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LINEARIZATION OF THE NONLINEAR SCHRÖDINGER EQUATION AND THE ISOTROPIC HEISENBERG SPIN CHAIN

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A new description in terms of one and the same linear inhomogeneous integral equation is proposed for the nonlinear Schrödinger equation (NLS), as well as for the equation of motion for the classical isotropic Heisenberg spin chain in the continuum limit (IHSC). From the integral equation which contains a two-fold integration over an arbitrary contour in the complex plane with an arbitrary measure one can obtain the various solutions of the NLS as well as of the IHSC in a direct way without going through the details of the inverse scattering formalism. Well-known properties such as the Miura transformation, the Gel'fand–Levitan equation and the Lax representations for NLS and IHSC can be derived as a corollary from the integral equation. The treatment leads also to a few more general (integrable) partial differential equations which contain the NLS and the IHSC as special cases.

1. Introduction

In recent years there has been a lot of interest in the Heisenberg spin chain in the continuum limit. For the classical isotropic case explicit solutions of the equation of motion

$$\partial_t \boldsymbol{S} = \boldsymbol{S} \times \partial_x^2 \boldsymbol{S} + \boldsymbol{S} \times \boldsymbol{B}, \qquad \boldsymbol{S} \cdot \boldsymbol{S} = 1, \tag{1.1}$$

where $\boldsymbol{S} = (S^x, S^y, S^z)$ denotes the spin vector and $\boldsymbol{B}$ an (external) magnetic field, were found by Lakshmanan, Ruijgrok and Thompson[1]) and by Tjon and Wright[2]). A description in terms of one real variable and the corresponding potential equation which is also valid in the case of axial symmetry have been given in refs. 3 and 4. An important development in the study of eq. (1.1) was the discovery of an inverse scattering scheme and thus the proof of complete integrability by Takhtadzhyan[5]). At an earlier stage the inverse scattering method had already been established for the nonlinear Schrödinger equation

$$\mathrm{i}\partial_t \phi + \partial_x^2 \phi + 2|\phi|^2 \phi = 0 \tag{1.2}$$

by Zakharov and Shabat[6]). The corresponding potential equation in terms of one real variable was given in refs. 7, 3 and 4.

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A close connection between the equation of motion (1.1) for the isotropic classical Heisenberg spin chain (IHSC) and the nonlinear Schrödinger equation (NLS) was found by Lakshmanan[8]), see also ref. 4 for the relation between the corresponding potential equations. A more profound understanding of this connection in the framework of the inverse scattering formalism was established by Zakharov and Takhtadzhyan[9]). Very recently a Miura transformation[10]), analogous to the well-known transformation[11]) between the modified Korteweg–de Vries equation (MKdV) and the Korteweg–de Vries equation (KdV), has been discovered. This transformation maps the solutions of the classical (continuous) Heisenberg spin chain with uniaxial symmetry (AHSC) which has also been proved to be an integrable system[12,13]), on solutions of the NLS. In the isotropic case this transformation reduces to the one given by Lakshmanan[8]). Explicit solutions of the NLS and the IHSC may be obtained using the inverse scattering formalism[5,6,9]). For a systematic treatment of the inverse scattering method for the NLS and IHSC, see also refs. 14 and 15 respectively.

Very recently Fokas and Ablowitz[16]) have proposed a linear inhomogeneous integral equation for the linearization of the Korteweg–de Vries equation. The essential feature is that this integral equation contains an integration over an arbitrary contour in the complex k-plane with an arbitrary measure. By an appropriate choice of contour and measure the various solutions of the KdV can be obtained directly without going through the details of the inverse scattering formalism. (The various ingredients of this formalism such as the Gel'fand–Levitan equation can be derived as a corollary from the integral equation.) Furthermore a general discussion of the Gel'fand-Levitan equation was given in a paper by Zakharov and Shabat[17]), see also ref. 18, and some other ways to extend or to bypass the inverse scattering formalism have been developed in refs. 19–21. The form of the integral equation in ref. 16 may be inferred from the treatment by Rosales[20]) on the basis of a power series expansion.

In the present paper the connections between the IHSC and the NLS are shown to arise in a natural way from a description in which solutions of both equations can be derived from one and the same linear inhomogeneous integral equation†. The integral equation has the form

$$\phi_k(x,t) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')(l'-l)}\,\phi_l(x,t) = \mathrm{e}^{\mathrm{i}(kx-k^2t)}, \tag{1.3}$$

† A similar connection has been found very recently by Hirota[22]) who showed that the NLS and the IHSC can be transformed into the same bilinear form. The relation of the treatment in ref. 22 with our method or with the usual inverse scattering method has not yet been clarified.

where $d\lambda(k)$ is an arbitrary measure over one complex variable k and the integrations are performed over an arbitrary contour and its complex conjugate C^*.

Assuming that the homogeneous integral equation, i.e. (1.3) with the right-hand side replaced by zero, has only the zero solution, it will be shown that

$$\phi(x,t) = \int_C d\lambda(k)\phi_k(x,t) \tag{1.4}$$

is a solution of the NLS and that

$$\tilde{\phi}(x,t) = \int_C d\lambda(k)k^{-1}\phi_k(x,t) \tag{1.5}$$

yields the spin vector $\boldsymbol{S}(x,t)$, associated with a solution of the IHSC. This means that solutions of (1.1) and (1.2) can be found choosing an appropriate contour and measure in (1.3). The form of the integral equation (1.3) may be inferred from the treatment in ref. 20 on the basis of a power series expansion, but this will not be treated in more detail here. A brief account of some of the considerations of the present paper can be found in ref. 23, where it was also shown that solutions of the MKdV can be obtained directly from the integral equation proposed by Fokas and Ablowitz[16]) for the linearization of the KdV, see also ref. 24 for further details.

In section 2 of this paper we investigate a slightly more general class of linear integral equations in which the source term $e^{i(kx-k^2t)}$ in the right-hand side is replaced by $k^{-n}\,e^{i(kx-k^2t)}$ (n integer), or more generally by $G(k)\,e^{i(kx-k^2t)}$, where $G(k) = \Sigma_n\,\lambda_n k^{-n}$ is an arbitrary meromorphic function of k. The integral equation with source term $k^{-n}\,e^{i(kx-k^2t)}$ leads to the so-called NLS–IHSC hierarchy consisting of the relations for the functions $\phi^{(n)}(x,t)$ which can be obtained from (1.4) in which $\phi_k^{(n)}$ is the solution of (1.3) with $k^{-n}\,e^{i(kx-k^2t)}$ in the right-hand side. The integral equation (1.3) with general meromorphic $G(k)$ leads to two coupled (integrable) partial differential equations (PDE's) for two fields $\phi(x,t)$ and $\psi(x,t)$ which are defined by (1.4) and by

$$\psi(x,t) = \int_C d\lambda(k)\psi_k(x,t),$$

$$\psi_k(x,t) = \int_C d\lambda^*(l')\,\frac{e^{i(kx-k^2t)}}{k-l'}\,\phi^*_{l'}(x,t), \tag{1.6}$$

where $\phi_{l'}(x,t)$ is any solution of (1.3) with a factor $G(k)$ in the right-hand side.

In particular, all linear combinations of functions $\phi^{(n)}(x, t)$ (and $\psi^{(n)}(x, t)$) for different values of n belonging to the NLS–IHSC hierarchy are solutions of the two coupled PDE's. From the two coupled PDE's we derive also one partial differential equation in terms of only $\psi(x, t)$.

In section 3 we derive the linear problem for the functions $\phi_k(x, t)$ (and the corresponding functions $\psi_k(x, t)$). We obtain differential equations which are linear in the functions $\phi_k(x, t)$ and $\psi_k(x, t)$, but the solutions $\phi(x, t)$ and $\psi(x, t)$ and their derivatives appear explicitly in the coefficients, as is the usual situation for the linear problem associated with a nonlinear integrable PDE. From the relations of the linear problem we derive (matrix) PDE's for the matrices $\mathbf{\Phi}$ and $\mathbf{\Psi}$ with the elements

$$\phi_{n,m}(x, t) = \int_C \mathrm{d}\lambda(k)\phi_k^{(n)}(x, t)k^{-m}, \tag{1.7}$$

$$\psi_{n,m}(x, t) = \int_C \mathrm{d}\lambda(k)\psi_k^{(n)}(x, t)k^{-m}. \tag{1.8}$$

In section 4 the special case $n = m = 0$ is treated and it is shown that $\phi_{0,0}(x, t)$, defined by (1.7) for $n = m = 0$ or by (1.4), is a solution of the NLS, whereas $\psi_{0,0}(x, t)$ is directly related to a solution of the potential nonlinear Schrödinger equation in terms of one real variable which has been introduced in refs. 7, 3 and 4. Finally there is a one to one correspondence between the solutions $\psi_{n,m}(x, t)$ for $n = m = 0$ on the one hand, and $n = -1$, $m = 0$ or $n = 0$, $m = -1$ on the other hand.

In section 5 we deal with the special cases $n = 1$, $m = 0$; $n = 0$, $m = 1$, and $n = m = 1$. The functions $\phi_{n,m}(x, t)$ (and also $\psi_{n,m}(x, t)$) in these cases are directly related to the components of the spin vector $\boldsymbol{S}(x, t)$ satisfying the IHSC. The Miura transformation between IHSC and NLS is obtained as a corollary of the transformation formulae in sections 2 and 3.

The linear integral equation (1.3), (or a more general one with $G(k)$ in the right-hand side), provides a general description of solutions of the NLS, the IHSC and related equations. Various features of the usual inverse scattering formalism can be obtained as a corollary and in appendices A, C and D we discuss the Gel'fand–Levitan equation, the Lax representation for the NLS found by Zakharov and Shabat[6]) and the Lax representation for the IHSC found by Takhtadzhyan[5]). In appendix E an explicit formulation of the gauge equivalence[9]) of both Lax representations is given. In appendix F we present a few details concerning the two-soliton solution of the IHSC.

2. The linear integral equation

2.1. *Derivation of the NLS–IHSC hierarchy*

We first consider a class of linear inhomogeneous integral equations of the form

$$\phi_k^{(n)}(x,t) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')(l'-l)}\,\phi_l^{(n)}(x,t)$$
$$= \frac{1}{k^n}\,\mathrm{e}^{\mathrm{i}(kx-k^2t)}, \tag{2.1}$$

i.e. eq. (1.3) with an extra factor k^{-n} in the right-hand side. Here n can be taken to be an integer, $\mathrm{d}\lambda(k)$ is an arbitrary measure over one complex variable k and the integrations are performed over an arbitrary contour C and its complex conjugate C^*. The measure and the contour are chosen such that the homogeneous integral equation has only the zero solution, i.e.

$$f_k(x,t) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')(l'-l)}\,f_l(x,t) = 0 \Rightarrow f_k(x,t) = 0. \tag{2.2}$$

Furthermore we assume that the differentiations with respect to x and t may be shifted through the integrals.

For later purpose we introduce the conjugate functions

$$\psi_k^{(n)}(x,t) = \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}}{k-l'}\,\phi_{l'}^{(n)*}(x,t), \tag{2.3}$$

cf. eq. (1.6) in the special case that $\phi_{l'}^* = \phi_{l'}^{(n)*}$, $\psi_k = \psi_k^{(n)}$. From the complex conjugate of (2.1), i.e.

$$\phi_{k'}^{(n)*}(x,t) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{-\mathrm{i}(k'x-k'^2t)}\,\mathrm{e}^{\mathrm{i}(lx-l^2t)}}{(k'-l)(l-l')}\,\phi_{l'}^{(n)*}(x,t)$$
$$= \frac{1}{k'^n}\,\mathrm{e}^{-\mathrm{i}(k'x-k'^2t)}, \tag{2.4}$$

and the definition (2.3), it is straightforward to derive the integral equation

$$\psi_k^{(n)}(x,t) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')(l'-l)}\,\psi_l^{(n)}(x,t)$$

$$= \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{l'^n(k-l')} \tag{2.5}$$

for the functions $\psi_k^{(n)}(x,t)$. Note that both integral equations (2.1) and (2.5) have the same kernel and in the following use will be made of (2.2) in order to derive various relations between the functions $\phi_k^{(n)}(x,t)$, $\psi_k^{(n)}(x,t)$ and their partial derivatives with respect to x and t. For that purpose we consider first the time derivatives and next the derivatives with respect to x.

