Physics Letters A 170 (1992) 379–383
North-Holland

PHYSICS LETTERS A

Continuous symmetries of differential–difference equations: the Kac–van Moerbeke equation and Painlevé reduction

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Received 6 March 1992; revised manuscript received 7 September 1992; accepted for publication 9 September 1992
Communicated by A.P. Fordy

A method is given to derive the point symmetries of partial differential–difference equations. Applying the method to the Kac–van Moerbeke equation we find its symmetries form a Kac–Moody–Virasoro algebra. Using the symmetries, a similarity reduction of the Kac–van Moerbeke equation to an ordinary differential–difference equation is obtained. This reduced equation possesses a Lax pair, reduces to the first Painlevé equation in the continuum limit, and satisfies a recently proposed discrete version of the Painlevé property.

1. In recent years much effort has been invested in taking concepts we know and love in the theory of differential equations, and finding the analogous concepts (and methods and theorems) for difference equations [1,2]. For example, integrable partial differential equations were generalized to integrable partial difference equations [3] (see also ref. [4]), and integrable ordinary differential equations were generalized to integrable mappings [5–8]. Attempts were also made to find discrete analogs of the Painlevé equations and of the Painlevé property [9–14].

In this Letter we show how the method of Lie symmetries which is so useful for finding similarity reductions of partial differential equations (PDEs) [15–21] can also be applied to partial differential–difference equations (PDΔE's) [#1]. In both cases the number of independent variables can be reduced. The difference is that, for example, a PDE in two independent variables reduces to an ordinary differential equation, whereas a PDΔE in two independent variables reduces to an ordinary differential–difference equation (our convention is to call an equation in one independent variable "ordinary", and an equation in more than one independent variable "partial"). One of the reasons similarity reductions have received a lot of attention is the Ablowitz–Ramani–Segur conjecture [22], which says that similarity reductions of integrable PDEs have the Painlevé property. Here we show that the similarity reduction of the Kac–van Moerbeke PDΔE possesses the discrete Painlevé property [10]. The approach for finding Lie symmetries of ref. [23] does not lead to the most general similarity reductions.

2. An equation for a scalar function u ($\mathbb{R}^n \to \mathbb{R}^n$) is said to possess a Lie point symmetry if it is invariant under the infinitesimal transformation

$$\boldsymbol{x}^* = \boldsymbol{x} + \epsilon \boldsymbol{\xi}(\boldsymbol{x}, u(\boldsymbol{x})) + \mathrm{O}(\epsilon^2) , \tag{1}$$

$$u^*(\boldsymbol{x}^*) = u(\boldsymbol{x}) + \epsilon v(\boldsymbol{x}, u(\boldsymbol{x})) + \mathrm{O}(\epsilon^2) . \tag{2}$$

For (differential–)difference equations the question now arises what the expression for $u^*(\boldsymbol{x}^* + \boldsymbol{\omega})$ is, where $\boldsymbol{\omega} \in \mathbb{R}^n$ is some given fixed span appearing in the equation. To answer this question, we first ex-

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[#1] J.M. Hill has kindly pointed out to us that an application to ordinary differential–difference equations is given in ref. [21].

press the right-hand-side of (2) in terms of $\boldsymbol{x}^*$, using (1)

$$u^*(\boldsymbol{x}^*)=u(\boldsymbol{x}^*-\epsilon\boldsymbol{\xi}(\boldsymbol{x}^*,u^*(\boldsymbol{x}^*))) +\epsilon v(\boldsymbol{x}^*,u^*(\boldsymbol{x}^*))+O(\epsilon^2) . \tag{3}$$

We now shift $\boldsymbol{x}^*\to\boldsymbol{x}^*+\boldsymbol{\omega}$:

$$u^*(\boldsymbol{x}^*+\boldsymbol{\omega})=u(\boldsymbol{x}^*+\boldsymbol{\omega}-\epsilon\boldsymbol{\xi}(\boldsymbol{x}^*+\boldsymbol{\omega},u^*(\boldsymbol{x}^*+\boldsymbol{\omega}))) +\epsilon v(\boldsymbol{x}^*+\boldsymbol{\omega},u^*(\boldsymbol{x}^*+\boldsymbol{\omega}))+O(\epsilon^2) . \tag{4}$$

Having done this, we express the right-hand-side in terms of $\boldsymbol{x}$ again (using (1)):

$$u^*(\boldsymbol{x}^*+\boldsymbol{\omega})=u(\boldsymbol{x}+\boldsymbol{\omega}+\epsilon\boldsymbol{\xi}(\boldsymbol{x},u(\boldsymbol{x})) -\epsilon\boldsymbol{\xi}(\boldsymbol{x}+\boldsymbol{\omega},u(\boldsymbol{x}+\boldsymbol{\omega}))) +\epsilon v(\boldsymbol{x}+\boldsymbol{\omega},u(\boldsymbol{x}+\boldsymbol{\omega}))+O(\epsilon^2) . \tag{5}$$

