1}{8}\eta^2 W - \tfrac{1}{2}\eta W^2 - \widetilde{\gamma}W^{-1}\,, \qquad (24)$$

and the inverse relation

$$W = \tfrac{1}{2}(Z - \eta Z' + 2Z'')\,[\tfrac{1}{3}(1+\Omega) - 2Z']^{-1}\,, \qquad (25)$$

and the identification of the integration constants

$$\gamma = \widetilde{\gamma} - \tfrac{1}{24}(1+\Omega)^2\,,$$

$$\gamma' = -\tfrac{1}{6}\widetilde{\gamma}(1+\Omega) - \tfrac{1}{432}(1+\Omega)^3\,.$$

In an analogous way as for Painlevé II, using (25), we have obtained a two-parameter family of solutions of Painlevé IV for $\widetilde{\gamma} = -\frac{1}{8}(1+\Omega)^2$, from which one can find solutions for other values of the parameters, using symmetry properties and Bäcklund transformations [19]. In ref. [18] special solutions of Painlevé IV have also been obtained, using the Miura transformation between the IHSC and the NLS. A more extended version of the considerations in this letter is in preparation [9].

This investigation is part of the research programme of the Stichting voor Fundamenteel Onderzoek der Materie (FOM) which is financially supported by the Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek (ZWO).

References

[1] R. Hirota and J. Satsuma, Prog. Theor. Phys. 57 (1977) 797.
[2] E.L. Ince, Ordinary differential equations (Dover Publications, New York, 1956).
[3] A.S. Fokas and M.J. Ablowitz, Phys. Rev. Lett. 47 (1981) 1096.
[4] F.W. Nijhoff, J. van der Linden, G.R.W. Quispel and H.W. Capel, Phys. Lett. 89A (1982) 106.
[5] F.W. Nijhoff, J. van der Linden, G.R.W. Quispel, H.W. Capel and J. Velthuizen, Physica A, to be published.
[6] F.W. Nijhoff, G.R.W. Quispel, J. van der Linden and H.W. Capel, Physica A, to be published.
[7] R.R. Rosales, Stud. Appl. Math. 59 (1978) 117.
[8] V.E. Zakharov, Sov. Phys. JETP 38 (1974) 108.
[9] G.R.W. Quispel, F.W. Nijhoff and H.W. Capel, to be published.
[10] H.H. Chen, in: Bäcklund transformations, Lecture notes in mathematics No. 515, ed. R.M. Miura (Springer, Berlin, 1976) p. 241.
[11] M. Boiti and F. Pempinelli, Nuovo Cimento 56B (1980) 148.
[12] T. Nishitani and M. Tajiri, Phys. Lett. 89A (1982) 379.
[13] J. Chazy, Acta Math. 34 (1911) 317.
[14] E.T. Whittaker and G.N. Watson, A course of modern analysis (Cambridge University Press, 1958).
[15] M. Boiti and F. Pempinelli, Nuovo Cimento 51B (1979) 70.
[16] A.S. Fokas and M.J. Ablowitz, preprint 1981.
[17] G.R.W. Quispel and H.W. Capel, Phys. Lett. 88A (1982) 371.
[18] G.R.W. Quispel and H.W. Capel, Physica A, to be published.
[19] N.A. Lukashevich, Differential Equations 3 (1967) 395, Diff. Urav. 3 (1967) 771.