The time evolution of $\phi_k^{(n)}$ and $\psi_k^{(n)}$ can be described in terms of the operator

$$M = \mathrm{i}\partial_t + \partial_x^2, \tag{2.6}$$

which has been chosen in such a way that the right-hand side of (2.1) vanishes under the action of M. Applying the operator M to eqs. (2.1) and (2.5) we find

$$M\phi_k^{(n)} + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')(l'-l)}\,M\phi_l^{(n)}$$

$$= -2\mathrm{i}(\partial_x\psi^{(n)*})\,\mathrm{e}^{\mathrm{i}(kx-k^2t)}, \tag{2.7}$$

and

$$M\psi_k^{(n)} + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')(l'-l)}\,M\psi_l^{(n)}$$

$$= 2\mathrm{i}(\partial_x\phi^{(n)*})\,\mathrm{e}^{\mathrm{i}(kx-k^2t)}, \tag{2.8}$$

where

$$\phi^{(n)}(x,t) = \int_C \mathrm{d}\lambda(k)\phi_k^{(n)}(x,t), \tag{2.9}$$

$$\psi^{(n)}(x,t) = \int_C \mathrm{d}\lambda(k)\psi_k^{(n)}(x,t). \tag{2.10}$$

Eq. (2.7) follows from (2.1), (2.3) and (2.10). The differentiations with respect to the exponentials in (2.5) give rise to the factor

$$-2\mathrm{i}\partial_x\left[\int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{l'-l}\,\psi_l^{(n)}\right] + 2\int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{l'^{n-1}}$$

occurring in the right-hand side of (2.8). Using the definition (2.3) for $\psi_l^{(n)}$ and the integral equation (2.4) for $\phi_{k'}^{(n)*}$ this factor can be written

$$-2\mathrm{i}\partial_x\left[\iint_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \int_{C^*} \mathrm{d}\lambda^*(k') \frac{\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}\,\mathrm{e}^{\mathrm{i}(lx-l^2t)}}{(l'-l)(l-k')}\,\phi_{k'}^{(n)*}\right]$$

$$+2\int_{C^*} \mathrm{d}\lambda^*(l')\frac{\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{l'^{n-1}} = 2\mathrm{i}(\partial_x\phi^{(n)*}). \tag{2.11}$$

Subtracting (2.7) from (2.1) we find that the function

$$f_k(x,t) = M\phi_k^{(n)} + 2\mathrm{i}(\partial_x\psi^{(n)*})\phi_k^{(0)} \tag{2.12}$$

satisfies eq. (2.2) and therefore

$$(\mathrm{i}\partial_t + \partial_x^2)\phi_k^{(n)} = -2\mathrm{i}(\partial_x\psi^{(n)*})\phi_k^{(0)}. \tag{2.13}$$

In the same way we find by comparing (2.8) and (2.1) and applying (2.2)

$$(\mathrm{i}\partial_t + \partial_x^2)\psi_k^{(n)} = 2\mathrm{i}(\partial_x\phi^{(n)*})\phi_k^{(0)}. \tag{2.14}$$

Eqs. (2.13) and (2.14) determine the time evolution of the functions $\phi_k^{(n)}$ and $\psi_k^{(n)}$. We now investigate the spatial derivatives. Differentiating (2.1) with respect to x we find

$$-\mathrm{i}\partial_x\phi_k^{(n)} + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l')\frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')(l'-l)}(-\mathrm{i}\partial_x\phi_l^{(n)})$$

$$= \left(\frac{1}{k^{n-1}} - \psi^{(n)*}\right)\mathrm{e}^{\mathrm{i}(kx-k^2t)}, \tag{2.15}$$

and from (2.5) using an argument similar to the one leading to (2.11), but without the derivatives ∂_x, it follows that

$$-\mathrm{i}\partial_x\psi_k^{(n)} + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l')\frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')(l'-l)}(-\mathrm{i}\partial_x\psi_l^{(n)})$$

$$= \phi^{(n)*}\,\mathrm{e}^{\mathrm{i}(kx-k^2t)}. \tag{2.16}$$

Comparing (2.15) with the integral equation (2.1) and using again (2.2), we obtain the relation

$$-\mathrm{i}\partial_x\phi_k^{(n)} = \phi_k^{(n-1)} - \psi^{(n)*}\phi_k^{(0)}, \tag{2.17}$$

and from (2.16), (2.1) and (2.2) we have

$$-\mathrm{i}\partial_x\psi_k^{(n)} = \phi^{(n)*}\phi_k^{(0)}. \tag{2.18}$$

Eqs. (2.13), (2.14), (2.17) and (2.18) are satisfied by every solution $\phi_k^{(n)}(x, t)$, $\psi_k^{(n)}(x, t)$ of the set of linear integral equations (2.1), (2.5). These four equations may be regarded as the constitutive relations of the NLS–IHSC hierarchy. Using the definitions (2.9) and (2.10) for the functions $\phi^{(n)}(x, t)$ and $\psi^{(n)}(x, t)$ it is clear that the corresponding relations for $\phi^{(n)}(x, t)$ and $\psi^{(n)}(x, t)$ can be obtained omitting the subscripts k in (2.13), (2.14), (2.17) and (2.18). Eqs. (2.13), (2.14), (2.17) and (2.18) will be further used in section 3 in the derivation of PDE's for the matrices $\mathbf{\Phi}$ and $\mathbf{\Psi}$, cf. eqs. (1.7) and (1.8). Furthermore (2.13), (2.14), (2.17) and (2.18) may be regarded as special cases of more general relations which will be derived from an integral equation of the type (1.3) with source term $G(k)\,\mathrm{e}^{\mathrm{i}(kx-k^2t)}$.

2.2. *General source term; coupled partial differential equations*

In this subsection we consider the more general integral equation

$$\phi_k(x, t) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')(l'-l)}\,\phi_l(x, t) = G(k)\,\mathrm{e}^{\mathrm{i}(kx-k^2t)}, \tag{2.19}$$

in which

$$G(k) = \sum_n \lambda_n k^{-n} \tag{2.20}$$

is an arbitrary meromorphic function of k, the coefficients λ_n being complex numbers, independent of k, x and t.

Comparing (2.19) with (2.1) and using (2.2) it is clear that

$$\phi_k(x, t) = \sum_n \lambda_n \phi_k^{(n)}(x, t) \tag{2.21}$$

and from (2.4) and (2.5) it follows that the function

$$\psi_k(x, t) = \sum_n \lambda_n^* \psi_k^{(n)}(x, t) \tag{2.22}$$

satisfies the integral equation

$$\psi_k(x, t) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')(l'-l)}\,\psi_l(x, t) = \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{k-l'}\,G^*(l'), \tag{2.23}$$

in which $G^*(l') = \Sigma_n \lambda_n^* l'^{-n}$.

Since the relations (2.13), (2.14) and (2.18) are linear in the functions with superscript (n) we immediately have

$$(\mathrm{i}\partial_t + \partial_x^2)\phi_k = -2\mathrm{i}(\partial_x\psi^*)\phi_k^{(0)}, \tag{2.24}$$

$$(\mathrm{i}\partial_t + \partial_x^2)\psi_k = 2\mathrm{i}(\partial_x\phi^*)\phi_k^{(0)} \tag{2.25}$$

and

$$-\mathrm{i}\partial_x\psi_k = \phi^*\phi_k^{(0)}, \tag{2.26}$$

with

$$\phi(x, t) = \int_C \mathrm{d}\lambda(k)\phi_k(x, t) = \sum_n \lambda_n\phi^{(n)}(x, t),$$

$$\psi(x, t) = \int_C \mathrm{d}\lambda(k)\psi_k(x, t) = \sum_n \lambda_n^*\psi^{(n)}(x, t). \tag{2.27}$$

From (2.17) we also obtain the relation

$$-\mathrm{i}\partial_x\phi_k = \hat{\phi}_k - \psi^*\phi_k^{(0)}, \tag{2.28}$$

in which $\hat{\phi}_k = \Sigma_n \lambda_n\phi_k^{(n-1)}$ is the solution of the integral equation (2.19) with $\hat{G}(k) = G(k)k$ instead of $G(k)$.

Inserting (2.26) in (2.24) and (2.25) and integrating over C we obtain the two coupled partial differential equations

$$(\mathrm{i}\partial_t + \partial_x^2)\phi(x, t) = -2|\partial_x\psi(x, t)|^2(\phi^*(x, t))^{-1}, \tag{2.29}$$

$$(\mathrm{i}\partial_t + \partial_x^2)\psi(x, t) = 2(\partial_x \ln \phi^*(x, t))\partial_x\psi(x, t). \tag{2.30}$$

Eqs. (2.29) and (2.30) are integrable in the sense that solutions can be obtained from the linear integral equations (2.19) and (2.23) with arbitrary meromorphic $G(k)$. (All ϕ and ψ in (2.27) are solutions of (2.29) and (2.30).)

Integrating (2.26) and (2.28) over C we obtain the additional relations

$$-\mathrm{i}\partial_x\psi = \phi^*\phi^{(0)}, \tag{2.31}$$

$$-\mathrm{i}\partial_x\phi = \hat{\phi} - \psi^*\phi^{(0)}, \tag{2.32}$$

with $\hat{\phi} = \int \mathrm{d}\lambda(k)\hat{\phi}_k$. Eqs. (2.31) and (2.32) relate the solutions ϕ and ψ which can be found from (2.19) and (2.23) (with $G(k)$) to the solutions $\hat{\phi}$ and $\phi^{(0)}$ which can be found from (2.19) with $G(k)k$ and 1 instead of $G(k)$, respectively.

2.3. *Partial differential equation for $\psi(x, t)$*

From (2.29)–(2.31) one can also derive a PDE in terms of ψ. In fact, using

(2.31) to eliminate $\partial_x \ln \phi^*$ in the right-hand side of (2.30) we obtain

$$A = \frac{-\mathrm{i}\partial_t \psi + \partial_x^2 \psi}{2\partial_x \psi} = \partial_x \ln \phi^{(0)}. \tag{2.33}$$

From (2.29) and (2.31) in the special case $\phi = \phi^{(0)}$, $\psi = \psi^{(0)}$, cf. (2.19) and (2.23) with $G(k) = 1$, we find

$$(\mathrm{i}\partial_t + \partial_x^2)\phi^{(0)} = -2|\phi^{(0)}|^2 \phi^{(0)}, \tag{2.34}$$

which is the NLS (1.1), cf. (1.4). Dividing by $\phi^{(0)}$ and differentiating with respect to x, we have, cf. (2.33),

$$\mathrm{i}\partial_t A + \partial_x^2 A + 2A\partial_x A = -2\partial_x |\phi^{(0)}|^2. \tag{2.35}$$

Using

$$\partial_x |\phi^{(0)}|^2 = |\phi^{(0)}|^2 \partial_x \ln |\phi^{(0)}|^2 = (A + A^*)|\phi^{(0)}|^2 \tag{2.36}$$

in (2.35) one can express $|\phi^{(0)}|^2$ in terms of A and its derivatives. Inserting the result again in (2.35) we obtain the equation

$$\mathrm{i}\partial_t A + \partial_x^2 A + 2A\partial_x A - \partial_x[\mathrm{i}\partial_t A + \partial_x^2 A + 2A\partial_x A)(A + A^*)^{-1}] = 0 \tag{2.37}$$

for $A = \partial_x \ln \phi^{(0)}$, where $\phi^{(0)}$ satisfies the NLS.

By explicit calculation it can be shown that

$$A + A^* = \mathrm{Re}(\psi'' \psi'^{-1}) + \mathrm{Im}(\dot{\psi} \psi'^{-1}), \tag{2.38}$$

$$2(\mathrm{i}\partial_t A + \partial_x^2 A + 2A\partial_x A) = \frac{\ddot{\psi} + \psi''''}{\psi'} - \frac{2(\dot{\psi}\dot{\psi}' + \psi''\psi''')}{\psi'^2} + \frac{(\dot{\psi}^2 + \psi''^2)\psi''}{\psi'^3}, \tag{2.39}$$

where we have used dots and primes to denote the differentiations ∂_t and ∂_x with respect to t and x.

Inserting (2.38) and (2.39) in (2.37) we obtain a partial differential equation in terms of ψ, i.e.

$$\begin{aligned} &\frac{\ddot{\psi} + \psi''''}{\psi'} - \frac{2(\dot{\psi}\dot{\psi}' + \psi''\psi''')}{\psi'^2} + \frac{(\dot{\psi}^2 + \psi''^2)\psi''}{\psi'^3} \\ &\quad = \partial_x \left[\frac{(\ddot{\psi} + \psi'''')\psi'^{-1} - 2(\dot{\psi}\dot{\psi}' + \psi''\psi''')\psi'^{-2} + (\dot{\psi}^2 + \psi''^2)\psi''\psi'^{-3}}{\mathrm{Re}(\psi''\psi'^{-1}) + \mathrm{Im}(\dot{\psi}\psi'^{-1})} \right]. \end{aligned} \tag{2.40}$$

Eq. (2.40) is integrable in the sense that solutions can be obtained from the linear integral equation (2.23) with arbitrary meromorphic $G^*(l')$ and every linear combination of functions $\psi^{(n)}(x, t)$, cf. (2.5), (2.10), (2.22) and (2.27), is a solution of (2.40).