Expanding the right-hand-side in a Taylor series, we obtain the general result

$$u^*(\boldsymbol{x}^*+\boldsymbol{\omega})=u(\boldsymbol{x}+\boldsymbol{\omega})+\epsilon v(\boldsymbol{x}+\boldsymbol{\omega},u(\boldsymbol{x}+\boldsymbol{\omega})) +\epsilon\sum_{i=1}^{n}[\xi_i(\boldsymbol{x},u(\boldsymbol{x}))-\xi_i(\boldsymbol{x}+\boldsymbol{\omega},u(\boldsymbol{x}+\boldsymbol{\omega}))] \times\frac{\mathrm{d}u}{\mathrm{d}x_i}(\boldsymbol{x}+\boldsymbol{\omega})+O(\epsilon^2) . \tag{6}$$

We now apply this theory to a specific example. As our example we have chosen the Kac–van Moerbeke equation, however the same approach holds for numerous other PDΔE's and partial difference equations [24].

The Kac–van Moerbeke equation [25,26], also known as the discrete Korteweg–de Vries equation, is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}u(x,t)=u(x,t)[u(x+1,t)-u(x-1,t)] , \tag{7}$$

where $u(x,t)$ is a function $\mathbb{R}^2\to\mathbb{R}$. (The reader should realize that although usually the independent variable x in this equation is taken to be integer, here we allow x to be any real number.)

This equation occurs in the study of the spectra of Langmuir oscillations in a plasma [27]. To derive the symmetries of the Kac–van Moerbeke equation (7), we assume that eq. (7) is invariant under the infinitesimal point transformation (1), (2), with $\boldsymbol{x}=(x,t)$ and $\boldsymbol{\xi}=:(\nu,\tau)$. The transformed equation is

$$\frac{\mathrm{d}}{\mathrm{d}t^*}u^*(x^*,t^*)=u^*(x^*,t^*)[u^*(x^*+1,t^*) -u^*(x^*-1,t^*)] , \tag{8}$$

The left-hand-side of eq. (8) is know from the usual theory [15,17]. In the right-hand-side we insert (6), with $\boldsymbol{\omega}=(1,0)$ and $(-1,0)$, respectively.

It then follows immediately that

$$\nu(x+1,t,u(x+1,t))=\nu(x,t,u(x,t)) , \tag{9}$$

$$\tau(x+1,t,u(x+1,t))=\tau(x,t,u(x,t)) . \tag{10}$$

This, in turn, can only be true if ν and τ do not depend on u. We thus obtain

$$\frac{\partial v(x)}{\partial t}+\left(\frac{\partial v(x)}{\partial u(x)}-\frac{\partial\tau(x)}{\partial t}\right)\frac{\mathrm{d}u(x)}{\mathrm{d}t}-\frac{\partial\nu(x)}{\partial t}\frac{\mathrm{d}u(x)}{\mathrm{d}x} =v(x)[u(x+1)-u(x-1)] +u(x)[v(x+1)-v(x-1)] , \tag{11}$$

where we have not indicated the t and u dependence explicitly.

Substituting eq. (7) in (11) and solving, we obtain the symmetries of the Kac–van Moerbeke equation:

$$u^*(x^*,t^*)=u(x,t)+\epsilon\alpha(x)u(x,t) ,$$
$$t^*=t+\epsilon[\beta(x)-\alpha(x)t] ,$$
$$x^*=x+\epsilon\gamma(x) , \tag{12}$$

where $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are arbitrary unit periodic functions [28]

$$\alpha(x)=\alpha(x+1),\quad \beta(x)=\beta(x+1) ,$$
$$\gamma(x)=\gamma(x+1) . \tag{13}$$

Expanding the functions $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ in a Fourier series we obtain the generators of the symmetry algebra

$$G_{1,k}=-(2\pi\mathrm{i})^{-1}\mathrm{e}^{2\pi\mathrm{i}kx}\partial/\partial x,\quad G_{2,k}=\mathrm{e}^{2\pi\mathrm{i}kx}\partial/\partial t ,$$
$$G_{3,k}=\mathrm{e}^{2\pi\mathrm{i}kx}(t\partial/\partial t-u\partial/\partial u) . \tag{14}$$

These generators form a centreless Kac–Moody–Virasoro algebra [16] given by the non-zero commutation relations

$$[G_{1,k}, G_{1,l}] = (k-l)G_{1,k+l},$$
$$[G_{1,k}, G_{2,l}] = -lG_{2,k+l},$$
$$[G_{1,k}, G_{3,l}] = -lG_{3,k+l},$$
$$[G_{2,k}, G_{3,l}] = G_{2,k+l}. \qquad (15)$$

