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Lie symmetries and linearisation of the QRT mapping

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Abstract

We study the map $u(x+2) = [f_1(u(x+1)) - u(x)f_2(u(x+1))]/[f_2(u(x+1)) - u(x)f_3(u(x+1))]$, introduced by Quispel, Roberts and Thompson (QRT). We show, using Lie point symmetries under what conditions the QRT mapping can be linearised. Requiring that the QRT mapping is invariant under the symmetry vector field $X(x, u) = \alpha(x)\partial/\partial x + A(x)[B + Cu + Du^2]\partial/\partial u$, where B , C and D are constants and $\alpha(x)$ is an arbitrary unit periodic function in x , we derive conditions on the unknown functions f_i in the QRT mapping. Further for these cases of the QRT mapping we explicitly construct two independent integrals of motion ensuring its integrability. We also derive its exact solution.

1. Introduction

In recent years there has been increasing interest in the study of discrete nonlinear dynamical systems governed by mappings or difference equations particularly from the viewpoint of integrability, linearisability or exact solvability [1–4]. Given a system of nonlinear difference equations there exist no well-defined analytical techniques to deal with them in general and in particular to investigate its integrability or linearisability or exact solvability. Thus, much effort has been spent by several groups [4–8] in developing analytical techniques for nonlinear difference equations in recent years. As a result several methods have been developed for nonlinear difference equations *analogous* to the existing methods for differential equations. For example, the singularity confinement method [7] testing the integrability of differential-difference and pure difference equations is considered as the discrete

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analogue of the Painlevé analysis for continuous systems (absence of movable critical singularities). In fact, Ramani et al. [9] have derived discrete Painlevé transcendental equations dP_{III} to dP_V using the singularity confinement criterion reminiscent of the Painlevé–Gambier method for continuous ones. Recently, we have shown how Lie's theory of continuous symmetries of differential equations can be extended to differential-difference and difference equations [10, 11] (see also [12,13]). Also, we have illustrated how to integrate the given difference equation using the Lie symmetry. Further, we have demonstrated that an autonomous difference equation of arbitrary order with one or more independent variables can be linearized by a point transformation if and only if it admits a symmetry vector field whose coefficient is the product of two functions, one of the dependent variable and of the independent variables [14,15].

In Ref. [4] one of the authors with Roberts and Thompson (QRT) considered a mapping in the plane given by

$$u(x+2) = \frac{f_1(u(x+1)) - u(x)f_2(u(x+1))}{f_2(u(x+1)) - u(x)f_3(u(x+1))} \quad (1)$$

or

$$u'' = \frac{f_1(u') - u f_2(u')}{f_2(u') - u f_3(u')} \quad (2)$$

where $u = u(x)$, $u' = u(x+1)$, $u'' = u(x+2)$ and f_i 's, $i = 1, 2, 3$, are unknown functions of u' alone, and investigated its integrability. In fact, they have asked under what condition, on the functions f_i 's the mapping (1) admits an integral of motion $I(x, u, u')$ which can be expressed as a biquadratic polynomial in (u, u') . They have concluded, after cumbersome calculations, that this is possible only if f_i 's are certain quartic polynomial in the variable u' . The linearisability of QRT mapping (1) is investigated by Ramani et al. [16] using the method of singularity confinement criterion. Thus, it is important to investigate the nature of QRT mapping, whether it is linearisable or integrable, when f_i 's are polynomials of degree greater than 4 in u' . In this article, using the Lie symmetry analysis [14, 15], first we recall that the QRT mapping (1) can be transformed to a linear second-order difference equation with constant coefficients if the symmetry vector field is factorisable. Secondly, we wish to investigate under what conditions on the functions f_i 's the QRT mapping (1) is invariant under a factorisable symmetry vector field of the form

$$X(x, u) = \alpha(x) \frac{\partial}{\partial x} + A(x)[B + Cu + Du^2] \frac{\partial}{\partial u}, \quad (3)$$

where B , C and D are constants, $A(x)$ is an arbitrary function of x , and $\alpha(x)$ is a unit periodic function. We wish to note that although the independent variable x in the QRT mapping takes *integer values* we allow here x to take real values.

