LINEARIZATION OF THE NONLINEAR SCHRÖDINGER EQUATION AND THE ISOTROPIC HEISENBERG SPIN CHAIN

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A description in terms of one and the same inhomogeneous linear integral equation is proposed for the solutions of the nonlinear Schrödinger equation and the equation of motion of the isotropic classical Heisenberg spin chain. In addition it is shown that the integral equation introduced by Fokas and Ablowitz for the Korteweg–de Vries equation yields the solutions of the modified Korteweg–de Vries equation as well.

In this letter we propose a new linear inhomogeneous integral equation from which one can evaluate in a direct way the solutions of interest of the nonlinear Schrödinger equation (NLS) and of the equation of motion of the isotropic classical Heisenberg spin chain in the continuum limit (IHSC) without going through the details of the inverse scattering formalism [1–4]. For that purpose we introduce functions $\phi_k(x, t)$ satisfying the integral equation

$$\phi_k(x,t) + \int_L \mathrm{d}\lambda(l) \int_{L^*} \mathrm{d}\lambda^*(l') \times \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')(l'-l)}\,\phi_l(x,t) = \mathrm{e}^{\mathrm{i}(kx-k^2t)}. \tag{1}$$

Here $\mathrm{d}\lambda(k)$ is an arbitrary measure over the complex variable k and the integrations are performed over a contour L and its complex conjugate L^*. The measure and the contour are to be chosen such that the homogeneous integral equation has only the zero solution, i.e. if $f_k(x, t)$ satisfies the homogeneous equation, then $f_k(x, t) = 0$. Furthermore we assume that the differentiations with respect to x and t may be shifted through the integrals. An integral equation of the type (1), but with one integration over a contour L, has been introduced by Fokas and Ablowitz [5] for the linearization of the Korteweg–de Vries (KdV) equation. The form of the integral equations may be inferred from a treatment by Rosales [6] on the basis of a power series expansion.

We also need the functions $\psi_k(x, t)$ which are defined by

$$\psi_k(x,t) = \int_{L^*} \mathrm{d}\lambda^*(l')\,\frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}}{k-l'}\,\phi^*_{l'}(x,t). \tag{2}$$

From (2) and the complex conjugate of (1) it is straightforward to derive the integral equation

$$\psi_k(x,t) + \int_L \mathrm{d}\lambda(l) \int_{L^*} \mathrm{d}\lambda^*(l') \times \frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')(l'-l)}\,\psi_l(x,t) = \int_{L^*} \mathrm{d}\lambda^*(l')\,\frac{\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{e}^{-\mathrm{i}(l'x-l'^2t)}}{(k-l')}. \tag{3}$$

Starting from eqs. (1) and (3) one can derive integral equations for the temporal and spatial derivatives of the functions $\phi_k(x, t)$ and $\psi_k(x, t)$. Subtracting these integral equations from eq. (1) or (3) and taking into account that the homogeneous equation has only the zero solution, various results for the derivatives of $\phi_k(x, t)$ and $\psi_k(x, t)$ may be obtained. In this letter we use the relations

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$$(-k - \mathrm{i}\partial_x)\phi_k = \phi\psi_k, \tag{4}$$

$$-\mathrm{i}\partial_x \psi_k = \phi^* \phi_k, \tag{5}$$

$$(\mathrm{i}\partial_t + \partial_x^2)\phi_k = -2\mathrm{i}(\partial_x \psi^*)\phi_k, \tag{6}$$

$$(\mathrm{i}\partial_t + \partial_x^2)\psi_k = 2\mathrm{i}(\partial_x \phi^*)\phi_k, \tag{7}$$

in which the functions $\phi(x, t)$ and $\psi(x, t)$ are defined by

$$\phi(x, t) = \int_L \mathrm{d}\lambda(k)\phi_k(x, t), \quad \psi(x, t) = \int_L \mathrm{d}\lambda(k)\psi_k(x, t). \tag{8}$$

It is now an easy exercise to show that $\phi(x, t)$ satisfies the NLS. In fact, from (5) and (8) we have $-\mathrm{i}\partial_x \psi = |\phi|^2$, which in combination with (6) and (8) gives

$$(\mathrm{i}\partial_t + \partial_x^2)\phi + 2|\phi|^2\phi = 0, \tag{9}$$

which is the NLS. The linear eigenvalue problem of Zakharov–Shabat [1] may be inferred from (4) and (5) introducing the functions $X_{1k} = \phi_k \exp[-\frac{1}{2}\mathrm{i} \times (kx - k^2 t)]$ and $X_{2k} = \psi_k \exp[-\frac{1}{2}\mathrm{i}(kx - k^2 t)]$. Furthermore, the real quantity $y(x, t) \equiv -\frac{1}{2}\mathrm{i}\psi(x, t)$ satisfies the potential NLS, given in refs. [7–9], i.e.

$$\partial_t(-\dot{y}/2y') = \partial_x [y'''/2y' - (y''^2 + \dot{y}^2)/4y'^2 + 4y']. \tag{10}$$

In eq. (10) the dots and primes denote the differentiations ∂_t and ∂_x with respect to t and x. The solutions of (10) can be found directly choosing appropriate L and $\lambda(l)$ in eq. (3) and from every solution of (10) one can obtain a solution of the NLS, cf. refs. [9,10].

