THE NONLINEAR SCHRÖDINGER EQUATION AND THE ANISOTROPIC HEISENBERG SPIN CHAIN

G.R.W. QUISPEL and H.W. CAPEL
Instituut-Lorentz voor Theoretische Natuurkunde, 2311 SB Leiden, The Netherlands

Received 7 January 1982

We derive a Miura transformation expressing solutions of the nonlinear Schrödinger equation in terms of solutions of the equation of motion of the classical *anisotropic* Heisenberg spin chain in the continuum description.

In this letter we shall derive a Miura transformation which maps the solutions of the (axially symmetric) *anisotropic* classical Heisenberg spin chain (AHSC) on the solutions of the nonlinear Schrödinger equation (NLS).

From every solution of the equation of motion

$$\dot{S} = S \times S'' + S \times (AS^z + B)e^z \quad (S \cdot S = 1, \quad A \geqslant 0)\,, \tag{1}$$

of the AHSC in the continuum description, we construct a solution of the NLS

$$\mathrm{i}\dot{\psi} + \psi'' + \alpha|\psi|^2\psi = 0 \quad (\alpha \text{ real})\,. \tag{2}$$

In eq. (1) S denotes the spin density $S(x, t)$ as a function of the time t and the position x on the chain, S^z is the z-component of S and e^z is a unit vector in the z-direction. The first term $S \times S''$ is due to an isotropic ferromagnetic exchange interaction, the term with A arises from a single-ion anisotropy of the Ising type, i.e. a term involving $-\frac{1}{2}AS^{z^2}$ with $A \geqslant 0$ in the hamiltonian, and the term with B describes the contribution of a magentic field B in the z-direction. The S and ψ in eqs. (1) and (2) are functions of x and t and the dot and the prime denote the differentiations with respect to t and x, i.e. e.g. $\dot{S} = \partial S/\partial t$ and $S' = \partial S/\partial x$.

Lakshmanan [1] has shown that to each solution of the equations of motion of the *isotropic* classical Heisenberg spin chain (IHSC), i.e. eq. (1) with $A = B = 0$, there corresponds a solution of the NLS, cf. also ref. [2]. In ref. [3] the transformation has been given in terms of the real variables $q(x, t)$ and $y(x, t)$ of the equivalent one-variable description of the IHSC [3] and the NLS [3,4]. On the other hand, in the limit of very large anisotropy ($A \to \infty$), it has been shown that the AHSC equation reduces to the NLS [5].

In order to derive the Miura transformation for the general case of Ising anisotropy ($A \geqslant 0$), we consider the auto-Bäcklund transformation of the NLS, as given in refs. [6,7], i.e.

$$\dot{\psi} - \dot{\widetilde{\psi}} = 2\xi(\psi' - \widetilde{\psi}') - \tfrac{1}{2}\mathrm{i}(\psi' + \widetilde{\psi}')(16\sigma^2 - 2\alpha|\psi - \widetilde{\psi}|^2)^{1/2} + \tfrac{1}{2}\mathrm{i}\alpha(\psi - \widetilde{\psi})(|\psi|^2 + |\widetilde{\psi}|^2)\,, \tag{3}$$

$$\psi' - \widetilde{\psi}' = -2\mathrm{i}\xi(\psi - \widetilde{\psi}) - \tfrac{1}{2}(\psi + \widetilde{\psi})(16\sigma^2 - 2\alpha|\psi - \widetilde{\psi}|^2)^{1/2}\,. \tag{4}$$

Defining

$$\psi - \widetilde{\psi} \equiv (2A/|\alpha|)^{1/2} a(x, t)\,, \tag{5}$$

we have from eqs. (3) and (4) with $\xi = 0$,

$$\dot{a} = -\tfrac{1}{2}\mathrm{i}[2|\alpha|^{1/2}\psi' - (2A)^{1/2}a'](2s^2 - 2\nu|a|^2)^{1/2} + \tfrac{1}{2}\mathrm{i}\alpha a[|\psi|^2 + |\psi - (2A/|\alpha|)^{1/2}a|^2]\,, \tag{6}$$

$$a' = -\tfrac{1}{2}[2|\alpha|^{1/2}\psi - (2A)^{1/2}a](2s^2 - 2\nu|a|^2)^{1/2} \quad (s^2 \equiv 4\sigma^2/A, \quad \nu \equiv \alpha/|\alpha|)\,. \tag{7}$$