The outline of the paper is as follows. We show first, using the Lie symmetry analysis in Section 2, how the QRT mapping (1) can be transformed to a linear second-order difference equation with constant coefficients, provided it preserves a *factorisable vector field*. Also, we investigate under what condition, on the functions f_i 's in (1), it is invariant under the symmetry vector field (3). In Section 3, for the derived condition on f_i 's we transform the QRT mapping into a linear second-order equation as well as derive explicitly two independent integrals of motion of (1) ensuring its integrability. Also we derive the exact solution of (1) for these cases.

2. Lie symmetries

Consider a one-parameter (ε) Lie group of continuous point transformations:

$$x^* = x + \varepsilon \xi(x, u) + O(\varepsilon^2), \quad (4)$$

$$u^* = u + \varepsilon v(x, u) + O(\varepsilon^2), \quad (5)$$

where $\xi(x, u)$ and $v(x, u)$ are infinitesimal transformations of x and u , respectively. Eq. (1) is invariant under the transformations (4) and (5) if

$$u^*(x+2) = (u^{**}) = \frac{f_1(u^{**}) - u^* f_2(u^{**})}{f_2(u^{**}) - u^* f_3(u^{**})} \quad (6)$$

provided $u = u(x)$ is a solution of Eq. (1). We know that [10]

$$(u^{**}) = u'' + \varepsilon [\xi(x, u) - \xi(x+2, u'')] \frac{du''}{dx} + \varepsilon v(x+2, u'') + O(\varepsilon^2). \quad (7)$$

Substituting Eqs. (4), (5) and (7) into the transformed equation (6) we find

$$\xi(x, u) = \alpha(x), \quad (8)$$

and the following invariant equation:

$$2[F - u]^2 v(x+2, u'') \\ = F[G - F]v(x, u) + \left[F^2 \frac{\partial}{\partial u'} G - u \frac{\partial}{\partial u'} FG + u^2 \frac{\partial}{\partial u'} F \right] v(x+1, u'), \quad (9)$$

where $\alpha(x)$ is an arbitrary unit periodic function, that is $\alpha(x+1) = \alpha(x)$, $x \in \mathbb{R}$, $F = f_2/f_3$ and $G = f_1/f_2$. Henceforth, we write $f_1(u') = f_1$, $f_2(u') = f_2$ and $f_3(u') = f_3$ for convenience.

The functional equation (9) cannot be solved for $v(x, u)$, in general, with the existing methods applicable to difference equations [17]. In order to solve Eq. (9) we make the ansatz that the infinitesimal symmetry $v(x, u)$ is factorisable in x and u ,

$$v(x, u) = A(x)G(u), \quad (10)$$

so that the symmetry vector field becomes

$$X(x, u) = \alpha(x) \frac{\partial}{\partial x} + A(x)G(u) \frac{\partial}{\partial u}. \tag{11}$$

Using Eq. (10) in the invariant equation (9) we find, after straightforward calculations, that the function $A(x)$ must satisfy [14, 15]

$$A(x+2) - kA(x+1) + lA(x) = 0, \tag{12}$$

where k and l are arbitrary constants and the QRT mapping (1) can be transformed to a linear second-order difference equation with constant coefficients given by

$$w(u'') - kw(u') + lw(u) = p, \tag{13}$$

where p is a constant and the homogenising variable $w(u)$ is given by

$$w(u) = \int \frac{du}{G(u)}. \tag{14}$$

In this article, we assume that $G(u)$ is a quadratic polynomial in u given by

$$G(u) = B + Cu + Du^2, \tag{15}$$

where B, C and D are constants, and find under what condition on the functions f_1, f_2 and f_3 the QRT mapping (1) is invariant under Eqs. (4) and (5). Substituting Eq. (10) with (15) into the invariant equation (9) and equating the powers of u^2, u and u^0 equal to zero we obtain the following determining equations:

$$[B + CF + DF^2]A(x+2) = D[FG - F^2]A(x) + [B + Cu' + Du'^2]A(x+1) \frac{\partial}{\partial u'} F, \tag{16}$$

$$F[2B + C(F + G) + 2DFG]A(x+2) = CF[F - G]A(x) + [B + Cu' + Du'^2]A(x+1) \frac{\partial}{\partial u'} FG, \tag{17}$$

$$[B + CG + DG^2]A(x+2) = B \left[\frac{G}{F} - 1 \right] A(x) + [B + Cu' + Du'^2]A(x+1) \frac{\partial}{\partial u'} G. \tag{18}$$

