Integrable Mappings and Soliton Lattices

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1 Introduction

Solitons are found as solutions of integrable partial differential equations (PDE's) which have been extensively studied during the last three decades. More recently there has been increasing interest in partial difference equations with fields defined at the sites of 2- and 3-dimensional (2D and 3D) lattices. They are the discrete-time analogues of PDE's as well as of partial difference equations depending on time as well as on 1 or 2 discrete variables. During the last years a variety of integrable 2D and 3D lattices has been found, by the direct linearization method (DLM) which is based on a linear integral equation with arbitrary measure and contour [1]-[4], as well as by the bilinearization method [5] and the τ-function approach [6]. By taking continuum limits the lattice equations yield hierarchies of integrable partial difference equations (or PDE's) with one or more continuous variables together with a Poisson structure and an infinite number of conserved quantities in involution [7,8], see also ref. [9].

On the other hand there has been a widespread interest in dynamical mappings with $1, 2, 3, \cdots$ degrees of freedom (dimensions) as time-discrete analogues of sets of coupled ordinary differential equations, mainly in connection with chaotic phenomena. A very special type of mappings are the *integrable* mappings [10]-[15]. These mappings are *symplectic*, i.e. they have a Poisson-bracket structure that is invariant under the mapping, and $2N$ dimensional mappings have N integrals that are in involution with respect to the Poisson bracket. In spite of the progress on the theory of integrable mappings there are still many open problems. An 18-parameter family of 2 dimensional mappings with 1 integral generalizing the McMillan mapping [10] has been given in ref. [16], but the symplectic structure of this family has not been investigated so far. In this report we show how integrable mappings with $2P$ degrees of freedom, $P = 1, 2, \cdots$, are obtained by systematic reduction of 2D lattices [17]-[19] taking the lattice version of the Korteweg-de Vries (KdV) equation as an example.

In section 2 we consider a rather general class of difference equations with solutions that are found from periodic initial conditions on a staircase consisting of subsequent horizontal and vertical steps and which can be described in terms of $2P$ dimensional mappings. In section 3 we introduce the $2P$ dimensional mappings associated with the lattice KdV and the integrals of the mapping are evaluated in section 4 from the Lax representation of the lattice KdV. In section 5 we give an action principle for the lattice KdV which leads in a natural way to the Poisson-bracket structure for the KdV-type of mappings. The involution property of the integrals is treated in section 6 and some concluding remarks are given in section 7.

2 Periodic solutions of difference equations

We consider a difference equation on the 2D lattice of the type

$$\Phi\left(u, Hu, Vu, HVu\right) = 0 \tag{2.1}$$

where Φ is a function of the variable u at the 4 sites of an elementary square on the 2D (square) lattice. At some site of the lattice we have the field u, the field at the site that is obtained by a horizontal (vertical) shift is denoted as Hu (Vu). We assume that from the function Φ in (2.1) any of the 4 fields in the argument can be solved as function of the other 3 ones. Then a complete initial condition for eq. (2.1) is provided by specifying the fields $\cdots, a_0, a_1, a_2, \cdots$ at the sites of a staircase consisting of successive horizontal and vertical steps as indicated in Figure 1.

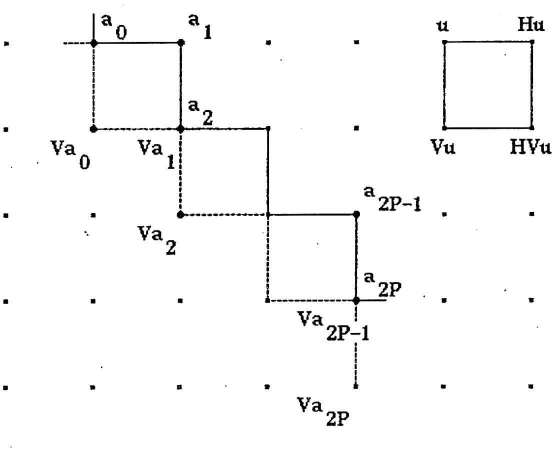

Figure 1. Difference equation, staircase (—–) and shifted staircase (- - -) on the 2D lattice

By completing squares and using (2.1) one can solve the difference equation iteratively at all sites at the left and below the staircase.