2.4. *Remarks*

i) The relations (2.31) and (2.32) can be used to obtain a transformation for the solutions of (2.40). We define a function $\hat{\psi}(x, t)$ by

$$\hat{\psi}(x, t) = \int_C \mathrm{d}\lambda(k)\hat{\psi}_k(x, t), \quad \hat{\psi}_k(x, t) = \sum_n \lambda_n^* \hat{\psi}_k^{(n-1)}(x, t), \tag{2.41}$$

where $\hat{\psi}_k$ is the solution of (2.23) with $\hat{G}^*(l') = l'G^*(l')$ instead of $G^*(l')$. From (2.31) for $\hat{\psi}$, (2.32) to eliminate $\hat{\phi}^*$, and again (2.31), we have

$$-\mathrm{i}\partial_x\hat{\psi} = \hat{\phi}^*\phi^{(0)} = (\partial_x\psi)\partial_x \ln \phi^* + \psi|\phi^{(0)}|^2. \tag{2.42}$$

Using (2.30) and also (2.33), (2.35) and (2.36) to express $\partial_x \ln \phi^*$ and $|\phi^{(0)}|^2$ in terms of ψ, we arrive at

$$-\mathrm{i}\partial_x\hat{\psi} = \tfrac{1}{2}(\mathrm{i}\dot{\psi} + \psi'') - \tfrac{1}{4}\psi\left\{\frac{(\ddot{\psi} + \psi'''')\psi'^{-1} - 2(\dot{\psi}\dot{\psi}' + \psi''\psi''')\psi'^{-2} + (\dot{\psi}^2 + \psi''^2)\psi''\psi'^{-3}}{\mathrm{Re}(\psi''\psi'^{-1}) + \mathrm{Im}(\dot{\psi}\psi'^{-1})}\right\} \tag{2.43}$$

From (2.33) for $\hat{\psi}$ and again (2.33) to express $\partial_x \ln \phi^{(0)}$ in terms of ψ we have

$$-\mathrm{i}\partial_t\hat{\psi} = -\mathrm{i}[\partial_x - (-\mathrm{i}\dot{\psi} + \psi'')\psi'^{-1}](-\mathrm{i}\partial_x\hat{\psi}), \tag{2.44}$$

in which $-\mathrm{i}\partial_x\hat{\psi}$ is given by (2.43). Eqs. (2.43) and (2.44) form a transformation mapping a solution ψ of (2.40) on another solution $\hat{\psi}$ of (2.40).

ii) Using (2.29)–(2.31) and in particular also (2.40) with $\psi = \psi^{(0)}$, one may also derive a PDE in terms of only $\phi(x, t)$. The result, however, is much more complicated than eq. (2.40) and will not be given here.

iii) From the integral equations presented in this section it is straightforward to derive the corresponding Gel'fand–Levitan equations. These equations may be regarded as general linear integral equations in coordinate space. (The reflection coefficient appears only after an explicit connection with the usual inverse scattering method has been made.) For some details we refer to appendix A.

3. The linear eigenvalue problem and matrix partial differential equations

3.1. *The linear eigenvalue problem*

In this subsection we show how to derive a linear eigenvalue problem associated with the equations (2.29) and (2.30). If we apply the operator $(-k - \mathrm{i}\partial_x)$ to the integral equation (2.1), we find

$$(-k-\mathrm{i}\partial_x)\phi_k^{(n)} + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')(l'-l)}(-l-\mathrm{i}\partial_x)\phi_l^{(n)}$$

$$= \phi^{(n)} \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{k-l'}. \tag{3.1}$$

Comparing eq. (3.1) with (2.5) for $n = 0$ and using (2.2) we have

$$(-k-\mathrm{i}\partial_x)\phi_k^{(n)} = \phi^{(n)}\psi_k^{(0)}. \tag{3.2}$$

A similar expression for $(-k-\mathrm{i}\partial_x)\psi_k^{(n)}$ can be found either by applying the operator $(-k-\mathrm{i}\partial_x)$ to the integral equation (2.5) or by using (2.17) and the definition (2.3). The result is

$$(-k-\mathrm{i}\partial_x)\psi_k^{(n)} = \psi^{(n)}\psi_k^{(0)} - \psi_k^{(n-1)}. \tag{3.3}$$

For the functions $\phi_k(x, t)$ and $\psi_k(x, t)$ which are defined by (2.21) and (2.22) and which are the solutions of the integral equations (2.19) and (2.23) we have the relations

$$(-k-\mathrm{i}\partial_x)\phi_k = \phi\psi_k^{(0)} \tag{3.4}$$

and

$$(-k-\mathrm{i}\partial_x)\psi_k = \psi\psi_k^{(0)} - \hat{\psi}_k, \tag{3.5}$$

where $\hat{\psi}_k$ and $\psi_k^{(0)}$ are the solutions of (2.23) with $l'G^*(l')$ and 1 instead of $G^*(l')$.

From (3.4) and (3.5) it is straightforward to derive a linear eigenvalue problem associated with the coupled PDE's (2.29) and (2.30). In fact, inserting (2.18) for $\partial_x\psi_k^{(0)}$ and eqs. (2.26) and (2.31) for $\phi_k^{(0)}$ and $\phi^{(0)*}$, we have

$$-\mathrm{i}\partial_x[\phi^{-1}(-k-\mathrm{i}\partial_x)\phi_k] = -\mathrm{i}\partial_x\psi_k^{(0)} = \phi^{(0)*}\phi_k^{(0)} = |\phi|^{-2}(\partial_x\psi^*)(\partial_x\psi_k). \tag{3.6}$$

From (3.5), applying (2.26) and (2.18) for $\partial_x\hat{\psi}_k$ and $\partial_x\psi_k^{(0)}$, eq. (2.32) to eliminate $\hat{\phi}^*$, and finally eq. (3.4) and (2.26) to express $\psi_k^{(0)}$ and $\phi_k^{(0)}$ in terms of ϕ, ψ, ϕ_k and ψ_k, we find

$$\begin{aligned}\mathrm{i}(-k-\mathrm{i}\partial_x)\partial_x\psi_k &= \mathrm{i}(\partial_x\psi)\psi_k^{(0)} - (\psi\phi^{(0)*} - \hat{\phi}^*)\phi_k^{(0)} \\ &= \mathrm{i}(\partial_x\psi)\phi^{-1}(-k-\mathrm{i}\partial_x)\phi_k + (\partial_x \ln \phi^*)\partial_x\psi_k.\end{aligned} \tag{3.7}$$

The time evolution of ϕ_k and ψ_k is given by

$$\mathrm{i}\partial_t\phi_k = -\partial_x^2\phi_k - 2(\partial_x\psi^*)\phi^{*-1}\partial_x\psi_k, \tag{3.8}$$

$$\mathrm{i}\partial_t\psi_k = -\partial_x^2\psi_k + 2(\partial_x \ln \phi^*)\partial_x\psi_k, \tag{3.9}$$

as follows from (2.24), (2.25) inserting (2.26) for $\phi_k^{(0)}$.

Eqs. (3.6) and (3.7) supplemented by eqs. (3.8) and (3.9) can be regarded as the linear eigenvalue problem associated with the PDE's (2.29) and (2.30). In (3.6)–(3.9) the ϕ_k and ψ_k are the eigenfunctions for the eigenvalue k, and ϕ and ψ play the role of the potentials. Note that as a consequence of our treatment eqs. (2.9) and (2.10) provide a simple basic relation between the eigenfunctions on the one hand and the potentials on the other hand.

From (3.6) and (3.7) it is easy to eliminate ϕ_k to obtain an eigenvalue problem containing only $\partial_x\psi_k$. The result is

$$\mathrm{i}\partial_x[(-\mathrm{i}\partial_x - \mathrm{i}(\partial_x \ln \partial_x\psi)]\eta_k] + |\phi^{(0)}|^2\eta_k - \partial_x[(\partial_x \ln \phi^*)\eta_k] = \mathrm{i}k\partial_x\eta_k, \tag{3.10}$$

where

$$\eta_k = (\partial_x\psi)^{-1}\partial_x\psi_k, \tag{3.11}$$

and $|\phi^{(0)}|^2$ and $\partial_x \ln \phi^*$ can be expressed in terms of ψ and its derivatives using (2.35), (2.36) and (2.30) respectively.

The linear eigenvalue problem will be further investigated in appendices C and D, where we derive the Lax representation of Zakharov and Shabat[6]) for the NLS, and of Takhtadzhyan[5]) for the IHSC as special cases.

3.2. *Matrix partial differential equations*

In this subsection we derive PDE's for the matrices $\mathbf{\Phi}$ and $\mathbf{\Psi}$, the elements of which are defined by

$$\phi_{n,m}(x,t) = \int_C \mathrm{d}\lambda(k)\phi_k^{(n)}(x,t)k^{-m}, \quad \psi_{n,m}(x,t) = \int_C \mathrm{d}\lambda(k)\psi_k^{(n)}(x,t)k^{-m}, \tag{3.12}$$

in which $\phi_k^{(n)}(x,t)$ and $\psi_k^{(n)}(x,t)$ are the solutions of (2.1) and (2.5). Comparing (3.12) with (2.9) and (2.10) we have in particular

$$\phi^{(n)}(x,t) = \phi_{n,0}(x,t), \ \psi^{(n)}(x,t) = \psi_{n,0}(x,t). \tag{3.13}$$

Dividing (2.13), (2.14), (2.17), (2.18), (3.2) and (3.3) by k^m and integrating over C, we have

$$(\mathrm{i}\partial_t + \partial_x^2)\phi_{n,m} = -2\mathrm{i}(\partial_x\psi_{n,0}^*)\phi_{0,m}, \tag{3.14}$$

$$(\mathrm{i}\partial_t + \partial_x^2)\psi_{n,m} = 2\mathrm{i}(\partial_x\phi_{n,0}^*)\phi_{0,m}, \tag{3.15}$$

$$-\mathrm{i}\partial_x\phi_{n,m} = \phi_{n-1,m} - \psi_{n,0}^*\phi_{0,m}, \tag{3.16}$$

$$-\mathrm{i}\partial_x\psi_{n,m} = \phi_{n,0}^*\phi_{0,m}, \tag{3.17}$$

$$-\phi_{n,m-1} - \mathrm{i}\partial_x\phi_{n,m} = \phi_{n,0}\psi_{0,m}, \tag{3.18}$$

$$-\psi_{n,m-1} - \mathrm{i}\partial_x\psi_{n,m} = \psi_{n,0}\psi_{0,m} - \psi_{n-1,m}. \tag{3.19}$$

With the special matrices $\mathbf{O}$ and $\mathbf{J}$ with elements

$$O_{m,n} = \delta_{m,0}\delta_{n,0}, \quad J_{m,n} = \delta_{n,m+1}, \tag{3.20}$$

eqs. (3.14)–(3.19) can be expressed as coupled partial differential equations in terms of the matrices $\mathbf{\Phi}$ and $\mathbf{\Psi}$ with elements $\phi_{m,n}$ and $\psi_{m,n}$, cf. (3.12). The result is

$$(\mathrm{i}\partial_t + \partial_x^2)\mathbf{\Phi} = -2\mathrm{i}(\partial_x\mathbf{\Psi}^*)\cdot\mathbf{O}\cdot\mathbf{\Phi}, \tag{3.21}$$

$$(\mathrm{i}\partial_t + \partial_x^2)\mathbf{\Psi} = 2\mathrm{i}(\partial_x\mathbf{\Phi}^*)\cdot\mathbf{O}\cdot\mathbf{\Phi}, \tag{3.22}$$

$$-\mathrm{i}\partial_x\mathbf{\Phi} = \mathbf{J}^{\mathrm{T}}\cdot\mathbf{\Phi} - \mathbf{\Psi}^*\cdot\mathbf{O}\cdot\mathbf{\Phi}, \tag{3.23}$$

$$-\mathrm{i}\partial_x\mathbf{\Psi} = \mathbf{\Phi}^*\cdot\mathbf{O}\cdot\mathbf{\Phi}, \tag{3.24}$$

$$-\mathbf{\Phi}\cdot\mathbf{J} - \mathrm{i}\partial_x\mathbf{\Phi} = \mathbf{\Phi}\cdot\mathbf{O}\cdot\mathbf{\Psi}, \tag{3.25}$$

$$-\mathbf{\Psi}\cdot\mathbf{J} - \mathrm{i}\partial_x\mathbf{\Psi} = \mathbf{\Psi}\cdot\mathbf{O}\cdot\mathbf{\Psi} - \mathbf{J}^{\mathrm{T}}\cdot\mathbf{\Psi}, \tag{3.26}$$

where $\mathbf{J}^{\mathrm{T}}$ is the transposed matrix of $\mathbf{J}$ with elements $J^{\mathrm{T}}_{m,n} = \delta_{m,n+1}$.

