CONTINUOUS SYMMETRIES AND PAINLEVÉ REDUCTION OF THE KAC-VAN MOERBEKE EQUATION

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ABSTRACT. A method is given to derive the point symmetries of partial differential-difference equations. Applying the method to the Kac-van Moerbeke equation we find its symmetries form a Kac-Moody-Virasoro algebra. Using the symmetries, a similarity reduction of the Kac-van Moerbeke equation to an ordinary differential-difference equation is obtained. This reduced equation possesses a Lax pair, reduces to the first Painlevé equation in the continuum limit, and satisfies a recently proposed discrete version of the Painlevé property.

1. Introduction

In recent years much effort has been invested in taking concepts we know and love in the theory of differential equations, and finding the analogous concepts (and methods and theorems) for difference equations [1, 2]. For example, integrable partial differential equations were generalized to integrable partial difference equations [3] (see also [4]), and integrable ordinary differential equations were generalized to integrable mappings [5-8]. Attempts were also made to find discrete analogues of the Painlevé equations and of the Painlevé property [9-14].

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In this paper we show how the method of Lie-symmetries which is so useful for finding similarity reductions of partial differential equations (PDEs) [15-21] can also be applied to partial differential-difference equations (PDΔEs).[1] In both cases the number of independent variables can be reduced. The difference is that, for example, a PDE in two independent variables reduces to an ordinary differential equation, whereas a PDΔE in two independent variables reduces to an ordinary differential-difference equation (our convention is to call an equation in one independent variable "ordinary", and an equation in more than one independent variable "partial"). One of the reasons similarity reductions have received a lot of attention is the Ablowitz-Ramani-Segur conjecture [22], which says that similarity reductions of integrable PDEs have the Painlevé property. Here we show that the similarity reduction of the Kac-van Moerbeke PDΔE possesses the discrete Painlevé property [10]. An alternative approach is given in [23].

2. Symmetries

An equation for a scalar function u ($\mathbb{R}^n \to \mathbb{R}^n$) is said to possess a Lie point symmetry if it is invariant under the infinitesimal transformation

$$\begin{aligned} \boldsymbol{x}^* &= \boldsymbol{x} + \varepsilon\boldsymbol{\xi}(\boldsymbol{x}, u(\boldsymbol{x})) + O(\varepsilon^2), & (1)\\ u^*(\boldsymbol{x}^*) &= u(\boldsymbol{x}) + \varepsilon v(\boldsymbol{x}, u(\boldsymbol{x})) + O(\varepsilon^2). & (2) \end{aligned}$$

For (differential-)difference equations the question now arises what the expression for $u^*(\boldsymbol{x}^*+\boldsymbol{\omega})$ is, where $\boldsymbol{\omega} \in \mathbb{R}^n$ is some given fixed span appearing in the equation. To answer this question, we first express the right-hand-side of (2) in terms of $\boldsymbol{x}^*$, using (1):

$$u^*(\boldsymbol{x}^*) = u(\boldsymbol{x}^* - \varepsilon\boldsymbol{\xi}(\boldsymbol{x}^*, u^*(\boldsymbol{x}^*))) + \varepsilon v(\boldsymbol{x}^*, u^*(\boldsymbol{x}^*)) + O(\varepsilon^2). \tag{3}$$

We now shift $\boldsymbol{x}^* \to \boldsymbol{x}^* + \boldsymbol{\omega}$:

$$u^*(\boldsymbol{x}^* + \boldsymbol{\omega}) = u(\boldsymbol{x}^* + \boldsymbol{\omega} - \varepsilon\boldsymbol{\xi}(\boldsymbol{x}^* + \boldsymbol{\omega}, u^*(\boldsymbol{x}^* + \boldsymbol{\omega}))) + \varepsilon v(\boldsymbol{x}^* + \boldsymbol{\omega}, u^*(\boldsymbol{x}^* + \boldsymbol{\omega})) + O(\varepsilon^2). \tag{4}$$