From (7) we find

$$|\alpha|^{1/2}\psi = (A/2)^{1/2}a - a'(2s^2 - 2\nu|a|^2)^{-1/2}\,, \tag{8}$$

and substituting (8) in (6) we find that a must obey the equation

$$\mathrm{i}\dot{a} + a'' + (a'^2a^* + 2aa'a^{*\prime})\,(2s^2\nu - 2|a|^2)^{-1} + \tfrac{1}{2}\nu A|a|^2a = 0\,. \tag{9}$$

Eq. (9) can also be derived from eqs. (3)–(5) in the more general case with $\xi \neq 0$, using the change of variables $a(x,t) \to a(x + 4\xi t, t)\exp[-4\mathrm{i}\xi^2 t - 2\mathrm{i}\xi x]$ and identifying the dots and the primes with the derivatives with respect to t and $x + 4\xi t$, respectively.

By taking the difference between the two solutions ψ and $\widetilde{\psi}$ in the auto-Bäcklund transformation, we have shown that this quantity must obey another partial differential equation, i.e. eq. (9), which we shall show to be equivalent to the AHSC. Note that eqs. (6) and (7) provide a Bäcklund-transformation relating the solutions of eq. (9) with those of the NLS and eq. (8) can be regarded as a Miura transformation [8] mapping the solutions of (9) on the NLS. This procedure of taking the difference between two solutions occurring in an auto-Bäcklund transformation of a partial differential equation, may also be used to derive Bäcklund transformations and Miura transformations for other cases in a very simple way, e.g. the transformation between the modified Korteweg-de Vries (MKdV) and the Korteweg-de Vries (KdV) equation [8,9], the transformation between eq. (4.4.9) of ref. [10] and the MKdV, cf. also ref. [11], the Miura transformation of the cylindrical KdV [12] etc.

We now proceed with the derivation of the AHSC. Substituting $a = \kappa\exp(\mathrm{i}\gamma)$, ($\kappa \geqslant 0$, γ real), eq. (9) is equivalent to

$$-\frac{\partial}{\partial t}(s^2 - \nu\kappa^2)^{1/2} + \frac{\partial}{\partial x}[\nu\kappa^2\gamma'(s^2 - \nu\kappa^2)^{-1/2}] = 0\,, \tag{10}$$

$$\dot{\gamma} = \kappa''\kappa^{-1} - \gamma'^2 + (3\kappa'^2 + \kappa^2\gamma'^2)\,(2\nu s^2 - 2\kappa^2)^{-1} + \tfrac{1}{2}\nu A\kappa^2\,. \tag{11}$$

Eq. (10) can be formally solved introducing a potential function $q(x,t)$, cf. also ref. [3], such that

$$\kappa = (\nu s^2 - \nu q'^2)^{1/2}, \quad \gamma' = \dot{q}q'(s^2 - q'^2)^{-1}\,. \tag{12,13}$$

Using (12) and (13), the right-hand side of eq. (11) can be expressed in terms of q and from the compatibility relation $\partial\dot{\gamma}/\partial x = \partial\gamma'/\partial t$, we find that q must obey the equation ($q' \not\equiv 0$),

$$(\ddot{q} + q'''')(s^2 - q'^2)^{-1} + 4q'(\dot{q}\dot{q}' + q''q''')(s^2 - q'^2)^{-2} + q''(s^2 + 3q'^2)(\dot{q}^2 + q''^2)(s^2 - q'^2)^{-3} + Aq'' = 0\,. \tag{14}$$

For $s \neq 0$, we can choose $s = 1$ without loss of generality, as can be shown by a simple scaling $q \to qs$, $A \to As^{-2}$. Eq. (14) with $s = 1$ is identical to the potential anisotropic Heisenberg spin chain (potential AHSC), given in eq. (9) of ref. [3], in the special case that $F(q') = \frac{1}{2}Aq'^2 + Bq'$ and from the treatment in ref. [13] it follows that eq. (14) is completely integrable. (Physical solutions for the AHSC are obtained for $q'^2 \leqslant s^2$ and hence $\nu = 1$.)