Using (3.21)–(3.26) it is straightforward to write down the corresponding relations for the matrices $\mathbf{\Phi}^{\mathrm{T}}$ and $\mathbf{\Psi}\dagger$, where $\mathbf{\Phi}^{\mathrm{T}}$ is the transposed matrix with elements $\phi_{m,n} = \phi_{n,m}$ and $\mathbf{\Psi}\dagger$ the hermitean adjoint matrix with elements $\psi^{\dagger}_{m,n} = \psi^*_{n,m}$. Comparing these relations with (3.21)–(3.26) it is clear that $\mathbf{\Phi}^{\mathrm{T}}$ and $-\mathbf{\Psi}\dagger$ satisfy the same set of equations as $\mathbf{\Phi}$ and $\mathbf{\Psi}$. Therefore,

$$\mathbf{\Phi}^{\mathrm{T}} = \mathbf{\Phi}, \quad \mathbf{\Psi}\dagger = -\mathbf{\Psi}. \tag{3.27}$$

A more explicit proof of (3.27) which is based on the integral equation (2.1) and eq. (2.3), and which leads also to bilinear expressions for the functions $\phi_{n,m}$ and $\psi_{n,m}$ in terms of the eigenfunctions of the linear problem, is given in appendix B.

In view of (3.27), eq. (3.25) is equivalent to eq. (3.23); (3.24) and (3.26) can be replaced by (3.24) and the algebraic relation

$$-\mathbf{\Psi}\cdot\mathbf{J} + \mathbf{J}^{\mathrm{T}}\cdot\mathbf{\Psi} = \mathbf{\Psi}\cdot\mathbf{O}\cdot\mathbf{\Psi} - \mathbf{\Phi}^*\cdot\mathbf{O}\cdot\mathbf{\Phi}. \tag{3.28}$$

From (3.21) and (3.22) it is easy to derive partial differential equations containing only the matrix $\mathbf{\Phi}$ (or $\mathbf{\Psi}$). For the matrix $\mathbf{\Phi}$ we have

$$(\mathrm{i}\partial_t + \partial_x^2)\mathbf{\Phi} = -2\mathbf{\Phi}\cdot\mathbf{O}\cdot\mathbf{\Phi}^*\cdot\mathbf{O}\cdot\mathbf{\Phi}, \tag{3.29}$$

which may be regarded as a matrix generalization of the NLS and from (3.22)–(3.24) it can be shown that

$$(\mathrm{i}\partial_t + \partial_x^2)\mathbf{\Psi} = -2\mathrm{i}\mathbf{J}^{\mathrm{T}}\cdot\partial_x\mathbf{\Psi} + 2\mathrm{i}\mathbf{\Psi}\cdot\mathbf{O}\cdot\partial_x\mathbf{\Psi}. \tag{3.30}$$

Both PDE's (3.29) and (3.30) are integrable in the sense that solutions can be obtained from the linear integral equations (2.1) and (2.5) respectively, using (3.12).

4. The nonlinear Schrödinger equation

4.1. *Nonlinear Schrödinger equation and potential nonlinear Schrödinger equation (the case $n = m = 0$)*

From eqs. (2.29) and (2.31) taking $\psi = \psi^{(0)}$, $\phi = \phi^{(0)}$, or from eq. (3.29) taking the $(0, 0)$-element of the matrices in the left- and right-hand side we obtain eq. (2.34), i.e.

$$(\mathrm{i}\partial_t + \partial_x^2)\phi^{(0)} = -2|\phi^{(0)}|^2\phi^{(0)} \tag{4.1}$$

which is the nonlinear Schrödinger equation, cf. (1.2) and (1.4). Solutions of the NLS can be obtained using the linear integral equation (2.1) for $n = 0$, or (2.19) with $G(k) = 1$, or (1.3), for various choices of the contour C and the measure $\mathrm{d}\lambda(k)$.

We can also obtain a partial differential equation for $\psi^{(0)}$, using (2.33) with $\psi = \psi^{(0)}$ and the relation $-\mathrm{i}\partial_x\psi^{(0)} = |\phi^{(0)}|^2$, cf. (2.31), in (2.35). The result is, cf. (2.39) with $\psi = \psi^{(0)}$,

$$\frac{\ddot{\psi}^{(0)} + \psi^{(0)''''}}{\psi^{(0)'}} - \frac{2(\dot{\psi}^{(0)}\dot{\psi}^{(0)'} + \psi^{(0)''}\psi^{(0)'''})}{\psi^{(0)'2}} + \frac{(\dot{\psi}^{(0)2} + \psi^{(0)''2})\psi^{(0)''}}{\psi^{(0)'3}} - 4\mathrm{i}\psi^{(0)''} = 0. \tag{4.2}$$

On substituting $\psi^{(0)} = 2\mathrm{i}y$, eq. (4.2) becomes identical to the potential NLS equation in terms of one real variable y, derived in refs. 7 and 4. Solutions of (4.2) can be obtained using the integral equation (2.5) for $n = 0$, or (2.23) with $G^*(l') = 1$, for various choices of the contour C and the measure $\mathrm{d}\lambda(k)$.

From every solution of (4.2) one can obtain a solution of the NLS noting that

$$\phi^{(0)} = |\partial_x\psi^{(0)}|^{1/2} \exp\left[\mathrm{i}\int_\Gamma (\mathrm{d}l_x \operatorname{Im} \partial_x \ln \phi^{(0)} + \mathrm{d}l_t \operatorname{Im} \partial_t \ln \phi^{(0)})\right], \tag{4.3}$$

where Γ is an arbitrary curve going from $(0, 0)$ to (x, t) and $\mathrm{d}l = (\mathrm{d}l_x, \mathrm{d}l_t)$ is an infinitesimal two-dimensional vector tangent to Γ. The terms in the exponent on the right-hand side of eq. (4.3) can be evaluated in terms of $\psi^{(0)}$ as follows. From (2.33) with $\psi = \psi^{(0)}$ and the relation

$$-\mathrm{i}\partial_t \ln \phi^{(0)} = \partial_x^2 \ln \phi^{(0)} + (\partial_x \ln \phi^{(0)})^2 + 2|\phi^{(0)}|^2, \tag{4.4}$$

cf. (4.1), it can be shown that

$$\operatorname{Im} \partial_x \ln \phi^{(0)} = -\tfrac{1}{2}\dot{\psi}\psi^{(0)'-1}, \tag{4.5}$$

$$\operatorname{Im} \partial_t \ln \phi^{(0)} = \tfrac{1}{2}\partial_x(A + A^*) + \tfrac{1}{2}(A^2 + A^{*2}) + 2|\phi^{(0)}|^2$$
$$= \frac{\psi^{(0)'''}}{2\psi^{(0)'}} - \frac{(\dot{\psi}^{(0)2} + \psi^{(0)''2})}{4\psi^{(0)'2}} - 2\mathrm{i}\psi^{(0)'}, \tag{4.6}$$

in agreement with the results of ref. 4, ($\psi^{(0)} = 2\mathrm{i}y$). The compatibility relation for $\operatorname{Im} \ln \phi^{(0)}$, i.e. $\partial_t(\operatorname{Im} \partial_x \ln \phi^{(0)}) = \partial_x(\operatorname{Im} \partial_t \ln \phi^{(0)})$, leads again to (4.2).

It may be noted that every real or imaginary solution of the general PDE (2.40) is directly related to the potential NLS. In fact, for real or imaginary ψ, we have, integrating (2.40) with respect to x,

$$\frac{\ddot{\psi} + \psi''''}{\psi'} - \frac{2(\dot{\psi}\dot{\psi}' + \psi''\psi''')}{\psi'^2} + \frac{(\dot{\psi}^2 + \psi''^2)\psi''}{\psi'^3} = c(t)\psi'', \tag{4.7}$$

implying with (2.35), (2.39) and (2.31) that $\psi'' = 4\mathrm{i}\psi^{(0)''}c^{-1}(t)$. From the fact that ψ is real or imaginary together with (2.33), it can be inferred that $\psi = 4\mathrm{i}\psi^{(0)}c^{-1}$, where $c(t) = c$, independent of t. Therefore, real or imaginary solutions of (2.40) can be obtained from (2.23) in the special case that $G^*(l') = 4\mathrm{i}c^{-1}$, c being imaginary or real.

A linear eigenvalue problem for the functions $\phi_k^{(0)}(x, t)$, $\psi_k^{(0)}(x, t)$, which are solutions of (2.1) and (2.5) for $n = 0$, can be inferred from (2.13), (2.14), (2.18) and (3.2) substituting $n = 0$. As an example we give the linear relations for $\psi_k^{(0)}(x, t)$:

$$L\psi_k^{(0)} = \mathrm{i}k\partial_x\psi_k^{(0)}, \tag{4.8}$$

with

$$L = \partial_x^2 - \frac{\mathrm{i}\dot{\psi}^{(0)} + \psi^{(0)''}}{2\psi^{(0)'}}\,\partial_x - \mathrm{i}\psi^{(0)'} \tag{4.9}$$

and

$$\mathrm{i}\partial_t\psi_k^{(0)} = R\,\mathrm{i}\partial_x\psi_k^{(0)}, \tag{4.10}$$

with

$$R = \mathrm{i}\partial_x - \mathrm{i}\,\frac{\mathrm{i}\dot{\psi}^{(0)} + \psi^{(0)''}}{\psi^{(0)'}} \tag{4.11}$$

The compatibility condition, i.e.

$$\partial_t L = -[LR\partial_x - \partial_x RL], \tag{4.12}$$

leads again to the potential NLS (4.2). It is also straightforward to derive the Lax representation given by Zakharov and Shabat[6]). For details we refer to appendix C.

4.2. *Second potential nonlinear Schrödinger equation (the case $n=-1$, $m=0$, or $n=0$, $m=-1$)*

The transformation (2.43), (2.44) can be used to derive the relation between the functions $\psi^{(-1)}$ and $\psi^{(0)}$ which can be obtained using (2.10) from the integral equation (2.5) for $n=-1$ and $n=0$ respectively, or from (2.23) with $G^*(l')=l'$ and $G^*(l')=1$ respectively.

In fact, using $\psi=\psi^{(0)}$, the relation $A+A^*=\psi^{(0)''}\psi^{(0)'-1}$ and eq. (4.2) and (2.43) and (2.44), it can be shown that

$$-\mathrm{i}\partial_x\psi^{(-1)}=\tfrac{1}{2}\mathrm{i}\partial_t\psi^{(0)}+\tfrac{1}{2}\partial_x^2\psi^{(0)}-\mathrm{i}(\partial_x\psi^{(0)})\psi^{(0)} \tag{4.13}$$

and

$$-\mathrm{i}\partial_t\psi^{(-1)}=\tfrac{1}{2}\partial_x\partial_t\psi^{(0)}-\mathrm{i}\psi^{(0)}\partial_t\psi^{(0)}-\tfrac{1}{2}\mathrm{i}\partial_x^3\psi^{(0)}-(\partial_x\psi^{(0)})^2 \\ +\tfrac{1}{2}\mathrm{i}\{(\partial_t\psi^{(0)})^2+(\partial_x^2\psi^{(0)})^2\}(\partial_x\psi^{(0)})^{-1}, \tag{4.14}$$

implying in particular ($\psi^{(0)}=-\psi^{(0)*}$) that

$$\partial_x\,\mathrm{Im}\,\psi^{(-1)}=\tfrac{1}{2}\mathrm{i}\partial_t\psi^{(0)},$$

$$\partial_t\,\mathrm{Im}\,\psi^{(-1)}=-\tfrac{1}{2}\mathrm{i}\partial_x^3\psi^{(0)}-(\partial_x\psi^{(0)})^2+\tfrac{1}{2}\mathrm{i}\Big\{(\partial_t\psi^{(0)})^2+(\partial_x^2\psi^{(0)})^2\Big\}(\partial_x\psi^{(0)})^{-1}. \tag{4.16}$$

Substituting $\psi^{(0)}=2\mathrm{i}z'$, we have from (4.15)

$$\mathrm{Im}\,\psi^{(-1)}=-\dot{z} \tag{4.17}$$

and (4.16) gives the partial differential equation for z, i.e.

$$\ddot{z}+z''''-(z'''^2+\dot{z}'^2)z''^{-1}+4z''^2=0. \tag{4.18}$$

Eq. (4.18) may be called the second potential nonlinear Schrödinger equation, since $y=z'$ satisfies the potential NLS equation. The solutions of (4.18) may be obtained from (2.5) for $n=-1$ and $n=0$, noting that

$$z(x,t)-z(0,0)=\int_\Gamma\{\mathrm{d}l_x(-\tfrac{1}{2}\mathrm{i}\psi^{(0)})-\mathrm{d}l_t\,\mathrm{Im}\,\psi^{(-1)}\}, \tag{4.19}$$

in which Γ is an arbitrary curve going from $(0,0)$ to (x,t) and $\mathrm{d}\boldsymbol{l}=(\mathrm{d}l_x,\mathrm{d}l_t)$ an infinitesimal two-dimensional vector tangent to Γ.