Using (12) in Eqs. (16)–(18) we obtain the following equations:

$$k[B + CF + DF^2] = [B + Cu' + Du'^2] \frac{\partial}{\partial u'} F, \tag{19}$$

$$kF[2B + C(F + G) + 2DFG] = [B + Cu' + Du'^2] \frac{\partial}{\partial u'} (FG), \tag{20}$$

$$k[B + CG + DG^2] = [B + Cu' + Du'^2] \frac{\partial}{\partial u'} G, \tag{21}$$

$$l[B + CF + DF^2] = D[F^2 - FG], \tag{22}$$

$$l[2B + C(F + G) + 2DFG] = C[G - F], \tag{23}$$

$$lF[B + CG + DG^2] = B[F - G]. \tag{24}$$

Combining Eqs. (22) and (24), we get

$$(Bf_3 - Df_1)(Cf_2 + Bf_3 + Df_1) = 0. \tag{25}$$

Thus, we have two possibilities.

Case 1:

$$Bf_3 - Df_1 = 0 \text{ or equivalently } \frac{f_1}{f_3} = \frac{B}{D} = \text{constant}. \tag{26}$$

Case 2:

$$Cf_2 + Bf_3 + Df_1 = 0. \tag{27}$$

We consider cases 1 and 2 separately.

Case 1: Substituting Eq. (26) into Eq. (23), we obtain

$$4Df_1f_2 + C(f_1f_3 + f_2^2) = 0. \tag{28}$$

Using (28) in (23) we find

$$C(f_1f_3 - f_2^2) = 0. \tag{29}$$

Since both $C = 0$ and $(f_1f_3 - f_2^2) = 0$ are not interesting choices, we refrain from presenting further details.

Case 2: Substituting Eq. (27) into Eqs. (22)–(24) we find that

$$l = 1. \tag{30}$$

Then Eqs. (19)–(21) can be rewritten as

$$k[B + CF + DF^2] = [B + Cu' + Du'^2] \frac{\partial}{\partial u'} F, \tag{31}$$

$$k[BF + CG + DFG] = [B + Cu' + Du'^2] \frac{\partial}{\partial u'} (FG), \tag{32}$$

$$k[B + CG + DG^2] = [B + Cu' + Du'^2] \frac{\partial}{\partial u'} G, \tag{33}$$

where $F = (f_2/f_3)$ and $G = (f_1/f_2)$. Substituting Eqs. (31) and (33) into Eq. (32) we find that it is satisfied identically. Also making use of Eq. (27), $C = -(B/F) - DG$ in Eq. (33) we find that it reduces to Eq. (31). Thus, it is sufficient to consider either Eq. (31) or Eq. (33) (for further discussion). Now Eq. (31) can be written as

$$\frac{dF}{[B + CF + DF^2]} = \frac{kdu'}{[B + Cu' + Du'^2]}. \tag{34}$$

Let $C^2 - 4BD > 0$. Then the integration of Eq. (34) gives

$$\frac{F - \alpha_1}{F - \alpha_2} = Q_1 \frac{(u' - \alpha_1)^k}{(u' - \alpha_2)^k} \quad (35)$$

and so

$$F = \frac{f_2}{f_3} = \frac{\alpha_1 [u' - \alpha_2]^k - \alpha_2 Q_1 [u' - \alpha_1]^k}{[u' - \alpha_2]^k - Q_1 [u' - \alpha_1]^k}, \quad (36)$$

where Q_1 is an arbitrary constant and α_1 and α_2 are roots of

$$B + C\alpha + D\alpha^2 = 0. \quad (37)$$

Substituting the function F , Eq. (36), into Eq. (27) we find

$$G = \frac{(\alpha_1 + \alpha_2)[\alpha_1(u' - \alpha_2)^k - \alpha_2 Q_1(u' - \alpha_1)^k] - \alpha_1 \alpha_2 [(u' - \alpha_2)^k - Q_1(u' - \alpha_1)^k]}{[\alpha_1(u' - \alpha_2)^k - \alpha_2 Q_1(u' - \alpha_1)^k]}. \quad (38)$$