Having done this, we express the right-hand-side in terms of $\boldsymbol{x}$ again (using (1)):

$$\begin{aligned} u^*(\boldsymbol{x}^* + \boldsymbol{\omega}) &= u(\boldsymbol{x} + \boldsymbol{\omega} + \varepsilon\boldsymbol{\xi}(\boldsymbol{x}, u(\boldsymbol{x})) - \varepsilon\boldsymbol{\xi}(\boldsymbol{x} + \boldsymbol{\omega}, u(\boldsymbol{x} + \boldsymbol{\omega})))\\ &\quad + \varepsilon v(\boldsymbol{x} + \boldsymbol{\omega}, u(\boldsymbol{x} + \boldsymbol{\omega})) + O(\varepsilon^2). \end{aligned} \tag{5}$$

Expanding the right-hand-side in a Taylor series, we obtain the general result

$$\begin{aligned} u^*(\boldsymbol{x}^* + \boldsymbol{\omega}) &= u(\boldsymbol{x} + \boldsymbol{\omega}) + \varepsilon v(\boldsymbol{x} + \boldsymbol{\omega}, u(\boldsymbol{x} + \boldsymbol{\omega}))\\ &\quad + \varepsilon \sum_{i=1}^{n} [\boldsymbol{\xi}_i(\boldsymbol{x}, u(\boldsymbol{x})) - \boldsymbol{\xi}_i(\boldsymbol{x} + \boldsymbol{\omega}, u(\boldsymbol{x} + \boldsymbol{\omega}))] \frac{\mathrm{d}u}{\mathrm{d}x_i}(\boldsymbol{x} + \boldsymbol{\omega}) + O(\varepsilon^2). \end{aligned} \tag{6}$$

We now apply this theory to a specific example. As our example we have chosen the Kac-van Moerbeke equation, however the same approach holds for numerous other PDΔEs and partial difference equations [24].

The Kac-van Moerbeke equation [25,26], also known as the discrete Korteweg-de Vries equation, is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} u(x,t) = u(x,t)[u(x+1,t) - u(x-1,t)], \tag{7}$$

[1] J.M. Hill has kindly pointed out to us that an application to ordinary differential-difference equations is given in [21].

where $u(x,t)$ is a function $\mathbf{R}^2 \to \mathbf{R}$. (The reader should realize that although usually the independent variable x in this equation is taken to be integer, here we allow x to be any real number.) Equation (7) occurs in the study of the spectra of Langmuir oscillations in a plasma [27].

To derive the symmetries of the Kac-van Moerbeke equation (7), we assume that (7) is invariant under the infinitesimal point transformation (1)-(2), with $\boldsymbol{x} = (x,t)$ and $\boldsymbol{\xi} =: (\nu,\tau)$. The transformed equation is

$$\frac{\mathrm{d}}{\mathrm{d}t^*}u^*(x^*,t^*) = u^*(x^*,t^*)[u^*(x^*+1,t^*) - u^*(x^*-1,t^*)]. \tag{8}$$

The left-hand-side of (8) is known from the usual theory [15,17]. In the right-hand-side we insert (6), with $\boldsymbol{\omega} = (1,0)$ and $(-1,0)$, respectively.

It then follows immediately that

$$\begin{aligned} \nu(x+1,t,u(x+1,t)) &= \nu(x,t,u(x,t)), & (9)\\ \tau(x+1,t,u(x+1,t)) &= \tau(x,t,u(x,t)). & (10)\end{aligned}$$

This, in turn, can only be true if ν and τ do not depend on u. We thus obtain

$$\begin{aligned} \frac{\partial v(x)}{\partial t} &+ \left[\frac{\partial v(x)}{\partial u(x)} - \frac{\partial \tau(x)}{\partial t}\right]\frac{\mathrm{d}u(x)}{\mathrm{d}x} - \frac{\partial \nu(x)}{\partial t}\frac{\mathrm{d}u(x)}{\mathrm{d}x} \\ &= v(x)[u(x+1) - u(x-1)] + u(x)[v(x+1) - v(x-1)], \end{aligned} \tag{11}$$

where we have not indicated the t and u dependence of v and τ explicitly.

