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Discrete Painlevé Equations from Nonisospectral Soliton Equations

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1. Introduction

The standard method to obtain Painlevé equations from soliton equations is to apply a (point) symmetry reduction [**1, 2, 9**]. When one or more of the independent variables that are involved is discrete, however, this method does not work, and for a long time it was not clear in what way a discrete analog of the above construction might be found.

The solution to this problem was found in [**7**]. There, for the example of the Korteweg-de Vries equation, it was shown that symmetry reduction of an isospectral equation is equivalent to stationary reduction of a corresponding nonisospectral equation.

In this paper, the method of [**7**] is applied to the Nonlinear Schrödinger (NLS) equation. In Section 2.1, a symmetry reduction is applied to reduce the continuous NLS to an equation known as Ince XXXIV, which is equivalent to the second Painlevé equation [**5**]. In Section 2.2, we show that exactly the same equation is obtained by stationary reduction of a nonisospectral NLS equation. Finally, in Section 3 it is shown that the latter method (but not the former) has a straightforward generalization to the discrete case, and we obtain a discrete version of Ince XXXIV from the stationary reduction of the discrete (differential-difference) NLS equation ($D\Delta$NLS).

1.1. The isospectral NLS and its symmetry reduction. The NLS equation is the first member of a hierarchy of isospectral equations.

$$\sigma_3 \frac{d}{dt}\begin{pmatrix} r \\ q \end{pmatrix} = \omega(\mathcal{L}_1)\begin{pmatrix} r \\ q \end{pmatrix}, \tag{1.1}$$

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This is the final form of the paper.

where ω is an arbitrary polynomial,

$$\sigma_3 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.2}$$

and the recursion operator $\mathcal{L}_1$ is defined by

$$2i\mathcal{L}_1 \begin{pmatrix} a \\ b \end{pmatrix} := \begin{pmatrix} -a_x \\ b_x \end{pmatrix} + 2\begin{pmatrix} r \\ q \end{pmatrix} \int_x^\infty dx\big(q(x)a(x) - r(x)b(x)\big), \tag{1.3}$$

(here and below partial derivatives are denoted by subscripts, e.g. $a_x := (\partial a)/(\partial x)$).

Taking

$$r(x) = -q^*(x), \quad \omega(\mathcal{L}_1) = -4i\mathcal{L}_1^2, \tag{1.4}$$

(the asterisk denoting complex conjugation), we obtain the NLS equation:

$$-iq_t = q_{xx} + 2|q|^2 q. \tag{1.5}$$

In [**6**], Johnson et al give a list of infinitesimal Lie point symmetries of the NLS. One of the symmetries in their list is:

$$\begin{aligned} x &\to x + \varepsilon bt \\ t &\to t + \varepsilon d \\ q &\to 9 + \varepsilon \frac{1}{2} ibxq. \end{aligned} \tag{1.6}$$

By the usual methods [**1, 9**] this symmetry leads to the following reduction:

$$q(x,t) = e^{i(-\gamma xt - 2/3\gamma^2 t^3)} r(\eta), \tag{1.7}$$

where

$$\eta := x + \gamma t^2, \quad \gamma := -\frac{b}{2d}. \tag{1.8}$$

Substituting the similarity solution (1.7) in the NLS, we obtain the ODE

$$r_{\eta\eta} + 2|r|^2 r + \gamma\eta r = 0. \tag{1.9}$$

Defining

$$c := r^* r \tag{1.10}$$

we obtain Ince XXXIV:

$$c_{\eta\eta} = \frac{1}{2}\frac{c_\eta^2}{c} + 4c^2 - 2\gamma\eta c + \frac{\alpha^2}{c} \tag{1.11}$$

(where α is an integration constant). Note that Eq. (1.11) is related to Painlevé II.

1.2. The nonisospectral NLS and its stationary reduction. There is also a nonisospectral NLS hierarchy [**4**]

$$\sigma_3 \frac{d}{dt}\begin{pmatrix} r \\ q \end{pmatrix} = \omega(\mathcal{L}_1)\begin{pmatrix} r \\ q \end{pmatrix} + \widehat{\omega}(\mathcal{L}_1)x\begin{pmatrix} r \\ q \end{pmatrix}, \tag{1.12}$$

where ω and $\widehat{\omega}$ are 2 arbitrary polynomials.

The simplest nonlinear member of the double hierarchy (1.12) is given by:

$$\omega(\mathcal{L}_1) = -4i(\mathcal{L}_1^2), \quad \widehat{\omega}(\mathcal{L}_1) = i\mu, \tag{1.13}$$

where μ is an arbitrary constant. This gives the nonisospectral NLS:

$$-iq_t = q_{xx} + 2|q|^2 q + \mu x q. \tag{1.14}$$

Taking the stationary reduction $q_t = 0$, we get:

$$q_{xx} + 2|q|^2 q + \mu x q = 0. \tag{1.15}$$

Note that Eq. (1.15) is identical to Eq. (1.9)!