3. The characteristic equation associated with the symmetries (12) is

$$\frac{\mathrm{d}u}{\alpha(x)u} = \frac{\mathrm{d}t}{\beta(x)-\alpha(x)t} = \frac{\mathrm{d}x}{\gamma(x)}. \qquad (16)$$

Equation (16) represents a linear constraint leading to the similarity solution

$$u(x,t) = [b(x)-at]^{-1}v(\eta), \qquad (17a)$$

with

$$\eta = x + d(x) + \frac{c}{a}\log[b(x)-at], \qquad (17b)$$

where a and c are constants, and

$$b(x) = b(x+1), \quad d(x) = d(x+1).$$

The explicit expressions which we have derived for a, c, $b(x)$ and $d(x)$ in terms of integrals over x, involving the functions $\alpha(x)$, $\beta(x)$ and $\gamma(x)$, are somewhat complicated and will not be given here [29]. Note that the linear constraint (16) has been obtained directly from the Lie symmetries, without invoking the integrability of the Kac–van Moerbeke equation. Inserting (17a) with (17b) in (7), the Kac–van Moerbeke PDΔE reduces to the following ordinary differential–difference equation,

$$av(\eta) - c\frac{\mathrm{d}v}{\mathrm{d}\eta} = v(\eta)[v(\eta+1)-v(\eta-1)]. \qquad (18)$$

In the special case that $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are constants, we obtain

$$\frac{\alpha}{\gamma} = \frac{a}{c}, \quad \frac{\beta}{\gamma} = \frac{b}{c}, \quad d(x) = 0 \qquad (19)$$

with b being a constant as well. This corresponds to the special similarity reduction

$$u(x,t) = (b-at)^{-1}v(\eta), \qquad (20a)$$

with

$$\eta = x + \frac{c}{a}\log(b-at), \qquad (20b)$$

where a, b and c are constants. The function $v(\eta)$ would again satisfy eq. (18).

Equation (18) is unusual, in the sense that it is both "advanced" and "retarded". This makes its integration difficult. Here we restrict ourselves to exhibiting two particular solutions for the case $a=0$:

(i) Soliton solution

$$v(\eta) = \frac{c\kappa(1+\mathrm{e}^{\kappa(\eta+1)+\delta})(1+\mathrm{e}^{\kappa(\eta-2)+\delta})}{(\mathrm{e}^{-\kappa}-\mathrm{e}^{\kappa})(1+\mathrm{e}^{\kappa\eta+\delta})(1+\mathrm{e}^{\kappa(\eta-1)+\delta})}, \qquad (21)$$

where κ and δ are arbitrary parameters.

(ii) Rational solution

$$v(\eta) = -\frac{c}{2}\frac{(\eta+1+\delta)(\eta-2+\delta)}{(\eta+\delta)(\eta-1+\delta)}, \qquad (22)$$

where δ is an arbitrary parameter.

A continuum limit of eq. (18) is obtained by putting

$$v(\eta) = 1+\epsilon^2 w(\eta),$$
$$v(\eta+1)-v(\eta-1) = 2\epsilon^2\sinh(\epsilon\,\mathrm{d}/\mathrm{d}\eta)w(\eta),$$
$$a = \epsilon^5 A, \quad c = -2\epsilon. \qquad (23)$$

Then $w(\eta)$ satisfies

$$\frac{1}{3}\frac{\mathrm{d}^2 w}{\mathrm{d}\eta^2} + w^2 - A\eta = 0. \qquad (24)$$

Equation (24) is the first Painlevé equation [30], apart from a trivial transformation.

Up to this point, the fact that the Kac–van Moerbeke equation (7) is integrable has not played a special role. Now, however, we will use the fact that (7) is integrable, and has the following Lax representation [14],

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(x,t,z) = -\tfrac{1}{2}[u(x,t)+u(x+1,t)]Q(x,t,z) - u^{1/2}(x,t)u^{1/2}(x-1,t)Q(x-2,t,z),$$
$$zQ(x,t,z) = \tfrac{1}{2}u^{1/2}(x+1,t)Q(x+1,t,z) + \tfrac{1}{2}u^{1/2}(x,t)Q(x-1,t,z). \qquad (25)$$

This leads to the following Lax representation for the reduced equation (18):

$$\left(c\frac{\mathrm{d}}{\mathrm{d}\eta}+\frac{1}{2}a\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)R(\eta,\zeta)$$
$$=\tfrac{1}{2}[v(\eta)+v(\eta+1)]R(\eta,\zeta)$$
$$+v^{1/2}(\eta)v^{1/2}(\eta-1)R(\eta-2,\zeta)\,,$$
$$\zeta R(\eta,\zeta)=\tfrac{1}{2}v^{1/2}(\eta+1)R(\eta+1,\zeta)$$
$$+\tfrac{1}{2}v^{1/2}(\eta)R(\eta-1,\zeta)\,. \qquad (26)$$

(This equation is obtained from (25) putting $z=(b-at)^{-1/2}\zeta$ and $Q(x,t,z)=R(\eta,\zeta)$.) Indeed, after some algebra, the compatibility condition of eqs. (26) is just the reduced equation (18).