Substituting (7) in (11) and solving, we obtain the symmetries of the Kac-van Moerbeke equation:

$$\begin{aligned} u^*(x^*,t^*) &= u(x,t) + \varepsilon\alpha(x)u(x,t), & (12)\\ t^* &= t + \varepsilon[\beta(x) - \alpha(x)t], & (13)\\ x^* &= x + \varepsilon\gamma(x), & (14)\end{aligned}$$

where $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are arbitrary unit periodic functions [28]

$$\alpha(x) = \alpha(x+1), \quad \beta(x) = \beta(x+1), \quad \gamma(x) = \gamma(x+1). \tag{15}$$

Expanding the functions $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ in a Fourier series we obtain the generators of the symmetry algebra

$$\mathrm{G}_{1,k} = -\frac{\mathrm{e}^{2\pi ikx}}{2\pi\mathrm{i}}\frac{\partial}{\partial x}, \quad \mathrm{G}_{2,k} = \mathrm{e}^{2\pi ikx}\frac{\partial}{\partial t}, \quad \mathrm{G}_{3,k} = \mathrm{e}^{2\pi ikx}\left[t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}\right]. \tag{16}$$

These generators form a centreless Kac-Moody-Virasoro algebra [16] given by the non-zero commutation relations

$$\begin{aligned} [\mathrm{G}_{1,k},\mathrm{G}_{1,\ell}] &= (k-\ell)\mathrm{G}_{1,k+\ell}, \quad [\mathrm{G}_{1,k},\mathrm{G}_{2,\ell}] = -\ell\mathrm{G}_{2,k+\ell}, \\ [\mathrm{G}_{1,k},\mathrm{G}_{3,\ell}] &= -\ell\mathrm{G}_{3,k+\ell}, \quad [\mathrm{G}_{2,k},\mathrm{G}_{3,\ell}] = \mathrm{G}_{2,k+\ell}. \end{aligned} \tag{17}$$

3. Invariant reduction

The characteristic equation associated with the symmetries (12-14) is

$$\frac{\mathrm{d}u}{\alpha(x)u} = \frac{\mathrm{d}t}{\beta(x) - \alpha(x)t} = \frac{\mathrm{d}x}{\gamma(x)}. \tag{18}$$

Equation (18) represents a linear constraint leading to the similarity solution

$$u(x,t) = v(\eta)/[b(x) - at] \tag{19}$$

with

$$\eta = x + d(x) + \frac{c}{a}\log[b(x) - at], \tag{20}$$

where a and c are constants, and

$$b(x) = b(x+1), \quad d(x) = d(x+1). \tag{21}$$

The explicit expressions which we have derived for a, c, $b(x)$ and $d(x)$ in terms of integrals over x, involving the functions $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are somewhat complicated and are given in the appendix. Note that the linear constraint (19) has been obtained directly from the Lie symmetries, without invoking the integrability of the Kac-van Moerbeke equation. Inserting (19-21) in (7), the Kac-van Moerbeke PDΔE reduces to the following ordinary differential-difference equation

$$av(\eta) - c\frac{\mathrm{d}v}{\mathrm{d}\eta} = v(\eta)[v(\eta+1) - v(\eta-1)]. \tag{22}$$

In the special case that $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are constants, we obtain

$$\frac{\alpha}{\gamma} = \frac{a}{c}, \quad \frac{\beta}{\gamma} = \frac{b}{c}, \quad d(x) = 0 \tag{23}$$

with b being a constant as well. This corresponds to the special similarity reduction

$$u(x,t) = v(\eta)/(b - at) \tag{24}$$

with

$$\eta = x + \frac{c}{a}\log(b - at), \tag{25}$$

where a, b and c are constants. The function $v(\eta)$ would again satisfy (22).

Equation (22) is unusual, in the sense that it is both "advanced" and "retarded". This makes its integration difficult. Here we restrict ourselves to exhibiting two particular solutions for the case $a = 0$, cf. also [29]:

(i) Soliton solution

$$v(\eta) = \frac{c\kappa[1 + \mathrm{e}^{\kappa(\eta+1)+\delta}][1 + \mathrm{e}^{\kappa(\eta-2)+\delta}]}{(\mathrm{e}^{-\kappa} - e^{\kappa})[1 + \mathrm{e}^{\kappa\eta+\delta}][1 + \mathrm{e}^{\kappa(\eta-1)+\delta}]}, \tag{26}$$

where κ and δ are arbitrary parameters.

(ii) Rational solution

$$v(\eta) = -\frac{c}{2}\frac{(\eta+1+\delta)(\eta-2+\delta)}{(\eta+\delta)(\eta-1+\delta)}, \tag{27}$$

where δ is an arbitrary parameter.