In order to establish the connection with the IHSC we introduce the functions

$$A(x, t) = \int_L \mathrm{d}\lambda(k)\phi_k(x, t)k^{-1},$$

$$B(x, t) = \int_L \mathrm{d}\lambda(k)\psi_k(x, t)k^{-1}. \tag{11}$$

From (4), (8) and (11) we find

$$\phi(x, t) = -\mathrm{i}(1 + B)^{-1}\partial_x A \tag{12}$$

and using the relation $-\mathrm{i}\partial_x B = \phi^* A$, cf. (5) and (11), it is easy to show that $(1 + B)\partial_x B^* = -A^*\partial_x A$ and hence

$$|1 + B|^2 + |A|^2 = C, \tag{13}$$

with C independent of x.

From eqs. (5)–(7), (11) and the derivation of (13) it follows that A and $1 + B^*$ both satisfy the same partial differential equation

$$(\mathrm{i}\partial_t + \partial_x^2)\Phi = -[2|\partial_x \Phi|^2/(C - |\Phi|^2)]\ \Phi, \tag{14}$$

implying in particular that C is a constant of the motion, i.e. $\partial_t C = 0$.

Further insight in eq. (14) can be obtained from the corresponding potential equation. In fact, substituting $\Phi = \kappa \exp(\mathrm{i}\gamma)$ with κ and γ real and introducing a real variable $q(x, t)$ such that $\kappa^2 = \frac{1}{2}(1 - \partial_x q)$, we have, taking $C = 1$,

$$\gamma' = \tfrac{1}{2}\dot{q}(1 - q')^{-1},$$

$$\dot{\gamma} = -\tfrac{1}{2}q'''(1 - q')^{-1} - \tfrac{1}{4}(q''^2 + \dot{q}^2)(1 - q')^{-2} + \tfrac{1}{2}(q''^2 + \dot{q}^2)(1 - q'^2)^{-1}, \tag{15}$$

and the compatibility condition $\partial_t \gamma' = \partial_x \dot{\gamma}$ gives

$$(\ddot{q} + q'''')(1 - q'^2)^{-1} + 4(\dot{q}\dot{q}' + q''q''')(1 - q'^2)^{-2} + (q''^2 + \dot{q}^2)(1 + 3q'^2)q''(1 - q'^2)^{-3} = 0. \tag{16}$$

Eq. (16) is identical with the potential equation for the IHSC (in an external magnetic field $\boldsymbol{b} = b\boldsymbol{e}_z$), i.e.

$$\partial_t \boldsymbol{S} = \boldsymbol{S} \times \partial_x^2 \boldsymbol{S} + \boldsymbol{S} \times b\boldsymbol{e}_z, \tag{17}$$

as follows from eq. (9) of ref. [8] in the isotropic case with $c = 0$, $F(q') = bq'$. Using eqs. (3), (6) and (7) of ref. [8] one can express κ, γ' and $\dot{\gamma}$ in terms of the polar angles θ and ϕ of the spin vector $\boldsymbol{S}$. The result is

$$\kappa = \sin\tfrac{1}{2}\theta, \quad \gamma' = \phi'\cos^2\tfrac{1}{2}\theta,$$

$$\dot{\gamma} = \tfrac{1}{2}\theta''\ \mathrm{cotg}\,\tfrac{1}{2}\theta + \tfrac{1}{4}\theta'^2 + \gamma'^2(2\tan^2\tfrac{1}{2}\theta - 1).$$

The connection between the solutions A and $1 + B^*$ of eq. (14) can be inferred using (7), (11) and the derivation of (13). Writing $A = \kappa \exp(\mathrm{i}\gamma)$ and $1 + B^* = \widetilde{\kappa} \exp(\mathrm{i}\widetilde{\gamma})$, we have

$$\widetilde{\kappa} = (1 - \kappa^2)^{1/2}, \quad \widetilde{\gamma}' = \kappa^2\gamma'(1 - \kappa^2)^{-1},$$

$$\dot{\widetilde{\gamma}} = -(1 - \kappa^2)^{-1}(\kappa\kappa'' + \kappa'^2) + (1 - \kappa^2)^{-2}(2 - 3\kappa^2)(\kappa'^2 + \kappa^2\gamma'^2). \tag{18}$$