The real part of $\psi^{(-1)}$ can be found from (3.17) and (3.19), or by taking the (n,n)-element of the matrix relation (3.28), i.e.

$$2\mathrm{Re}\,\psi_{n-1,n}=-(|\psi^{(n)}|^2+|\phi^{(n)}|^2). \tag{4.20}$$

Eq. (4.20) implies in particular that

$$\mathrm{Re}\,\psi^{(-1)}=-\tfrac{1}{2}(|\psi^{(0)}|^2-\mathrm{i}\partial_x\psi^{(0)})=-2z'^2-z'' \tag{4.21}$$

and

$$\psi^{(-1)} = -2z'^2 - z'' - \mathrm{i}\dot{z}. \tag{4.22}$$

Taking the derivative

$$\partial_t \operatorname{Re} \psi^{(-1)} = \psi^{(0)}\partial_t\psi^{(0)} + \tfrac{1}{2}\mathrm{i}\partial_x\partial_t\psi^{(0)} \tag{4.23}$$

and inserting (4.15) for $\partial_t\psi^{(0)}$, we obtain $\psi^{(0)}$ as function of $\psi^{(-1)}$, i.e.

$$\psi^{(0)} = \frac{\partial_x^2 \operatorname{Im} \psi^{(-1)} - \partial_t \operatorname{Re} \psi^{(-1)}}{2\mathrm{i}\partial_x \operatorname{Im} \psi^{(-1)}}. \tag{4.24}$$

5. The Heisenberg spin chain

In this section we show how the equation of motion of the classical Heisenberg spin chain in the continuum limit (IHSC) can be obtained from the general equations derived in sections 2 and 3.

5.1. *The isotropic Heisenberg spin chain (the case $n = 1$, $m = 0$, or $n = 0$, $m = 1$)*

From (2.31), (2.32) with $\phi = \phi^{(1)}$, $\psi = \psi^{(1)}$, $\hat{\phi} = \phi^{(0)}$, or taking the $(1, 0)$ element of the matrix equations (3.23), (3.24) we have

$$\phi^{(0)} = -\mathrm{i}(1 - \psi^{(1)*})^{-1}\partial_x\phi^{(1)} = -\mathrm{i}(\phi^{(1)*})^{-1}\partial_x\psi^{(1)}. \tag{5.1}$$

From the equations of motion (2.29) and (2.30) with $\phi = \phi^{(1)}$, $\psi = \psi^{(1)}$, or taking the $(1, 0)$ element of the matrix equations (3.21), (3.22) we obtain

$$(\mathrm{i}\partial_t + \partial_x^2)\phi^{(1)} = -2(\phi^{(1)*})^{-1}|\partial_x\psi^{(1)}|^2 = -2|\phi^{(0)}|^2\phi^{(1)}, \tag{5.2}$$

$$(\mathrm{i}\partial_t + \partial_x^2)\psi^{(1)} = 2(\partial_x \ln \phi^{(1)*})\partial_x\psi^{(1)} = 2(1 - \psi^{(1)})|\phi^{(0)}|^2. \tag{5.3}$$

Furthermore, taking the $(1, 0)$ element of (3.28), or considering (4.20) for $\psi^{(1)} = \psi_{1,0} = -\psi^*_{0,1}$, it follows that

$$|1 - \psi^{(1)}|^2 + |\phi^{(1)}|^2 = 1. \tag{5.4}$$

Inserting (5.1) and (5.4) in (5.2) and (5.3) it is clear that $\phi^{(1)}$ and $1 - \psi^{(1)}$ both are solutions of the same partial differential equation

$$(\mathrm{i}\partial_t + \partial_x^2)\Phi = \frac{-2|\partial_x\Phi|^2}{1 - |\Phi|^2}\,\Phi. \tag{5.5}$$

Eq. (5.5) is an integrable PDE in the sense that solutions can be obtained from the linear integral equations (2.1) and (2.5) for $n = 1$.

Further insight in (5.5) can be obtained from the corresponding potential equation. In fact, using the substitution $\Phi = \kappa\, e^{i\gamma}$ in (5.5) and decomposing the result into a real and an imaginary part, we have

$$\partial_t(\tfrac{1}{2}\kappa^2) + \partial_x(\kappa^2\gamma') = 0 \tag{5.6}$$

and

$$\dot\gamma = \frac{\kappa''}{\kappa} - \gamma'^2 + \frac{2(\kappa'^2 + \kappa^2\gamma'^2)}{1-\kappa^2}, \tag{5.7}$$

where we have also used dots and primes to denote the differentiations ∂_t and ∂_x with respect to t and x, respectively.

Eq. (5.6) can be formally solved introducing a real variable $q(x, t)$ such that

$$\kappa^2 = \tfrac{1}{2}(1 - q'). \tag{5.8}$$

From (5.6) we then have

$$\gamma' = \frac{\dot q}{2(1-q')} \tag{5.9}$$

and inserting (5.8) and (5.9) in (5.7) it follows that

$$\dot\gamma = -\tfrac{1}{2}\frac{q'''}{1-q'} - \tfrac{1}{4}\frac{\dot q^2 + q''^2}{(1-q')^2} + \tfrac{1}{2}\frac{\dot q^2 + q''^2}{1-q'^2}. \tag{5.10}$$

The compatibility condition for γ, i.e.

$$\partial_t\left(\frac{\dot q}{2(1-q')}\right) + \partial_x\left(\tfrac{1}{2}\frac{q'''}{1-q'} + \tfrac{1}{4}\frac{\dot q^2 + q''^2}{(1-q')^2} - \tfrac{1}{2}\frac{\dot q^2 + q''^2}{1-q'^2}\right) = 0 \tag{5.11}$$

can be expressed as

$$\frac{\ddot q + q''''}{1-q'^2} + \frac{4(\dot q\dot q' + q''q''')q'}{(1-q'^2)^2} + \frac{(\dot q^2 + q''^2)(1+3q'^2)q''}{(1-q'^2)^3} = 0 \tag{5.12}$$

after dividing the left-hand side by $\tfrac{1}{2}(1+q')$.

Eq. (5.12) is the potential equation for eq. (5.5), but it is also completely identical to the isotropic limit of the potential equation which was derived in ref. 3 for the classical Heisenberg spin chain with axial symmetry. In fact, taking the external magnetic field $\boldsymbol{B}$ in the z-direction $\boldsymbol{B} = B\boldsymbol{e}^z$ and expressing the spin components in terms of polar angles, i.e.

$$\boldsymbol{S} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{5.13}$$

the equation of motion (1.1) for the IHSC can be rewritten as

$$\partial_t(\cos\theta) = \partial_x(\phi'\sin^2\theta), \tag{5.14}$$

$$\dot{\phi} = \frac{\theta''}{\sin\theta} - \phi'^2 \cos\theta - B. \tag{5.15}$$

Eq. (5.14) can be formally solved introducing a real variable $q(x, t)$ such that

$$\cos\theta = q'(x, t). \tag{5.16}$$

From (5.14) we then have

$$\phi' = \frac{\dot{q}}{1 - q'^2} \tag{5.17}$$

and inserting (5.16) and (5.17) in (5.15) it follows that

$$\dot{\phi} = \frac{-q'''}{1 - q'^2} - \frac{(q''^2 + \dot{q}^2)q'}{(1 - q'^2)^2} - B. \tag{5.18}$$

The compatibility condition for ϕ, i.e. $\partial_t \phi' = \partial_x \dot{\phi}$, gives the potential IHSC, i.e.

$$\partial_t \left(\frac{\dot{q}}{1 - q'^2} \right) + \partial_x \left(\frac{q'''}{1 - q'^2} + \frac{(q''^2 + \dot{q}^2)q'}{(1 - q'^2)^2} \right) = 0, \tag{5.19}$$

which is identical to (5.12).

Therefore (5.5) can be regarded as the equation of motion for the classical isotropic Heisenberg spin chain and its solutions can be expressed in the form

$$\Phi(x, t) = (\sin\tfrac{1}{2}\theta) \exp\left[\mathrm{i} \int_\Gamma \{ \mathrm{d}l_x \gamma' + \mathrm{d}l_t \dot{\gamma} \} \right], \tag{5.20}$$

where we have used (5.8) and (5.16) and where Γ is an arbitrary curve going from (0, 0) to (x, t) and $\mathrm{d}l = (\mathrm{d}l_x, \mathrm{d}l_t)$ is an infinitesimal two-dimensional vector tangent to Γ. Using (5.9), (5.10), (5.16) and (5.17) it is straightforward to derive the explicit expressions

$$\begin{aligned} \gamma' &= \phi' \cos^2 \tfrac{1}{2}\theta, \\ \dot{\gamma} &= \tfrac{1}{2}\theta'' \operatorname{cotg} \tfrac{1}{2}\theta + \tfrac{1}{4}\theta'^2 + \phi'^2 \cos^4 \tfrac{1}{2}\theta (2 \tan^2 \tfrac{1}{2}\theta - 1) \\ &= (\dot{\phi} + B) \cos^2 \tfrac{1}{2}\theta + \tfrac{1}{4}(\theta'^2 + \phi'^2 \sin^2\theta). \end{aligned} \tag{5.21}$$

So far we have considered the PDE for $\phi_{1,0}$ corresponding to the case $n = 1$, $m = 0$. In the other case $n = 0$, $m = 1$ we obtain exactly the same PDE, because of the relation $\phi_{0,1} = \phi_{1,0}$, cf. (3.27). Since $\tilde{\phi} = \phi_{0,1}$, it is clear that the solutions of the IHSC can be obtained from eqs. (1.3) and (1.5) given in the introduction.

5.2. *Miura transformation*

In order to derive the transformation between $\phi^{(1)}$ and $\phi^{(0)}$, we evaluate the explicit relation between the solutions $\phi^{(1)}$ and $1-\psi^{(1)}$ of (5.5), the result of which will be substituted in (5.1). Inserting $1-\psi^{(1)}=\tilde{\kappa}\,\mathrm{e}^{\mathrm{i}\tilde{\gamma}}$ in (5.4) and (5.1) we have

$$\tilde{\kappa}^2 = 1-|\phi^{(1)}|^2, \tag{5.22}$$

$$2\mathrm{i}\tilde{\kappa}^2\tilde{\gamma}' = -\phi^{(1)}\partial_x\phi^{(1)*} + \phi^{(1)*}\partial_x\phi^{(1)} \tag{5.23}$$

and from (5.3) it follows that

$$-\tilde{\kappa}\dot{\tilde{\gamma}} + \tilde{\kappa}'' - \tilde{\kappa}\tilde{\gamma}'^2 - 2\mathrm{Re}[(\tilde{\kappa}'+\mathrm{i}\tilde{\kappa}\tilde{\gamma}')\partial_x \ln \phi^{(1)*}] = 0. \tag{5.24}$$

Substituting $\phi^{(1)}=\kappa\,\mathrm{e}^{\mathrm{i}\gamma}$ in (5.22)–(5.24) leads to the relations

$$\tilde{\kappa} = (1-\kappa^2)^{1/2}, \tag{5.25}$$

$$\tilde{\gamma}' = -\kappa^2(1-\kappa^2)^{-1}\gamma', \tag{5.26}$$

$$\dot{\tilde{\gamma}} = -\kappa^2(1-\kappa^2)^{-1}\dot{\gamma} + (1-\kappa^2)^{-2}(\kappa'^2+\kappa^2\gamma'^2), \tag{5.27}$$

where

$$\kappa = |\phi^{(1)}|, \quad \gamma = -\tfrac{1}{2}\mathrm{i}\ln(\phi^{(1)}/\phi^{(1)*}),$$
$$\tilde{\kappa} = |1-\psi^{(1)}|, \quad \tilde{\gamma} = -\tfrac{1}{2}\mathrm{i}\ln[(1-\psi^{(1)})(1-\psi^{(1)*})^{-1}]. \tag{5.28}$$

From (5.25)–(5.27) it is clear that the transformation between κ, γ', $\dot{\gamma}$ and $\tilde{\kappa}$, $\tilde{\gamma}'$, $\dot{\tilde{\gamma}}$ is an involution, i.e. it is equal to its inverse. (The expressions for κ, γ', $\dot{\gamma}$ in terms of $\tilde{\kappa}$, $\tilde{\gamma}'$, $\dot{\tilde{\gamma}}$ can be found from (5.25)–(5.27) omitting the tildes in the left-hand side and adding them in the right-hand side.)