Using polar angles for the spin components, i.e. $\boldsymbol{S} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, we have, cf. eqs. (6) and (7) of ref. [3], taking $\alpha > 0$,

$$q' = \cos\theta, \quad \dot{q} = \phi'\sin^2\theta\,, \tag{15}$$

and from eqs. (15), (11)–(13) and (8) it can be shown that

$$(2\alpha)^{1/2}\psi = (A^{1/2}\sin\theta - \theta' - \mathrm{i}\phi'\sin\theta)\mathrm{e}^{\mathrm{i}\gamma}\,, \tag{16}$$

where γ is determined by

$$\gamma' = \phi'\cos\theta\,, \quad \dot{\gamma} = \theta''\operatorname{cotan}\theta + \tfrac{1}{2}\theta'^2 + \tfrac{1}{2}\phi'^2(1 - 3\cos^2\theta) + \tfrac{1}{2}A\sin^2\theta\,, \tag{17}$$

or

$$\gamma = \int_{C} (\mathrm{d}l_x \gamma' + \mathrm{d}l_t \dot{\gamma}) + \gamma(0,0) \,, \tag{18}$$

in which C denotes an arbitrary curve in the xt-plane going from (0,0) to (x, t) and where $(\mathrm{d}l_x, \mathrm{d}l_t)$ denotes an infinitesimal two-dimensional vector tangent to C.

Eq. (16) is an explicit expression for the Miura transformation relating the AHSC, i.e. eq. (1), to the NLS, with $\alpha > 0$.

In analogy to eq. (14), there is also a potential equation with one real variable for the NLS [3,4]. In fact, the solutions of the NLS can be expressed in the form

$$\psi(x,t) = (2y')^{1/2} \exp\left[\mathrm{i} \int_{C} \{\mathrm{d}l_x (-\dot{y}/2y') + \mathrm{d}l_t [y'''/2y' - (y''^2 + \dot{y}^2)/4y'^2 + 2\alpha y']\}\right] , \tag{19}$$

in which the real variable $y(x, t)$ obeys the equation

$$\ddot{y} + y'''' + y''(\dot{y}^2 + y''^2)y'^{-2} - 2(\dot{y}\dot{y}' + y''y''')y'^{-1} + 4\alpha y''y' = 0 \,. \tag{20}$$

From (19) it follows that y is determined by

$$y' = \tfrac{1}{2}\psi^*\psi \,, \quad \dot{y} = -\tfrac{1}{2}\mathrm{i}(\psi'^*\psi - \psi'\psi^*) \,, \tag{21}$$

and from eqs. (8), (11)–(13), one obtains,

$$\begin{aligned} 2\alpha y' &= \tfrac{1}{2}(\dot{q}^2 + q''^2)(s^2 - q'^2)^{-1} + \tfrac{1}{2}A(s^2 - q'^2) + A^{1/2}q'' \,, \\ 2\alpha\dot{y} &= (\dot{q}'q'' - \dot{q}q''')(s^2 - q'^2)^{-1} - \dot{q}q'(\dot{q}^2 + q''^2)(s^2 - q'^2)^{-2} - A\dot{q}q' + A^{1/2}\dot{q}' \,, \end{aligned} \tag{22}$$

which, for $s \neq 0$, α real, is the Miura transformation between the potential AHSC in eq. (14) and the potential NLS in eq. (20). Eq. (22) in the special case of the IHSC, i.e. $A = B = 0$, was given in ref. [3] on the basis of straightforward, but rather tedious algebra.

In terms of the spin components, as given in eq. (15), and using eq. (1), eq. (22) with $s = 1$ and $\alpha > 0$ can be expressed as,

$$2\alpha y' = \tfrac{1}{2}\boldsymbol{S}' \cdot \boldsymbol{S}' - \tfrac{1}{2}A(S^{z^2} - 1) + A^{1/2}S^{z'} \,, \quad 2\alpha\dot{y} = \boldsymbol{S}' \cdot \dot{\boldsymbol{S}} + B(\boldsymbol{S} \times \boldsymbol{S}') \cdot \boldsymbol{e}^z + A^{1/2}\dot{S}^z \,, \tag{23}$$

which in view of the considerations given above is equivalent to eqs. (16) and (17). In the special case of the IHSC with $A = B = 0$, the expressions for $2\alpha y'$ and $-2\alpha\dot{y}$ can be interpreted as the energy density and the current density and for $A = B = 0$ eq. (23) is equivalent to the result derived in ref. [1].