A continuum limit of equation (22) is obtained by putting

$$v(\eta) = 1 + \varepsilon^2 w(\eta), \quad v(\eta+1) - v(\eta-1) = 2\varepsilon^2 \sinh\left(\varepsilon \frac{\mathrm{d}}{\mathrm{d}\eta}\right) w(\eta), \tag{28}$$

where $a\varepsilon^5 A$ and $c = -2\varepsilon$. Then $w(\eta)$ satisfies

$$\frac{1}{3}\frac{\mathrm{d}^2 w}{\mathrm{d}\eta^2} + w^2 - A\eta = 0. \tag{29}$$

Equation (29) is the first Painlevé equation [30], apart from a trivial transformation.

Up to this point, the fact that the Kac-van Moerbeke equation (7) is integrable has not played a special role. Now, however, we will use the fact that (7) is integrable, and has the following Lax representation [14]:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} Q(x,t,z) &= -\tfrac{1}{2}[u(x,t) + u(x+1,t)]Q(x,t,z) - u^{1/2}(x,t)u^{1/2}(x-1,t)Q(x-2,t,z), \\
zQ(x,t,z) &= \tfrac{1}{2}u^{1/2}(x+1,t)Q(x+1,t,z) + \tfrac{1}{2}u^{1/2}(x,t)Q(x-1,t,z).
\end{aligned} \tag{30}$$

This leads to the following Lax representation for the reduced equation (22):

$$\begin{aligned}
\left[c\frac{\mathrm{d}}{\mathrm{d}\eta} + \tfrac{1}{2}a\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\right] R(\eta,\zeta) &= \tfrac{1}{2}[v(\eta) + v(\eta+1)]R(\eta,\zeta) + v^{1/2}(\eta)v^{1/2}(\eta-1)R(\eta-2,\zeta), \\
\zeta R(\eta,\zeta) &= \tfrac{1}{2}v^{1/2}(\eta+1)R(\eta+1,\zeta) + \tfrac{1}{2}v^{1/2}(\eta)R(\eta-1,\zeta).
\end{aligned} \tag{31}$$

(This equation is obtained from (30) putting $z = (b-at)^{-1/2}\zeta$ and $Q(x,t,z) = R(\eta,\zeta)$.) Indeed, after some algebra, the compatibility condition of (31) is just the reduced equation (22).

4. Singularity confinement

Recently, a discrete version of the Painlevé property has been proposed [10].[2] Stated simply it says that difference equations have the Painlevé property if their singularities are confined. We assume the same definition for differential-difference equations. Rewriting (22) as

$$v(\eta+1) = -c\frac{v'(\eta)}{v(\eta)} + a + v(\eta-1), \quad v'(\eta) := \frac{\mathrm{d}v}{\mathrm{d}\eta}, \tag{32}$$

we see that equation (32) has a singularity if $v(\eta_0) = 0$ and $v'(\eta_0) \neq 0$. We have expanded (shifted) equations for $v(\eta_0+2)$, $v(\eta_0+3)$ and $v(\eta_0+4)$ in a Laurent series in $v(\eta_0)$. It turns out to be necessary to keep the leading three orders. Then, if $v(\eta_0-1)$, $v'(\eta_0-1)$, $v'(\eta_0)$, $v''(\eta_0)$ and $v'''(\eta_0)$ are in general position, we obtain

$$\begin{aligned}
v(\eta_0+1) &= -c\frac{v'(\eta_0)}{v(\eta_0)} + a + v(\eta_0-1) = O\left(1/v(\eta_0)\right) \\
v(\eta_0+2) &= c\frac{v'(\eta_0)}{v(\eta_0)} + v(\eta_0-1) + 2a - c\frac{v''(\eta_0)}{v'(\eta_0)} + v(\eta_0) \\
&\quad - \left\{(a + v(\eta_0-1))\frac{v''(\eta_0)}{v'(\eta_0)} - \frac{1}{c}(a+v(\eta_0-1))^2 - v'(\eta_0-1)\right\}\frac{v(\eta_0)}{v'(\eta_0)}
\end{aligned}$$

[2] For an alternative discrete version of the Painlevé property see [9].