Inserting (5.26) and (5.27) in the relation

$$\phi^{(0)} = -\mathrm{i}(1-|\phi^{(1)}|^2)^{-1/2}(\partial_x\phi^{(1)})\exp\left[\mathrm{i}\int_\Gamma \{\mathrm{d}l_x\tilde{\gamma}' + \mathrm{d}l_t\dot{\tilde{\gamma}}\}\right], \tag{5.29}$$

cf. (5.1), we obtain an explicit expression for the Miura transformation which maps a solution of (5.5), (or equivalently a solution of the IHSC), on a solution of the NLS. This transformation is equivalent to the one given by Lakshmanan[8]), cf. also refs. 3 and 4, and also to the isotropic limit of a more general transformation[10]) which is valid in the case of uniaxial anisotropy.

A formal inversion of the Miura transformation can be obtained using the relations

$$\partial_x^2\psi^{(n)} = -\mathrm{i}\partial_x\psi^{(n-1)} + \mathrm{i}\psi^{(n)}\partial_x\psi^{(0)} + A\partial_x\psi^{(n)}, \tag{5.30}$$

$$\partial_t\psi^{(n)} = -\partial_x\psi^{(n-1)} + \psi^{(n)}\partial_x\psi^{(0)} + \mathrm{i}A\partial_x\psi^{(n)}, \tag{5.31}$$

in which

$$A = \partial_x \ln\phi^{(0)} = [(-\mathrm{i}\partial_t + \partial_x^2)\psi^{(0)}](2\partial_x\psi^{(0)})^{-1}, \tag{5.32}$$

cf. (2.33). Eq. (5.30) can be derived using (2.31) for $-i\partial_x\psi^{(n)}$, eq. (2.32) for $i\partial_x\phi^{(n)*}$, and again (2.31) for $\phi^{(n)*}\phi^{(0)}$ and for $|\phi^{(0)}|^2$. Eq. (5.31) follows from (5.30) and eq. (2.33) for $\partial_t\psi^{(n)}$.

In the special case $n = 1$ eqs. (5.30) and (5.31) reduce to

$$-\partial_x^2\psi^{(1)} = i(1-\psi^{(1)})\partial_x\psi^{(0)} - A\partial_x\psi^{(1)}, \tag{5.33}$$

$$-\partial_t\psi^{(1)} = (1-\psi^{(1)})\partial_x\psi^{(0)} - iA\partial_x\psi^{(1)}. \tag{5.34}$$

Substituting

$$Y = -(1-\psi^{(1)})^{-1}\partial_x\psi^{(1)} \tag{5.35}$$

in (5.33) we obtain

$$\partial_x Y = -Y^2 + AY + i\partial_x\psi^{(0)} \tag{5.36}$$

which is a Riccati equation†. From (5.36) $1-\psi^{(1)}$ can be formally solved as a function of the solution $\psi^{(0)}$ of the potential NLS, the integration constants being determined by (5.34) and (5.5). From (2.31) and (4.3) one can obtain $\phi^{(1)}$ as a function of $\phi^{(0)}$. This means that the Miura transformation (5.29) from IHSC to NLS can be formally inverted, (in agreement with the treatment in ref. 9, see also appendix E), but for more explicit results the Riccati equation should be solved.

5.3. *The isotropic Heisenberg spin chain* (*the case* $n = m = 1$)

In this subsection we derive a partial differential equation for the function $\phi_{1,1}$ defined by (3.12). Taking $n = 1$, $m = 1$ in eqs. (3.16), (3.17) and (3.14) we have

$$-i\partial_x\phi_{1,1} = (1-\psi^{(1)*})\phi^{(1)}, \tag{5.37}$$

$$-i\partial_x\psi_{1,1} = |\phi^{(1)}|^2, \tag{5.38}$$

$$(i\partial_t + \partial_x^2)\phi_{1,1} = -2i(\partial_x\psi^{(1)*})\phi^{(1)}. \tag{5.39}$$

Using the relation (5.1) between $\partial_x\phi^{(1)}$ and $\partial_x\psi^{(1)}$ and also (5.37) we have

$$-i\partial_x^2\phi_{1,1} = \phi^{(1)}(1-\psi^{(1)})^{-1}(\partial_x \ln\phi^{(1)} - \partial_x|\phi^{(1)}|^2), \tag{5.40}$$

from which one can solve

$$\partial_x \ln\phi^{(1)} = (\partial_x^2\phi_{1,1})\left[\frac{1+i\partial_x\psi_{1,1}}{\partial_x\phi_{1,1}}\right] - i\partial_x^2\psi_{1,1} \tag{5.41}$$

† In ref. 1 another Riccati equation has been derived.

and

$$\phi^{(1)}\partial_x\psi^{(1)*} = \mathrm{i}\partial_x^2\phi_{1,1} + \phi^{(1)}(1-\psi^{(1)*})\partial_x \ln \phi^{(1)}$$
$$= (\partial_x^2\phi_{1,1})(\partial_x\psi_{1,1}) - (\partial_x^2\psi_{1,1})(\partial_x\phi_{1,1}). \tag{5.42}$$

Inserting (5.42) in (5.39) we obtain

$$\mathrm{i}\partial_t\phi_{1,1} = (\partial_x\phi_{1,1})\partial_x(1+2\mathrm{i}\partial_x\psi_{1,1}) - (1+2\mathrm{i}\partial_x\psi_{1,1})\partial_x^2\phi_{1,1}. \tag{5.43}$$

Using the relation

$$4|\partial_x\phi_{1,1}|^2 + (1+2\mathrm{i}\partial_x\psi_{1,1})^2 = 1, \tag{5.44}$$

which can be derived from (5.4), (5.37) and (5.38), and the substitution

$$\chi = -\mathrm{i}\sqrt{2}(\partial_x\phi_{1,1})\,\mathrm{e}^{-\mathrm{i}Bt} \tag{5.45}$$

we obtain

$$-\mathrm{i}\partial_t\chi = (\partial_x^2\chi)(1-2|\chi|^2)^{1/2} - \chi\partial_x^2(1-2|\chi|^2)^{1/2} - B\chi. \tag{5.46}$$

Eq. (5.46) is identical to the equation of motion (8.2) derived in ref. 4 for the quantity $\chi = S^+/\sqrt{2} = (S^x + \mathrm{i}S^y)/\sqrt{2}$ of the classical Heisenberg spin chain in the continuum limit in the isotropic case with $c = \alpha = 0$, $\mu = 1$, $\beta = B$.

We therefore have a direct relation with the spin components, $(\boldsymbol{S}\cdot\boldsymbol{S} = 1)$,

$$S^+ = \sin\theta\,\mathrm{e}^{\mathrm{i}\phi} = -2\mathrm{i}(\partial_x\phi_{1,1})\,\mathrm{e}^{-\mathrm{i}Bt}, \tag{5.47}$$

$$S^z = \cos\theta = (1-4|\partial_x\phi_{1,1}|^2)^{1/2} = 1 + 2\mathrm{i}\partial_x\psi_{1,1}. \tag{5.48}$$

Eq. (5.47) may also be inferred from (5.21) and (5.26), (5.27). In fact, using (5.37) and (5.8), (5.16), (5.25) we have

$$S^+ = 2(1-\psi^{(1)*})\phi^{(1)}\,\mathrm{e}^{-\mathrm{i}Bt} = \sin\theta\,\mathrm{e}^{\mathrm{i}(\gamma-\tilde{\gamma}-Bt)} \tag{5.49}$$

and from (5.21), (5.26) and (5.27) it can be shown that

$$\gamma' - \tilde{\gamma}' = \phi', \quad \dot{\gamma} - \dot{\tilde{\gamma}} = \dot{\phi} + B, \tag{5.50}$$

so that indeed $\phi = \gamma - \tilde{\gamma} - Bt$, apart from a constant phase.

Remarks.

i) From (3.4) and (3.5) one can derive the linear eigenvalue problem for the functions $\phi_k^{(1)}$ and $\psi_k^{(1)}$ and establish a direct connection with the Lax representation of Takhtadzhyan[5]) for the IHSC. Some details of the derivation will be presented in appendix D, and in appendix E we discuss the gauge equivalence[9]) between the Lax representations of the NLS and the IHSC.

ii) From (5.47) and (5.48) it is clear that the solutions of the spin vector $\boldsymbol{S}(x,t)$ satisfying eq. (1.1) can be found directly from eqs. (2.1) and (2.5) using

(1.7) and (1.8) for $m = 1$. As an example some results for the two-soliton solution of the IHSC will be given in appendix F. The integral equations (2.1) and (2.5) can also be used to investigate similarity solutions, in analogy with ref. 16, in which the similarity solutions of the KdV have been studied, cf. also ref. 25 for the similarity solutions of the Heisenberg spin chain.

6. Conclusion

In this paper we have presented an inhomogeneous linear integral equation (1.3) which can be used for the linearization of the nonlinear Schrödinger equation (NLS), as well as the equation of motion for the classical isotropic Heisenberg spin chain in the continuum limit (IHSC). The integral equation is quite general in the sense that it contains a (two-fold) integration over an arbitrary contour (and its complex conjugate) in the complex plane with an arbitrary measure. The various solutions of the NLS and the IHSC can be obtained directly by an appropriate choice of contour and measure without going through the details of the inverse scattering formalism. The treatment in this paper is based on a slightly more general integral equation in which the source term $e^{i(kx-k^2t)}$ in the right-hand side of (1.3) is replaced by a more general term $k^{-n}\, e^{i(kx-k^2t)}$, or $G(k)\, e^{i(kx-k^2t)}$, where $G(k) = \Sigma_n \lambda_n k^{-n}$ is an arbitrary meromorphic function of k. The linear integral equation with source term $k^{-n}\, e^{i(kx-k^2t)}$ leads to the so-called NLS–IHSC hierarchy consisting of the relations between the solutions $\phi^{(n)}(x, t)$, $\psi^{(n)}(x, t)$ originating from the integral equation. The integral equation with source term $G(k)\, e^{i(kx-k^2t)}$ leads to two coupled partial differential equations for the functions $\phi(x, t)$, $\psi(x, t)$ which contain all linear combinations of functions of the NLS-IHSC hierarchy for various values of n as special solutions. Using eqs. (1.7) and (1.8) we have also defined matrices $\mathbf{\Phi}$ and $\mathbf{\Psi}$ with elements $\phi_{n,m}$ and $\psi_{n,m}$ and we have derived (matrix) partial differential equations for $\mathbf{\Phi}$ and $\mathbf{\Psi}$. All these equations contain the NLS and the IHSC as special cases. Some well-known properties of the usual inverse scattering scheme, such as the Gel'fand–Levitan equation and the Lax representations of Zakharov and Shabat[6]) and of Takhtadzhyan[5]) for the NLS and the IHSC respectively, have been derived as a corollary from the integral equation.

We have also investigated the hierarchies arising from other linear integral equations which can be used e.g. for the linearization of the Korteweg–de Vries equation, the modified Korteweg–de Vries equation and the sine-Gordon equation. These results will be reported in a separate publication[24]).

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Appendix A

In this appendix we show how the Gel'fand–Levitan equations arise as a specification of the integral equations (2.19), (2.23). If we introduce the quantity

$$K(x, y; t) = \int_C \mathrm{d}\lambda(k)\, \mathrm{e}^{(1/2)\mathrm{i}k(y-x)} \phi_k(x, t) \tag{A.1}$$

one can derive from (2.19) an integral equation for $K(x, y; t)$ which can be inverted in order to regain (2.19), provided that a proper choice of the measure $\mathrm{d}\lambda(k)$ and the contour C allow a representation of the δ-function.