Thus, the functions f_1, f_2 and f_3 here take the following form:

$$f_1(u') = (\alpha_1 + \alpha_2)[\alpha_1(u' - \alpha_2)^k - \alpha_2 Q_1(u' - \alpha_1)^k] - \alpha_1 \alpha_2 [(u' - \alpha_2)^k - Q_1(u' - \alpha_1)^k], \quad (39)$$

$$f_2(u') = \alpha_1(u' - \alpha_2)^k - \alpha_2 Q_1(u' - \alpha_1)^k \quad (40)$$

and

$$f_3(u') = (u' - \alpha_2)^k - Q_1(u' - \alpha_1)^k. \quad (41)$$

Let $C^2 = 4BD$. Then the integration of Eq. (34) gives

$$\frac{1}{2DF + C} = \frac{k}{(2Du' + C)} + Q_2 \quad (42)$$

and so

$$F = \frac{f_2}{f_3} = \frac{1}{2D} \frac{(2Du' + C)(1 - CQ_2) - Ck}{[k + Q_2(2Du' + C)]}, \quad (43)$$

where Q_2 is an arbitrary constant. Substituting the function F into Eq. (27) we obtain

$$G = \frac{f_1}{f_2} = \frac{(2Du' + C)[2\alpha_1(1 - CQ_2) - 2BQ_2] - 2k(B + C\alpha_1)}{[(2Du' + C)(1 - CQ_2) - Ck]}, \quad (44)$$

where $\alpha_1 = -\frac{C}{2D}$. Therefore, the functions f_1, f_2 and f_3 in this case take the following form:

$$f_1(u') = (2Du' + C)[2\alpha_1(1 - CQ_2) - 2BQ_2] - 2k(B + C\alpha_1), \quad (45)$$

$$f_2(u') = (2Du' + C)(1 - CQ_2) - Ck \quad (46)$$

and

$$f_3(u') = 2D[k + Q_2(2Du' + C)]. \quad (47)$$

Let $C^2 - 4BD < 0$. Then the integration of Eq. (33) gives

$$G = \frac{f_1}{f_2} = \frac{1}{2D} \left[-C + \delta \tan \left[k \tan^{-1} \left(\frac{(2Du' + C)}{\delta} \right) + Q_3 \right] \right], \quad (48)$$

where Q_3 is an arbitrary constant and

$$\delta = \sqrt{4BD - C^2}. \quad (49)$$

Substituting the function G in Eq. (27) we find

$$F = \frac{f_2}{f_3} = \frac{-2B}{C + \delta \tan \left[k \tan^{-1} \left(\frac{(2Du' + C)}{\delta} \right) + Q_3 \right]}. \quad (50)$$

Thus, the functions f_1, f_2 and f_3 here take the following form:

$$f_1(u') = \frac{1}{D} \left[-C + \delta \tan \left[k \tan^{-1} \left(\frac{(2Du' + C)}{\delta} \right) + Q_3 \right] \right], \quad (51)$$

$$f_2(u') = 2 \quad (52)$$

and

$$f_3(u') = -\frac{1}{B} \left[C + \delta \tan \left[k \tan^{-1} \left(\frac{(2Du' + C)}{\delta} \right) + Q_3 \right] \right]. \quad (53)$$

Thus, the QRT mapping (1) is invariant under the following one-parameter (ε) transformations:

$$x^* = x + \varepsilon \alpha(x) + O(\varepsilon^2), \quad u^* = u + \varepsilon A(x)[B + Cu + Du^2] + O(\varepsilon^2), \quad (54)$$

provided f_1, f_2 and f_3 satisfy Eqs. (39)–(41), (45)–(47) and (51)–(53) for $C^2 - 4BD > 0$, $C^2 = 4BD$ and $C^2 - 4BD < 0$, respectively, and $A(x)$ satisfies Eq. (12) with $l = 1$.