$$+O\left(v^2(\eta_0)\right)=O\left(1/v(\eta_0)\right)$$

$$v(\eta_0+3) = \left\{c\frac{v'''(\eta_0)}{v'(\eta_0)}-c\left(\frac{v''(\eta_0)}{v'(\eta_0)}\right)^2-3v'(\eta_0-1)+v(\eta_0-1)\frac{v''(\eta_0)}{v'(\eta_0)}\right.$$
$$\left.+\frac{a^2}{c}-\frac{1}{c}\left(v(\eta_0-1)\right)^2\right\}\frac{v(\eta_0)}{v'(\eta_0)}+O\left(v^2(\eta_0)\right)=O(v(\eta_0))$$
$$v(\eta_0+4) = \left(v(\eta_0+2)-c\frac{v'(\eta_0+3)}{v(\eta_0+3)}+a\right)=O(1) \qquad (33)$$

Hence, the singularity is confined and (22) or (32) satisfies the singularity confinement criterion in a remarkably nontrivial way.

5. Final remarks

As we have indicated in §2, the above method applies to other PDΔEs as well, whether integrable or not. We here briefly mention the results of applying our method to the Reduced Two-Dimensional Toda Equation [31-33]

$$\frac{\partial^2 u(x,t)}{\partial x\partial t}=\exp\left\{u(x,t)-u(x-1,t-1)\right\}-\exp\left\{u(x+1,t+1)-u(x,t)\right\}. \qquad (34)$$

The symmetries of this equation are

$$x^* = x+\varepsilon\alpha(x)$$
$$t^* = t+\varepsilon\beta(t)$$
$$u^* = u+\varepsilon[qx-qt-x\alpha'(x)+\gamma(x)-t\beta'(t)+\delta(t)]. \qquad (35)$$

In (35) α,β,γ and δ are arbitrary unit periodic functions:

$$\alpha(x)=\alpha(x+1),\ \gamma(x)=\gamma(x+1),\ \beta(t)=\beta(t+1),\ \delta(t)=\delta(t+1), \qquad (36)$$

the primes denote derivatives, and q is an arbitrary constant. Expanding the functions $\alpha(x),\beta(x),\gamma(x)$ and $\delta(x)$ in a Fourier series we obtain the generators of the symmetry algebra

$$\mathrm{G}_{1,k} = \mathrm{e}^{2\pi ikx}\left[kx\frac{\partial}{\partial u}-\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial x}\right],\quad \mathrm{G}_{2,k}=\frac{\mathrm{e}^{2\pi ikx}}{2\pi\mathrm{i}}\frac{\partial}{\partial u},$$
$$\mathrm{G}_{3,k} = \mathrm{e}^{2\pi\mathrm{i}kt}\left[kt\frac{\partial}{\partial u}-\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial t}\right],\quad \mathrm{G}_{4,k}=\frac{\mathrm{e}^{2\pi ikx}}{2\pi\mathrm{i}}\frac{\partial}{\partial u},\quad \mathrm{G}_5=(x-t)\frac{\partial}{\partial u}. \qquad (37)$$

These generators form an infinite-dimensional algebra given by the non-zero commutation relations

$$[\mathrm{G}_{1,k},\mathrm{G}_{1,l}] = (k-l)\mathrm{G}_{1,k+l}+(k-l)\mathrm{G}_{2,k+l},$$
$$[\mathrm{G}_{1,k},\mathrm{G}_{2,l}] = -l\mathrm{G}_{2,k+l},$$
$$[\mathrm{G}_{1,k},\mathrm{G}_5] = -\mathrm{G}_{2,k},$$
$$[\mathrm{G}_{3,k},\mathrm{G}_{3,l}] = (k-l)\mathrm{G}_{3,k+l}+(k-l)\mathrm{G}_{4,k+l},$$
$$[\mathrm{G}_{3,k},\mathrm{G}_{4,l}] = -l\mathrm{G}_{4,k+l},$$
$$[\mathrm{G}_{3,k},\mathrm{G}_5] = -\mathrm{G}_{4,k}. \qquad (38)$$

Because G_5 cannot be written as the commutator of two elements in the algebra, it follows that the algebra described by (38) is not a Kac-Moody-Virasoro algebra.

More results about similarity solutions and symmetry algebra of (34), and its derivation from the Two-Dimensional Toda Equation are given in [24]. A shorter version of this paper is published in [34].