Making use of the identity

$$\frac{\mathrm{e}^{(1/2)\mathrm{i}(k-l')x}}{k-l'} = -\frac{\mathrm{i}}{2} \int_x^\infty \mathrm{e}^{(1/2)\mathrm{i}(k-l')\xi}\, \mathrm{d}\xi. \tag{A.2}$$

and assuming that $\mathrm{Im}\, k > 0$, $\mathrm{Im}\, l' < 0$, we find the following equation for $K(x, y; t)$:

$$K(x, y; t) + \tfrac{1}{4} \int_{x+y}^\infty \mathrm{d}\xi \int_{x-y}^\infty \mathrm{d}\eta F^{(0)}(\xi, t) F^{(0)*}(\xi + \eta, t) K(x, y + \eta; t)$$
$$= F(x + y, t), \tag{A.3}$$

where

$$F(z, t) = \int_C \mathrm{d}\lambda(k)\, \mathrm{e}^{(1/2)\mathrm{i}kz}\, \mathrm{e}^{-\mathrm{i}k^2 t} G(k), \tag{A.4}$$

$$F^{(0)}(z, t) = \int_C \mathrm{d}\lambda(k)\, \mathrm{e}^{(1/2)\mathrm{i}kz}\, \mathrm{e}^{-\mathrm{i}k^2 t}. \tag{A.5}$$

In a similar way eq. (2.23) can be rewritten as an integral equation for the function

$$\tilde{K}(x, y; t) = \int_C d\lambda(k)\, e^{(1/2)ik(y-x)} \psi_k(x, t). \tag{A.6}$$

The result is

$$\tilde{K}(x, y; t) + \tfrac{1}{4} \int_{x+y}^{\infty} d\xi \int_{x-y}^{\infty} d\eta F^{(0)}(\xi, t) F^{(0)*}(\xi + \eta, t) \tilde{K}(x, y + \eta; t)$$

$$= -\frac{i}{2} \int_{x+y}^{\infty} d\xi F^{(0)}(\xi, t) F^*(x - y + \xi, t) \tag{A.7}$$

and the relation between K and $\tilde{K}$ is given by

$$\tilde{K}(x, y; t) = -\frac{i}{2} \int_{x+y}^{\infty} d\xi F^{(0)}(\xi, t) K^*(x, \xi - y; t) \tag{A.8}$$

as follows from (1.6).

Eqs. (A.3) and (A.7) are linear integral equations in (real) coordinate space from which one can obtain the functions $\phi(x, t)$ and $\psi(x, t)$, cf. (A.1), (A.6) and (2.27), i.e.

$$\phi(x, t) = K(x, x; t), \quad \psi(x, t) = \tilde{K}(x, x; t). \tag{A.9}$$

The functions $\phi(x, t)$ and $\psi(x, t)$ are solutions of the coupled PDE's (2.29) and (2.30), cf. also eq. (2.40) containing only $\psi(x, t)$. Eqs. (A.3) and (A.7) may be regarded as a formal extension of the Gel'fand–Levitan equations, see also refs. 16 and 18 for considerations of a similar nature. (The usual Gel'fand–Levitan equations of the inverse scattering method for the NLS and the IHSC may be inferred from (A.3) and (A.7) choosing a measure which contains the reflection coefficient and a contour along the real axis and passing through the poles of the reflection coefficient.)

Appendix B

We give an explicit proof of (3.27) which is based on the integral equations (2.1) and (2.3). Inserting

$$\chi_{1k}^{(n)} = \phi_k^{(n)} e^{-(1/2)i(kx - k^2 t)}, \quad \chi_{2k}^{(n)} = \psi_k^{(n)} e^{-(1/2)i(kx - k^2 t)} \tag{B.1}$$

in (2.1) and (2.3), we obtain

$$\chi_{1k}^{(n)} + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{(1/2)\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}\,\mathrm{e}^{(1/2)\mathrm{i}(lx-l^2t)}}{(k-l')(l'-l)} \chi_{1l}^{(n)} = \frac{1}{k^n}\mathrm{e}^{(1/2)\mathrm{i}(kx-k^2t)} \tag{B.2}$$

and

$$\chi_{2k}^{(n)} - \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{(1/2)\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-(1/2)\mathrm{i}(l'x-l'^2t)}}{k-l'} \chi_{1l'}^{(n)*} = 0. \tag{B.3}$$

Eq. (B.2) can be rewritten as

$$\chi_{1k}^{(n)} + \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\mathrm{e}^{(1/2)\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-(1/2)\mathrm{i}(l'x-l'^2t)}}{k-l'} \chi_{2l'}^{(n)*} = \frac{1}{k^n}\mathrm{e}^{(1/2)\mathrm{i}(kx-k^2t)}. \tag{B.4}$$

From (3.12) with (B.3) and (B.4) we obtain

$$\begin{aligned}\phi_{n,m} &= \int_C \mathrm{d}\lambda(k)\chi_{1k}^{(n)}\,\mathrm{e}^{(1/2)\mathrm{i}(kx-k^2t)}k^{-m}\\ &= \int_C \mathrm{d}\lambda(k)\chi_{1k}^{(n)}\left[\chi_{1k}^{(m)} + \int_{C^*}\mathrm{d}\lambda^*(l')\frac{\mathrm{e}^{(1/2)\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-(1/2)\mathrm{i}(l'x-l'^2t)}}{k-l'}\chi_{2l'}^{(m)*}\right]\\ &= \int_C \mathrm{d}\lambda(k)\chi_{1k}^{(n)}\chi_{1k}^{(m)} - \int_{C^*}\mathrm{d}\lambda^*(k')\chi_{2k'}^{(n)*}\chi_{2k'}^{(m)*}\end{aligned} \tag{B.5}$$

and

$$\begin{aligned}\psi_{n,m} &= \int_C \mathrm{d}\lambda(k)\chi_{2k}^{(n)}\,\mathrm{e}^{(1/2)\mathrm{i}(kx-k^2t)}k^{-m}\\ &= \int_C \mathrm{d}\lambda(k)\int_{C^*}\mathrm{d}\lambda^*(k')(\chi_{1k}^{(m)}\chi_{1k'}^{(n)*} + \chi_{2k}^{(n)}\chi_{2k'}^{(m)*})\frac{\mathrm{e}^{(1/2)\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-(1/2)\mathrm{i}(k'x-k'^2t)}}{k-k'}.\end{aligned} \tag{B.6}$$

Eqs. (B.5) and (B.6) for $\phi_{n,m}$ and $\psi_{n,m}$ are (integrals of) bilinear expressions in the eigenfunctions of the linear problem and eq. (3.27), i.e. $\phi_{m,n} = \phi_{n,m}$ and $\psi_{m,n} = -\psi^*_{n,m}$, is obvious from (B.5) and (B.6).

Appendix C

In order to derive the Lax representation of Zakharov and Shabat[6]) we introduce the functions

$$\chi_{1k}^{(0)} = \phi_k^{(0)}\,\mathrm{e}^{-(1/2)\mathrm{i}(kx-k^2t)}, \quad \chi_{2k}^{(0)} = \psi_k^{(0)}\,\mathrm{e}^{-(1/2)\mathrm{i}(kx-k^2t)}, \tag{C.1}$$

cf. eq. (B.1) for $n = 0$. From (3.2) and (2.18) for $n = 0$ it is straightforward to show that

$$\mathscr{L}^{(0)} \cdot \boldsymbol{\chi}_k^{(0)} = \tfrac{1}{2} k \boldsymbol{\chi}_k^{(0)}, \tag{C.2}$$

in which the 2×2 matrix $\mathscr{L}^{(0)}$ and the two-dimensional vector $\boldsymbol{\chi}_k^{(0)}$ are given by

$$\mathscr{L}^{(0)} = \begin{pmatrix} -\mathrm{i}\partial_x & -\phi^{(0)} \\ \phi^{(0)*} & \mathrm{i}\partial_x \end{pmatrix}, \quad \boldsymbol{\chi}_k^{(0)} = \begin{pmatrix} \chi_{1k}^{(0)} \\ \chi_{2k}^{(0)} \end{pmatrix}. \tag{C.3}$$

From (2.29) and (2.30) we obtain

$$(\mathrm{i}\partial_t + \partial_x^2)\chi_{1k}^{(0)} + \mathrm{i}k\partial_x\chi_{1k}^{(0)} + \tfrac{1}{4}k^2\chi_{1k}^{(0)} = -2|\phi^{(0)}|^2\chi_{1k}^{(0)},$$

$$(\mathrm{i}\partial_t + \partial_x^2)\chi_{2k}^{(0)} + \mathrm{i}k\partial_x\chi_{2k}^{(0)} + \tfrac{1}{4}k^2\chi_{2k}^{(0)} = 2\mathrm{i}(\partial_x\phi^{(0)*})\chi_{1k}^{(0)} \tag{C.4}$$

and eliminating the k-dependent terms with (C.2) we have

$$\mathrm{i}\partial_t\boldsymbol{\chi}_k^{(0)} = \mathscr{M}^{(0)} \cdot \boldsymbol{\chi}_k^{(0)}, \tag{C.5}$$

in which the 2×2 matrix $\mathscr{M}^{(0)}$ is given by

$$\mathscr{M}^{(0)} = \begin{pmatrix} -2\partial_x^2 - |\phi^{(0)}|^2 & \mathrm{i}\phi^{(0)}\partial_x + \mathrm{i}\partial_x\phi^{(0)} \\ -\mathrm{i}\partial_x\phi^{(0)*} - \mathrm{i}\phi^{(0)*}\partial_x & 2\partial_x^2 + |\phi^{(0)}|^2 \end{pmatrix}, \tag{C.6}$$

where the operator ∂_x acts on everything at the right.

The compatibility relation for (C.2) and (C.6), i.e.

$$\partial_t\mathscr{L}^{(0)} = \mathrm{i}[\mathscr{L}^{(0)}, \mathscr{M}^{(0)}], \tag{C.7}$$

is fulfilled, if and only if $\phi^{(0)}$ satisfies the NLS. The matrices $\mathscr{L}^{(0)}$ and $\mathscr{M}^{(0)}$ form the Lax representation for the NLS obtained in ref. 6.

Appendix D

We consider the linear eigenvalue problem for the functions $\phi_k^{(1)}$ and $\psi_k^{(1)}$. From (3.4) and (3.5) for $n = 1$ we have

$$(-k - \mathrm{i}\partial_x)\phi_k^{(1)} = \phi^{(1)}\psi_k^{(0)}, \tag{D.1}$$

$$(-k - \mathrm{i}\partial_x)\psi_k^{(1)} = (\psi^{(1)} - 1)\psi_k^{(0)} \tag{D.2}$$

and from (2.17) and (2.18) for $n = 1$

$$-\mathrm{i}\partial_x\phi_k^{(1)} = (1 - \psi^{(1)*})\phi_k^{(0)}, \tag{D.3}$$

$$-\mathrm{i}\partial_x\psi_k^{(1)} = \phi^{(1)*}\phi_k^{(0)}. \tag{D.4}$$

Introducing the functions

$$\chi_{1k}^{(1)} = \phi_k^{(1)} \mathrm{e}^{-(1/2)\mathrm{i}(kx-k^2t)}, \quad \chi_{2k}^{(1)} = \psi_k^{(1)} \mathrm{e}^{-(1/2)\mathrm{i}(kx-k^2t)}, \tag{D.5}$$

cf. (B.1), and eliminating $\phi_k^{(0)}$ and $\psi_k^{(0)}$ from (D.1)–(D.4) we obtain the linear eigenvalue problem of Takhtadzhyan[5]), i.e.

$$\begin{aligned}(1-\psi^{(1)})(-\tfrac{1}{2}k - \mathrm{i}\partial_x)\chi_{1k}^{(1)} &= -\phi^{(1)}(-\tfrac{1}{2}k - \mathrm{i}\partial_x)\chi_{2k}^{(1)},\\ \phi^{(1)*}(\tfrac{1}{2}k - \mathrm{i}\partial_x)\chi_{1k}^{(1)} &= (1-\psi^{(1)*})(\tfrac{1}{2}k - \mathrm{i}\partial_x)\chi_{2k}^{(1)}.\end{aligned} \tag{D.6}$$

Eq. (D.6) can be expressed in matrix notation as

$$\tfrac{1}{2}k\mathbf{g}\cdot\boldsymbol{\chi}_k^{(1)} = \boldsymbol{\sigma}^z\cdot\mathbf{g}\cdot(\mathrm{i}\partial_x\boldsymbol{\chi}_k^{(1)}), \tag{D.7}$$

in which the 2×2 matrix $\mathbf{g}$ and the two-dimensional vector $\boldsymbol{\chi}_k^{(1)}$ are given by

$$\mathbf{g} = \begin{pmatrix}\phi^{(1)*} & -(1-\psi^{(1)*})\\ 1-\psi^{(1)} & \phi^{(1)}\end{pmatrix}, \quad \boldsymbol{\chi}_k^{(1)} = \begin{pmatrix}\chi_{1k}^{(1)}\\ \chi_{2k}^{(1)}\end{pmatrix}, \tag{D.8}$$

$\boldsymbol{\sigma}^z$ is a Pauli matrix. (From eq. (5.4) it is clear that $\mathbf{g}$ is a unitary matrix.) From (D.7) it follows that

$$\tfrac{1}{2}k\boldsymbol{\chi}_k = \mathscr{L}^{(1)}\cdot\boldsymbol{\chi}_k, \tag{D.9}$$

with

$$\mathscr{L}^{(1)} = \mathbf{g}^{-1}\cdot\boldsymbol{\sigma}^z\cdot\mathbf{g}\mathrm{i}\partial_x = -\mathbf{S}\mathrm{i}\partial_x, \tag{D.10}$$

where

$$\mathbf{S} = -\mathbf{g}^{-1}\cdot\boldsymbol{\sigma}^z\cdot\mathbf{g} = \begin{pmatrix}S^z & S^+\mathrm{e}^{\mathrm{i}Bt}\\ S^-\mathrm{e}^{-\mathrm{i}Bt} & -S^z\end{pmatrix}, \tag{D.11}$$

cf. (5.37), (5.38), (5.47) and (5.48) for the identification with the spin components.