3. Linearisation and integrals of motion

Now the homogenising variable $w(u)$ associated with the symmetry $v(x, u)$ is

$$w(u) = \int \frac{du}{B + Cu + Du^2}. \quad (55)$$

When $C^2 - 4BD > 0$ the homogenising variable $w(u)$ takes the following form:

$$w(u) = \frac{1}{D(\alpha_1 - \alpha_2)} \log \frac{(u - \alpha_1)}{(u - \alpha_2)}, \quad (56)$$

where α_1 and α_2 are roots of Eq. (37). Now

$$\frac{(u'' - \alpha_1)}{(u'' - \alpha_2)} = \frac{(f_2 - \alpha_1 f_3)(u - F_1)}{(f_2 - \alpha_2 f_3)(u - F_2)}, \quad (57)$$

where

$$F_1 = \frac{(f_1 - \alpha_1 f_2)}{(f_2 - \alpha_1 f_3)} \quad \text{and} \quad F_2 = \frac{(f_1 - \alpha_2 f_2)}{(f_2 - \alpha_2 f_3)}. \quad (58)$$

Making use of Eq. (35) in the above equation, we find

$$\frac{(u'' - \alpha_1)}{(u'' - \alpha_2)} = Q_1 \frac{(u' - \alpha_1)^k (u - \alpha_2)}{(u' - \alpha_2)^k (u - \alpha_1)}. \quad (59)$$

Using Eqs. (56) and (59) in the QRT mapping (1) together with constraint on the functions f_1, f_2 and f_3 given in (39)–(41) we find that it reduces to a linear second-order difference equation

$$w(u'') - kw(u') + w(u) = p, \quad (60)$$

where

$$p = \frac{1}{D(\alpha_1 - \alpha_2)} \log Q_1. \quad (61)$$

Then the general solution of Eq. (60) is

$$w(u) = c_1(x)\lambda_1^x + c_2(x)\lambda_2^x + \frac{p}{k-2}, \quad k \neq 2, \quad (62)$$

where λ_1 and λ_2 are roots of the equation

$$\lambda^2 - k\lambda + 1 = 0, \quad (63)$$

and $c_1(x)$ and $c_2(x)$ are arbitrary unit periodic functions and so

$$\log \frac{(u - \alpha_1)}{(u - \alpha_2)} = D(\alpha_1 - \alpha_2) \left[c_1(x)\lambda_1^x + c_2(x)\lambda_2^x + \frac{p}{k-2} \right], \quad k \neq 2 \quad (64)$$

and from which we can easily find the exact solution $u = u(x)$ of the QRT mapping (1). The integrals of motion for this case are

$$I_1(x, u, u') = \left[\log \frac{(u - \alpha_1)^{\lambda_2} (u' - \alpha_2)}{(u - \alpha_2)^{\lambda_2} (u' - \alpha_1)} + \frac{p}{2-k} (1 - \lambda_2) \right] \lambda_1^{-x}, \quad k \neq 2 \quad (65)$$

and

$$I_2(x, u, u') = \left[\log \frac{(u - \alpha_2)^{\lambda_1} (u' - \alpha_1)}{(u - \alpha_1)^{\lambda_1} (u' - \alpha_2)} + \frac{p}{2-k} (\lambda_1 - 1) \right] \lambda_2^{-x}, \quad k \neq 2, \quad (66)$$

where α_1 and α_2 satisfy (37), while λ_1 and λ_2 satisfy (63). Obviously, the above integrals of motion are independent. For the choice $C^2 = 4BD$, the homogenising variable $w(u)$ becomes

$$w(u) = -\frac{2}{2Du + C}. \quad (67)$$

Now

$$2Du'' + C = -\frac{(2Du' + C)(2Du + C)}{[(2Du' + C) - (2Du + C)(2Du' + C)Q_2 - k(2Du + C)]}. \quad (68)$$

Substituting the above equations (67) and (68), in the QRT mapping (1) along with f_1, f_2 and f_3 given in Eqs. (45)–(47) we find that it reduces to a linear second-order difference equation

$$w(u'') - kw(u') + w(u) = p, \quad (69)$$

where $p = -2Q_2$ and so the general solution is Eq. (62) and hence we obtain the corresponding exact solution of QRT mapping (1). The integrals of motion corresponding to this case are

$$I_1(x, u, u') = \left[\frac{u - \lambda_2 u' + \alpha(\lambda_2 - 1)}{D(u - \alpha)(u' - \alpha)} + \frac{p}{2-k} (\lambda_2 - 1) \right] \lambda_1^{-x}, \quad k \neq 2, \quad (70)$$

and

$$I_2(x, u, u') = \left[\frac{\lambda_1 u' - u + \alpha(1 - \lambda_1)}{D(u - \alpha)(u' - \alpha)} + \frac{p}{2-k} (1 - \lambda_1) \right] \lambda_2^{-x}, \quad k \neq 2. \quad (71)$$