So putting $s = q^* q$ we obtain again Ince XXXIV:

$$s_{xx} = \frac{1}{2}\frac{s_x^2}{s} + 4s^2 - 2\mu x s + \frac{\alpha^2}{s}. \tag{1.16}$$

2. The Nonisospectral $D\Delta$NLS and Its Stationary Reduction

Whereas the isospectral point symmetry reduction of Section 2.1 has no straightforward generalization to the discrete case, the nonisospectral stationary reduction does.

The nonisospectral discrete NLS hierarchy [3] is given by

$$\sigma_3 \frac{d}{dt}\begin{pmatrix} R_n \\ Q_n \end{pmatrix} = \omega(\mathcal{L}_2)\begin{pmatrix} R_n \\ Q_n \end{pmatrix} + \widehat{\omega}(\mathcal{L}_2)\Delta\left(n + \frac{1}{2}\right)\begin{pmatrix} R_n \\ Q_n \end{pmatrix}, \tag{2.1}$$

where Δ denotes the lattice spacing, and the discrete recursion operator $\mathcal{L}_2$ is defined by:

$$\mathcal{L}_2 \begin{pmatrix} A_n \\ B_n \end{pmatrix} := \begin{pmatrix} A_{n-1} + R_{n-1}(1 - R_n Q_n) D_n + R_n E_n \\ B_{n+1} + Q_{n+1}(1 - R_n Q_n) D_{n+1} + Q_n E_{n+1} \end{pmatrix}, \tag{2.2}$$

with

$$D_n := \sum_{j=n}^{\infty} \frac{R_j B_j - Q_j A_j}{1 - R_j Q_j}, \tag{2.3}$$

and

$$E_n := \sum_{j=n}^{\infty} (R_j B_{j+1} - Q_j A_{j-1}). \tag{2.4}$$

Taking $R_n = -Q_n^*$, and

$$\omega(\mathcal{L}_2) = -\frac{i}{\Delta^2}(\mathcal{L}_2 - 2 + \mathcal{L}_2^{-1}), \quad \widehat{\omega}(\mathcal{L}_2) = i\mu, \tag{2.5}$$

we get

$$-i\frac{d}{dt}Q_n = \frac{Q_{n+1} - 2Q_n + Q_{n-1}}{\Delta^2} + \frac{|Q_n|^2(Q_{n+1} + Q_{n-1})}{\Delta^2} + \mu\Delta\left(n + \frac{1}{2}\right)Q_n. \tag{2.6}$$

We take the stationary reduction $(dQ_n)/(dt) = 0$, and define

$$C_n := Q_{n+1}Q_n^* + Q_{n+1}^* Q_n.$$

This way we obtain a discrete analog of Ince XXXIV:[1]

$$\left(\frac{\zeta(n)}{C_n + C_{n-1}} - 1\right)\left(\frac{\zeta(n+1)}{C_n + C_{n+1}} - 1\right) = \frac{4}{C_n^2 + \beta^2}, \tag{2.7}$$

where β is an integration constant, and

$$\zeta(n) := 4 - \mu\Delta^3(2n + 1). \tag{2.8}$$

[1]Similar equations are studied in [8].

Equation (2.7) reduces to Ince XXXIV in the continuum limit:

$$C_n = \Delta^2 C(x), \quad \beta = \alpha\Delta^3, \; x = n\Delta. \tag{2.9}$$

References

1. M. J. Ablowitz and P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, Cambridge Univ. Press, Cambridge, 1991.
2. M. J. Ablowitz, A. Ramani and H. Segur, *Nonlinear evolution equations and ordinary differential equations of Painlevé type*, Lett. Nuovo Cim. **23** (1978) 333–338.
3. M. Bruschi, D. Levi, and O. Ragnisco, *Discrete version of the Nonlinear Schrödinger equation with linearly x-dependent coefficients*, Nuovo Cimento A **53** (1979), 21–30.
4. F. Calogero and A. Degasperis, *Exact solution via the spectral transform of a generalization with linearly x-dependent coefficients of the nonlinear Schrödinger equation*, Lett. Nuovo Cimento **22** (1978), 270–273.
5. E. L. Ince, *Ordinary differential equations*, Dover, New York, 1956.
6. S. F. Johnson, K. E. Lonngren, and D. R. Nicholson, *On the invariants of the Nonlinear Schrödinger equation*, Phys. Lett. A **74** (1979), 393–394.
7. D. Levi, O. Ragnisco, and M. A. Rodrigues, *On nonisospectral flows, Painlevé equations and symmetries of differential and difference equations*, Theoret. and Math. Phys. **93** (1992), 473–480.
8. F. Nijhoff, J. Satsuma, K. Kajiwara, B. Grammaticos, and A. Ramani, *A study of the alternate discrete Painlevé. II. Equation*, Inverse Problems **12** (1996), 697–716.
9. P. J. Olver, *Applications of Lie groups to differential equations*, Springer, Berlin, 1986.

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