The time evolution of $\phi_k^{(1)}$ and $\psi_k^{(1)}$ can be evaluated using the relations

$$(\mathrm{i}\partial_t + \partial_x^2)\phi_k^{(1)} = 2(\partial_x \ln(1-\psi^{(1)*}))\partial_x\phi_k^{(1)}, \tag{D.12}$$

$$(\mathrm{i}\partial_t + \partial_x^2)\psi_k^{(1)} = 2(\partial_x \ln\phi^{(1)*})\partial_x\psi_k^{(1)}, \tag{D.13}$$

which can be derived from (2.13) and (2.14) for $n = 1$, using (2.17) and (2.18) to express $\phi_k^{(0)}$ in terms of $\partial_x\phi_k^{(1)}$ and $\partial_x\psi_k^{(1)}$.

Using (D.10) to eliminate the k-dependent terms we have in matrix notation

$$\begin{aligned}&(\mathrm{i}\partial_t + \partial_x^2)\boldsymbol{\chi}_k^{(1)} + 2\partial_x(\mathbf{S}\cdot\partial_x\boldsymbol{\chi}_k^{(1)}) - \mathbf{S}\partial_x\cdot\mathbf{S}\cdot\partial_x\boldsymbol{\chi}_k^{(1)}\\ &\quad= 2\begin{pmatrix}(\partial_x \ln(1-\psi^{(1)*})) & 0\\ 0 & (\partial_x \ln\phi^{(1)*})\end{pmatrix}\cdot(\mathbb{1}+\mathbf{S})\cdot\partial_x\boldsymbol{\chi}_k^{(1)}.\end{aligned} \tag{D.14}$$

For $\partial_x \ln\phi^{(1)*}$ and $\partial_x \ln(1-\psi^{(1)*})$ we have the explicit relations

$$\partial_x \ln\phi^{(1)} = \tfrac{1}{2}[(1+S^z)S^{+^{-1}}\partial_x S^+ - \partial_x S^z], \tag{D.15}$$

$$\partial_x \ln(1-\psi^{(1)*}) = \tfrac{1}{2}[(1-S^Z)S^{+^{-1}}\partial_x S^{+^{-1}} + \partial_x S^z], \tag{D.16}$$

cf. (5.37), (5.41), (5.42), (5.47) and (5.48).

From the relation

$$2\begin{pmatrix} (\partial_x \ln(1-\psi^{(1)*})) & \\ & (\partial_x \ln \phi^{(1)*}) \end{pmatrix} \cdot (\mathbf{1}+\mathbf{S}) = (\partial_x \mathbf{S}) \cdot (\mathbf{1}+\mathbf{S}), \tag{D.17}$$

we finally obtain

$$\mathrm{i}\partial_t \boldsymbol{\chi}_k^{(1)} = \mathcal{M}^{(1)} \cdot \boldsymbol{\chi}_k^{(1)}, \tag{D.18}$$

with

$$\mathcal{M}^{(1)} = -\mathbf{S}\partial_x^2 - \partial_x \mathbf{S}\partial_x, \tag{D.19}$$

the operator ∂_x acting on everything at the right.

The relations (D.9) and (D.18) form the Lax representation for the IHSC as first obtained by Takhtadzhyan[5]) and the compatibility relation

$$\partial_t \mathcal{L}^{(1)} = \mathrm{i}[\mathcal{L}^{(1)}, \mathcal{M}^{(1)}] \tag{D.20}$$

leads to

$$\partial_t \mathbf{S} = \tfrac{1}{2}\mathrm{i}[\mathbf{S}, (\partial_x^2 \mathbf{S})] \tag{D.21}$$

which in view of (D.11) is equivalent to eq. (1.1) with $\boldsymbol{B} = B\mathbf{e}^z$ for the IHSC.

Appendix E

The gauge equivalence[9]) between the Lax representations for the NLS and the IHSC can be easily derived from the considerations in appendix C and D. From (D.1) and (D.3) we have

$$k\phi_k^{(1)} = (1-\psi^{(1)*})\phi_k^{(0)} - \phi^{(1)}\psi_k^{(0)} \tag{E.1}$$

and from (D.2) and (D.4)

$$k\psi_k^{(1)} = \phi^{(1)*}\phi_k^{(0)} + (1-\psi^{(1)})\psi_k^{(0)}. \tag{E.2}$$

Using the definitions (C.1) and (D.5) we have in matrix notation

$$k\boldsymbol{\chi}_k^{(1)} = \mathbf{U} \cdot \boldsymbol{\chi}_k^{(0)}, \tag{E.3}$$

where

$$\mathbf{U} = \begin{pmatrix} 1-\psi^{(1)*} & -\phi^{(1)} \\ \phi^{(1)*} & 1-\psi^{(1)} \end{pmatrix} = -\mathrm{i}\boldsymbol{\sigma}^y \cdot \mathbf{g}^{\mathrm{T}}, \tag{E.4}$$

in which $\boldsymbol{\sigma}^y$ is a Pauli matrix and $\mathbf{g}^{\mathrm{T}}$ is the transposed matrix of the matrix $\mathbf{g}$ in (D.8). It is clear that $\mathbf{U}$ is a unitary matrix.

From (C.2) and (C.5) on the one hand and eqs. (D.9) and (D.18) on the other hand, we obtain the relations

$$\mathscr{L}^{(1)} = \mathbf{U} \cdot \mathscr{L}^{(0)} \cdot \mathbf{U}^\dagger, \quad \mathscr{M}^{(1)} = \mathbf{U} \cdot \mathscr{M}^{(0)} \cdot \mathbf{U}^\dagger + \mathrm{i}(\partial_t \mathbf{U}) \cdot \mathbf{U}^\dagger, \tag{E.5}$$

which express the gauge equivalence of the Lax representations of NLS and IHSC and which have been given in slightly different form in ref. 9, cf. also ref. 26.

Appendix F

In this appendix we give a few results concerning soliton solutions for the quantity $\psi_{1,1}(x, t)$ which because of (5.48) determines the S^z component of the spin in the IHSC, (and also the function $q(x, t)$ which is the solution of the potential IHSC (5.12).)

The N-soliton solutions can be obtained inserting

$$\mathrm{d}\lambda(l) = \sum_{\alpha=1}^{N} c_\alpha \delta(l - k_\alpha)\, \mathrm{d}l \tag{F.1}$$

in the integral equation (2.5) and choosing a contour C which passes through the N "poles" of the delta functions.

From (F.1) and (2.5) for $n = 1$ we find

$$\begin{aligned} &c_\alpha^{1/2}\, \mathrm{e}^{-(1/2)\mathrm{i}(k_\alpha x - k_\alpha^2 t)} \psi_{k_\alpha}^{(1)} + c_\alpha^{1/2}\, \mathrm{e}^{(1/2)\mathrm{i}(k_\alpha x - k_\alpha^2 t)} \sum_{\beta,\gamma} \frac{c_\beta^*\, \mathrm{e}^{-\mathrm{i}(k_\beta^* x - k_\beta^{*2} t)}}{(k_\alpha - k_\beta^*)(k_\beta^* - k_\gamma)} c_\gamma \psi_{k_\gamma}^{(1)} \\ &\quad = c_\alpha^{1/2}\, \mathrm{e}^{(1/2)\mathrm{i}(k_\alpha x - k_\alpha^2 t)} \sum_\beta \frac{c_\beta^*\, \mathrm{e}^{-\mathrm{i}(k_\beta^* - k_\beta^{*2} t)}}{(k_\alpha - k_\beta^*) k_\beta^*}. \end{aligned} \tag{F.2}$$

If we define an $N \times N$ matrix $\mathbf{M}$ with elements

$$M_{\alpha\beta} = \frac{\mathrm{e}^{(1/2)\mathrm{i}(k_\alpha x - k_\alpha^2 t)}\, c_\alpha^{1/2} c_\beta^{*1/2}\, \mathrm{e}^{-(1/2)(k_\beta^* x - k_\beta^{*2} t)}}{k_\alpha - k_\beta^*} = -M_{\beta\alpha}^*, \tag{F.3}$$

and N-dimensional vectors $\boldsymbol{\xi}$ and f by

$$\begin{aligned} \xi_\alpha &= c_\alpha^{1/2}\, \mathrm{e}^{-(1/2)\mathrm{i}(k_\alpha x - k_\alpha^2 t)} \psi_{k_\alpha}^{(1)}, \\ f_\beta &= c_\beta^{1/2}\, \mathrm{e}^{(1/2)\mathrm{i}(k_\beta x - k_\beta^2 t)} k_\beta^{-1}, \end{aligned} \tag{F.4}$$

then we have in matrix notation

$$\boldsymbol{\xi} + \mathbf{M} \cdot \mathbf{M}^* \cdot \boldsymbol{\xi} = \mathbf{M} \cdot f^*. \tag{F.5}$$

The function $\psi_{1,1}$ is given by, cf. (3.12),

$$\psi_{1,1} = \sum_\alpha c_\alpha \psi_{k_\alpha}^{(1)} k_\alpha^{-1} \tag{F.6}$$

and in matrix notation we have

$$\psi_{1,1} = f \cdot \xi = f \cdot (\mathbf{1} + \mathbf{M} \cdot \mathbf{M}^*)^{-1} \cdot \mathbf{M} \cdot f^*. \tag{F.7}$$

Considering (F.7) for the special case $N = 2$ one can write down the explicit result for the two-soliton solution of the IHSC. We have after a straightforward but tedious calculation

$$\begin{aligned}\psi_{1,1} = {} & [G_1\, e^{-2x\, \mathrm{Im}(k_1+k_2)+2t\, \mathrm{Im}(k_1^2+k_2^2)} \\ & + G_2\, e^{-2x\, \mathrm{Im}\, k_1+2t\, \mathrm{Im}\, k_1^2} + G_3\, e^{-2x\, \mathrm{Im}\, k_2+2t\, \mathrm{Im}\, k_2^2} \\ & + 2\mathrm{i}\, \mathrm{Im}\{G_4\, e^{\mathrm{i}(k_1x-k_2^*x-k_1^2t+k_2^{*2}t)}\}] \\ & \times [1 + F_1\, e^{-2x\, \mathrm{Im}(k_1+k_2)+2t\, \mathrm{Im}(k_1^2+k_2^2)} \\ & + F_2\, e^{-2x\, \mathrm{Im}\, k_1+2t\, \mathrm{Im}\, k_1^2} + F_3\, e^{-2x\, \mathrm{Im}\, k_2+2t\, \mathrm{Im}\, k_2^2} \\ & + 2\, \mathrm{Re}\{F_4\, e^{\mathrm{i}(k_1x-k_2^*x-k_1^2t+k_2^{*2}t)}\}]^{-1},\end{aligned} \tag{F.8}$$

with

$$\begin{aligned} G_1 &= \frac{-\mathrm{i}|c_1|^2|c_2|^2}{8|k_1|^2|k_2|^2} \left|\frac{k_1 - k_2}{k_1 - k_2^*}\right|^4 \frac{(\mathrm{Im}\, k_1)|k_2|^2 + (\mathrm{Im}\, k_2)|k_1|^2}{(\mathrm{Im}\, k_1)^2\, (\mathrm{Im}\, k_2)^2}, \\ G_2 &= \frac{-\mathrm{i}|c_1|^2}{2|k_1|^2\, \mathrm{Im}\, k_1}, \quad G_3 = \frac{-\mathrm{i}|c_2|^2}{2|k_2|^2\, \mathrm{Im}\, k_2}, \quad G_4 = \frac{c_1 c_2^*}{k_1 k_2^*(k_1 - k_2^*)} \end{aligned} \tag{F.9}$$

and

$$\begin{aligned} F_1 &= \frac{|c_1|^2|c_2|^2}{16(\mathrm{Im}\, k_1)^2(\mathrm{Im}\, k_2)^2} \left|\frac{k_1 - k_2}{k_1 - k_2^*}\right|^4, \\ F_2 &= \frac{|c_1|^2}{4(\mathrm{Im}\, k_1)^2}, \quad F_3 = \frac{|c_2|^2}{4(\mathrm{Im}\, k_2)^2}, \quad F_4 = \frac{-c_1 c_2^*}{(k_2^* - k_1)^2}. \end{aligned} \tag{F.10}$$

At $x \to \pm\infty$ we have for all finite t $\psi_{1,1} \to$ constant, implying with (5.48) that $S^z \to 1$, which is the usual boundary condition for soliton solutions of the IHSC.

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