where $\alpha = -C/2D$ and λ_1 and λ_2 satisfy (63). For the choice $C^2 - 4BD < 0$, the homogenising variable reads

$$w(u) = \frac{2}{\delta} \tan^{-1} \left(\frac{(2Du + C)}{\delta} \right), \quad (72)$$

where δ is given in (49). Proceeding further with the constraints on the functions f_1, f_2 and f_3 given in Eqs. (51)–(53), we find that the QRT mapping (1) reduces to a linear second-order difference equation

$$w(u'') - kw(u') + w(u) = p. \quad (73)$$

From Eq. (73), we find $w(u)$, Eq. (62), and its inversion gives the associated general solution of QRT mapping (1). Here also one can construct two integrals of motion.

4. Discussion

In this paper we have demonstrated that the method of Lie symmetries provides an effective technique to study nonlinear difference equations or mappings towards its linearisability or integrability. By applying the Lie symmetry method to QRT mapping we have shown that it can be linearised in general if it admits factorisable symmetry. In particular, by demanding that the QRT mapping is invariant under the symmetry vector field

$$X(x, u) = \alpha(x) \frac{\partial}{\partial x} + A(x)[B + Cu + Du^2] \frac{\partial}{\partial u}, \quad (74)$$

where B , C , and D are constants and $A(x)$ is an arbitrary function in x satisfying Eq. (12) we have derived condition on the functions f_1, f_2 and f_3 . Further for the identified choice we have transformed the QRT mapping (1) into a linear second-order difference equation with constant coefficient. Also, for the identified choice, we have explicitly constructed two independent integrals of motion for the QRT mapping (1) ensuring its integrability. We have also derived the exact solution of the QRT mapping in this case.

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References

- [1] M.J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
- [2] G.R.W. Quispel, H.W. Capel, V.G. Papageorgiou and F.W. Nijhoff, *Physica A* 173 (1991) 243.
- [3] M. Bruschi, O. Ragnisco, P.M. Santini and T.G. Gui-Zhang, *Physica D* 49 (1991) 273.
- [4] G.R.W. Quispel, J.A.G. Roberts and C.J. Thompson, *Physica D* 34 (1989) 183.
- [5] M.J. Ablowitz and J.F. Ladik, *J. Math. Phys.* 17 (1976) 1011.
- [6] H.W. Capel, F.W. Nijhoff, V.G. Papageorgiou and G.R.W. Quispel, in: *Integrable Mappings and Soliton Lattices in Solitons and Chaos*, (ed.) F. Lambert (Springer, Berlin, 1991).
- [7] B. Grammaticos, A. Ramani and V.G. Papageorgiou, *Phys. Rev. Lett.* 67 (1991) 1825.
- [8] N. Joshi, D. Burtonclay and R.G. Habbard, *Lett. Math. Phys.* 26 (1992) 123; A. Ramani, B. Grammaticos and J. Satsuma, *J. Phys. A* 28 (1995) 4655.
- [9] A. Ramani, B. Grammaticos and J. Hietarinta, *Phys. Rev. Lett.* 67 (1991) 1829.
- [10] G.R.W. Quispel, H.W. Capel and R. Sahadevan, *Phys. Lett. A* 170 (1992) 379.
- [11] G.R.W. Quispel and R. Sahadevan, *Phys. Lett. A* 184 (1993) 64.
- [12] S. Maeda, *IMA J. Appl. Math.* 38 (1987) 129.
- [13] P. Winternitz and D. Levi, *Phys. Lett. A* 152 (1991) 335.
- [14] R. Sahadevan, G.B. Byrnes and G.R.W. Quispel, Linearisation of difference equations using factorisable Lie symmetries, in: *Proc. Workshop on Symmetries and Integrability of Difference Equations*, Esterl, eds., D. Levi, L. Vinet and P. Winternitz (CRM Proc. and Lecture Notes Series) (AMS, Providence, R Island, 1996).
- [15] G.B. Byrnes, R. Sahadevan and G.R.W. Quispel, *Nonlinearity* 8 (1995) 443.
- [16] A. Ramani, B. Grammaticos and G. Kara, *Physica A* 180 (1992) 115.
- [17] C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).