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Appendix A.

As a first step we integrate the second and third member involving dx and dt of (18). This gives

$$t \exp\left\{\int_X^x \frac{\alpha(x')}{\gamma(x')}\,\mathrm{d}x'\right\} = \int_X^x \mathrm{d}x\frac{\beta(x')}{\gamma(x')} \exp\left\{\int_X^{x'} \mathrm{d}x''\frac{\alpha(x'')}{\gamma(x'')}\right\} + \text{cst.} \tag{A.1}$$

Equation (A.1) can be expressed as

$$\log f(x) + \log\left(\frac{g(x)}{f(x)} - t\right) = \text{cst.} \tag{A.2}$$

in which

$$f(x) = \exp\left\{\int_X^x \frac{\alpha(x')}{\gamma(x')}\,\mathrm{d}x'\right\}, \quad g(x) = \int_X^x \frac{\beta(x')}{\gamma(x')} f(x')\,\mathrm{d}x' \tag{A.3}$$

For $\alpha(x)/\gamma(x)$ we use the decomposition

$$\frac{\alpha(x)}{\gamma(x)} = \frac{a}{c} + h(x) \tag{A.4}$$

with

$$h(x) = h(x+1), \quad \int_x^{x+1} h(x')\,\mathrm{d}x' = 0, \quad \int_x^{x+1} \frac{\alpha(x')}{\gamma(x')}\,\mathrm{d}x' = \frac{a}{c} \tag{A.5}$$

This yields

$$f(x) = \exp\left\{\frac{a[x + d(x)]}{c}\right\} \tag{A.6}$$

with

$$d(x) = \int_X^x \left(\frac{c}{a}\,\frac{\alpha(x')}{\gamma(x')} - 1\right)\mathrm{d}x' = d(x+1) \tag{A.7}$$

For the function $g(x)/f(x)$ we have

$$\frac{g(x+1)}{f(x+1)} = \frac{g(x)+\Gamma}{f(x)}, \quad \Gamma = \int_{X-1}^{X} \frac{\beta(x')}{\gamma(x')} \exp\left\{\frac{a[x'+d(x')]}{c}\right\} \mathrm{d}x' \tag{A.8}$$

From (A.2) we introduce a similarity variable ρ by identifying the constant with $ac^{-1}\rho + \log a$. Then ρ is given by

$$\rho = x + d(x) + \frac{c}{a}\log\left[a\frac{g(x)}{f(x)} - at\right] \tag{A.9}$$

Equation (A.9) differs from (20) by the fact that $g(x)/f(x)$ is not periodic. This can be repaired by considering the periodic function

$$\frac{g(x+1)}{f(x+1)} + \frac{\Gamma}{(1-\mathrm{e}^{a/c})\,f(x+1)} = \frac{g(x)}{f(x)} + \frac{\Gamma}{(1-\mathrm{e}^{a/c})\,f(x)} \tag{A.10}$$

and introducing the new variable

$$\eta = \rho + \frac{c}{a}\log\left[1 + a\Gamma\frac{\mathrm{e}^{-a\rho/c}}{1-\mathrm{e}^{-a/c}}\right] = x + d(x) + \frac{c}{a}\log[b(x) - at] \tag{A.11}$$

with

$$\begin{aligned}\frac{b(x)}{a} &= \frac{g(x)}{f(x)} + \frac{\Gamma}{(1-\mathrm{e}^{-a/c})\,f(x)} \\ &= \exp\left\{\frac{-a[x+d(x)]}{c}\right\}\int_{X}^{x}\frac{\beta(x')}{\gamma(x')}\exp\left\{\frac{a[x'+d(x')]}{c}\right\}\mathrm{d}x'\left(1+\frac{\Gamma}{1-\mathrm{e}^{-a/c}}\right)\end{aligned} \tag{A.12}$$

in which the constant Γ is given by (A.8).

Finally, from the first and second member of (18) we obtain

$$\log u = \log f(x) + \text{cst.} \tag{A.13}$$

Identifying the integration constant with $\log(v(\eta)\exp\{-a\eta/c\})$ and using (A.6), we show that

$$u = \frac{v(\eta)}{b(x) - at}, \tag{A.14}$$

which is (19).

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