Eq. (18) may be considered as an involution which maps a solution A of eq. (14) on another solution $1 + B^*$ of (14). Inserting (18) in (12) we obtain an explicit expression for the Miura transformation which maps a solution of the IHSC on a solution of the NLS. This transformation is equivalent to the one given by Lakshmanan [11], cf. also ref. [8], and also to the isotropic limit of a more general transformation which is valid in the case of uniaxial anisotropy [10]. The linear eigenvalue problem of the IHSC, as given by Takhtadzhyan [3] may be obtained as a corollary, but details will be given in a more extended publication [12].

We have shown that the solutions of the NLS and the IHSC can be obtained from the same integral equation (1), using eqs. (8) and (11), respectively. In order to find e.g. soliton solutions one can insert a measure $d\lambda(k)$ containing a linear combination of delta functions, and the integrations in (8) and (11) over a contour through some of the delta peaks are trivial. At the same time we have obtained the Miura transformation between the IHSC and the NLS.

Remark. An analogous treatment can be given to show that the solutions of the modified Korteweg–de Vries equation (MKdV) can be derived from the integral equation given by Fokas and Ablowitz [5] for the KdV equation. In fact, starting from the integral equation [5]

$$v_k(x,t) + \mathrm{i}\,\mathrm{e}^{\mathrm{i}(kx+k^3t)} \int_L d\lambda(l)\, \frac{v_l(x,t)}{k+l} = \mathrm{e}^{\mathrm{i}(kx+k^3t)}, \tag{19}$$

one can derive the relations

$$(-k - \mathrm{i}\partial_x)\,\mathrm{i}\partial_x v_k = (\partial_x v) v_k, \tag{20}$$

$$(\partial_t + \partial_x^3) v_k = 3(\partial_x v)(\partial_x v_k), \tag{21}$$

with $v = \int_L d\lambda(l) v_l$, which were presented in a slightly different form in ref. [5]. From (20) and (21) it is easy to show that $u = \partial_x v$ satisfies the KdV, i.e. $(\partial_t + \partial_x^3) u = 6u\partial_x u$. We now consider the function $a \equiv \int_L d\lambda(l) u_l l^{-1}$. From (20) and (21) we then have

$$\partial_x v = (a + \mathrm{i})^{-1} \partial_x^2 a, \tag{22}$$

$$(\partial_t + \partial_x^3) a = 3(a + \mathrm{i})^{-1} (\partial_x^2 a) \partial_x a. \tag{23}$$

Using the substitution $w = \partial_x \ln(\mathrm{i} + a)$, eqs. (23) and (22) reduce to $(\partial_t + \partial_x^3) w - 6w^2 \partial_x w = 0$, which is the MKdV, and $u = \partial_x w + w^2$, which is the Miura transformation [13] which maps solutions of the MKdV on solutions of the KdV.

A more extended version of these considerations in which also integral equations of the type (1) with a source term $k^{-n} \exp[\mathrm{i}(kx - k^2 t)]$ are taken into consideration will be published in the future [12].

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References

[1] V.E. Zakharov and A.B. Shabat, Sov. Phys. JETP 34 (1972) 62.
[2] V.E. Zakharov and A.B. Shabat, Funct. Anal. Appl. 8 (1974) 226.
[3] L.A. Takhtadzhyan, Phys. Lett. 64A (1977) 235.
[4] V.E. Zakharov and L.A. Takhtadzhyan, Theor. Math. Phys. 38 (1979) 17.
[5] A.S. Fokas and M.J. Ablowitz, Phys. Rev. Lett. 47 (1981) 1096.
[6] R.R. Rosales, Stud. Appl. Math. 59 (1978) 117.
[7] M. Boiti, C. Laddomada and F. Pempinelli, Nuovo Cimento 62B (1981) 315.
[8] G.R.W. Quispel and H.W. Capel, Phys. Lett. 85A (1981) 248.
[9] G.R.W. Quispel and H.W. Capel, Physica 110A (1982) 41.
[10] G.R.W. Quispel and H.W. Capel, Phys. Lett. 88A (1982) 371.
[11] M. Lakshmanan, Phys. Lett. 61A (1977) 53.
[12] F.W. Nijhoff, J.v.d. Linden, G.R.W. Quispel, H.W. Capel and J. Velthuizen, to be published.
[13] R.M. Miura, J. Math. Phys. 9 (1968) 1202.