For $s = 0$, eq. (22) provides a Miura transformation between eq. (14) for $s = 0$ and the NLS. Although in the case $s = 0$ eq. (14) is *not* related to the AHSC (or the IHSC for $A = 0$), this equation can be of interest for studying e.g. similarity solutions. In this respect it may be noted that eq. (14) for $s = 0$ has similarity solutions $q(x,t) = \delta \ln t + \hat{q}(\eta)$, $\eta = xt^{-1/2}$, where, for $\delta \neq 0$, $(\partial\hat{q}/\partial\eta)^{-1}$ satisfies the fourth Painlevé equation [14], cf. refs. [7, 15]. The similarity solutions $y(x,t) = t^{-1/2}\hat{y}(\eta)$, $\eta = xt^{-1/2}$, of eq. (18) obey the equation

$$(\mathrm{d}^2\hat{y}/\mathrm{d}\eta^2)^2 + \tfrac{1}{4}(\hat{y} - \eta\,\mathrm{d}\hat{y}/\mathrm{d}\eta)^2 + 4\alpha(\mathrm{d}\hat{y}/\mathrm{d}\eta)^3 - \beta(\mathrm{d}\hat{y}/\mathrm{d}\eta)^2 + \epsilon\,\mathrm{d}\hat{y}/\mathrm{d}\eta = 0 \,, \tag{24}$$

in which β and ϵ are integration constants. Eq. (24) is related to the equation for the similarity solutions $q(x,t) = t^{1/2}\hat{q}(\eta)$, $\eta = xt^{-1/2}$ of the IHSC with $A = 0$, given as eq. (18) in ref. [3]. Chazy proved that eq. (24) has no movable critical points [16], which confirms the conjecture by Ablowitz et al. in ref. [17]. Applying eq. (22) in the special case of similarity solutions one can derive various transformations [15] between the "Chazy" equation (24) and the fourth Painlevé equation.

This investigation is part of the research programme of the Stichting voor Fundamenteel Onderzoek der Materie (FOM) which is financially supported by the Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek (ZWO).

References

[1] M. Lakshmanan, Phys. Lett. 61A (1977) 53.
[2] V.E. Zakharov and L.A. Takhtadzhyan, Theor. Math. Phys. 38 (1979) 17.
[3] G.R.W. Quispel and H.W. Capel, Phys. Lett. 85A (1981) 248; Physica A, to be published.
[4] M. Boiti, C. Laddomada and F. Pempinelli, Nuovo Cimento 62B (1981) 315.
[5] E.K. Sklyanin, LOMI preprint E-3-1979.
[6] H.H. Chen, Phys. Rev. Lett. 33 (1974) 925.
[7] M. Boiti and F. Pempinelli, Nuovo Cimento 59 (1980) 40.
[8] R.M. Miura, J. Math. Phys. 9 (1968) 1202.
[9] H.D. Wahlquist and F.B. Estabrook, J. Math. Phys. 16 (1975) 1.
[10] F. Calogero and A. Degasperis, J. Math. Phys. 22 (1981) 23.
[11] A. Degasperis, preprint INFN, Rome, no. 175 (1979).
[12] A. Nakamura, Phys. Lett. 82A (1981) 111.
[13] A.E. Borovik, Sov. Phys. JETP Lett. 28 (1978) 581.
[14] H.T. Davis, Introduction to nonlinear differential and integral equations (Dover Publ., New York, 1962).
[15] G.R.W. Quispel and H.W. Capel, to be published.
[16] J. Chazy, Acta Math. 34 (1911) 317.
[17] M.J. Ablowitz, A. Ramani and H. Segur, Lett. Nuovo Cimento 23 (1978) 333.