4. Recently, a discrete version of the Painlevé property has been proposed [10] [#2]. Stated simply it says that difference equations have the Painlevé property if their singularities are confined. We assume the same definition for differential–difference equations. Rewriting eq. (18) as

$$v(\eta+1)=-c\frac{v'(\eta)}{v(\eta)}+a+v(\eta-1)\,,$$
$$v'(\eta):=\frac{\mathrm{d}v}{\mathrm{d}\eta}, \qquad (27)$$

we see that this equation (27) has a singularity if $v(\eta_0)=0$ and $v'(\eta_0)\neq 0$. We have expanded (shifted) equations for $v(\eta_0+2)$, $v(\eta_0+3)$ and $v(\eta_0+4)$ in a Laurent series in $v(\eta_0)$. It turns out to be necessary to keep the leading three orders. Then, if $v(\eta_0-1)$, $v'(\eta_0-1)$, $v'(\eta_0)$, $v''(\eta_0)$ and $v'''(\eta_0)$ are in general position [29], we obtain

$$v(\eta_0+1)=\mathrm{O}(1/v(\eta_0)),\quad v(\eta_0+2)=\mathrm{O}(1/v(\eta_0))\,,$$
$$v(\eta_0+3)=\mathrm{O}(v(\eta_0)),\quad v(\eta_0+4)=\mathrm{O}(1)\,. \qquad (28)$$

Hence the singularity is confined, and our equation (18) or (27) satisfies the singularity confinement criterion in a remarkably nontrivial way.

[#2] For an alternative discrete version of the Painlevé property see ref. [9].

5. A more general reduction of the Kac–van Moerbeke equation is given by

$$u(x,t)=[b(x)-at]^{-1}v(\eta,N)\,, \qquad (29)$$

where η is again give by (17b) and

$$N=x\ (\mathrm{mod}\ P)$$

with period $P\in\mathbb{N}$. This leads to a set of P coupled equations

$$av(\eta,N)-c\frac{\mathrm{d}v}{\mathrm{d}\eta}(\eta,N)$$
$$=v(\eta,N)[v(\eta+1,N+1)-v(\eta-1,N-1)]\,, \qquad (30)$$

where the variable N is taken mod P.

6. In this Letter we have derived the ordinary differential–difference equation (18) as a reduction of the integrable Kac–van Moerbeke equation on the basis of its Lie symmetries. The reduced equation admits a Lax representation, and reduces to the first Painlevé equation in a continuum limit. This suggests calling the reduced equation (18) integrable. This is corroborated by the fact that eq. (18) satisfies the discrete Painlevé property of singularity confinement of ref. [10] (cf. ref. [11] where discretised versions of various Painlevé equations were derived using the singularity confinement criterion).

As we have indicated in section 2, the above method applies to other PDΔE's as well, whether integrable or not. We here briefly mention the results of applying our method to the reduced two-dimensional Toda equation

$$\frac{\mathrm{d}^2u(x,t)}{\mathrm{d}x\mathrm{d}t}=\exp[u(x,t)-u(x-1,t-1)]$$
$$-\exp[u(x+1,t+1)-u(x,t)]\,. \qquad (31)$$

The symmetries of this equation are

$$x^*=x+\epsilon\alpha(x),\quad t^*=t+\epsilon\beta(t)\,,$$
$$u^*=u+\epsilon[qx-qt-x\alpha'(x)+\gamma(x)-t\beta'(t)+\delta(t)]\,. \qquad (32)$$

In (31) α, β, γ and δ are arbitrary unit periodic functions:

$$\alpha(x)=\alpha(x+1), \quad \gamma(x)=\gamma(x+1),$$
$$\beta(t)=\beta(t+1), \quad \delta(t)=\delta(t+1), \tag{33}$$

the primes denote derivatives, and q is an arbitrary constant. More results about similarity solutions and symmetry algebra of eq. (31), and its derivation from the two-dimensional Toda equation are given in ref. [24].

H.W.C. is grateful to the Mathematics Department of La Trobe University for supporting his visit to Melbourne during which time this paper was completed. This investigation is part of the research programme of the Australian Research Council.

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