We now consider the case that the initial data on the staircase satisfy the periodicity property $(HV)^P u = u$, $P = 1, 2, \cdots$. Then by completing squares it is clear that the same property holds for the complete solution of (2.1). In the periodic case it is sufficient to specify the fields $a_0, a_1, \cdots, a_{2P-1}$ on a finite part of the staircase consisting of P horizontal and P vertical steps $(a_j = a_{j+2P})$. To solve the difference equation we can consider the $2P$ dimensional mapping corresponding to the vertical shift V. The mapping is given by

$$Va_{2j+1} = a_{2j+2} \quad , \quad \Phi\left(a_{2j}, a_{2j+1}, Va_{2j}, a_{2j+2}\right) = 0 \quad , \tag{2.2}$$

$j = 0, 1, \cdots, P-1$, as is clear from (2.1) and Fig.1 considering the elementary square with $u = a_{2j}$, $Hu = a_{2j+1}$, $HVu = a_{2j+2}$.

3 Lattice KdV

The solution of the lattice KdV can be found from the linear integral equation

$$u_k + \rho_k \int_C d\lambda(l) \frac{u_l}{k+l} = \rho_k \tag{3.1}$$

in which ρ_k is a free-wave basis function satisfying [1]

$$H\rho_k = \frac{p+k}{p-k}\rho_k \quad , \quad V\rho_k = \frac{q+k}{q-k}\rho_k \tag{3.2}$$

(The integral equation with $\rho_k = \exp(kx + k^3 t)$ was introduced in ref. [20] to study the solutions of the KdV.) C is an arbitrary contour in the space of the complex spectral parameter k and $d\lambda(k)$ an arbitrary measure satisfying the *uniqueness condition*, i.e. for the given ρ_k C and $d\lambda(k)$ are such that the solution u_k as function of k on the 2D lattice is unique. Starting from (3.1) it can be shown that the potential $u = \int_C d\lambda(k) u_k$ obtained by an integration of u_k over the same contour C with the same measure satisfies

$$(p + q + u - HVu)(p - q + Vu - Hu) = p^2 - q^2 \quad , \tag{3.3}$$

which is the 2D lattice version of the KdV. In the DLM the solutions of (3.3) are found solving the linear integral equation (3.1). From eq. (3.1) with (3.2) and the uniqueness condition one finds the linear relation

$$(p-k)H \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} p - Hu & 1 \\ k^2 - p^2 + * & p + u \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} \tag{3.4}$$

corresponding to the horizontal shift, in which v_k is the solution of an integral equation similar to (3.1) but with the source term ρ_k replaced by $k\rho_k$ and $*$ is a short-hand notation for the product of the diagonal elements $(p - Hu)(p + u)$. For the vertical shift we have a similar relation that can be found from (3.4) by the replacements $H \to V$, $p \to q$. The compatibility of both relations (3.4) yields the lattice KdV (3.3).

Assuming the periodicity property $(HV)^P u = u$ one obtains from (3.3) and (2.2) an explicit expression for the $2P$ dimensional KdV mapping in terms of the fields a_j with $j = 0, 1, \cdots, 2P-1$ on the staircase. This mapping can be reduced to a $2P-2$ dimensional mapping in terms of the fields $v_j = \varepsilon + a_{j-1} - a_{j+1}$ with $\varepsilon = p + q$, $\delta = p - q$. We have [17]

$$\begin{aligned} Vv_{2j} &= v_{2j+1} \\ Vv_{2j+1} &= v_{2j+2} - \frac{\varepsilon\delta}{v_{2j+3}} + \frac{\varepsilon\delta}{v_{2j+1}} \quad , \quad j = 0, 1, \cdots, P-2 \end{aligned} \tag{3.5}$$

with $v_{2P-1} = P\varepsilon-(v_1+v_3+\cdots+v_{2P-3})$, $v_{2P-2} = P\varepsilon-(v_0+v_2+\cdots+v_{2P-4})$. For simplicity we consider here only periodic conditions ($a_j = a_{j+2P}$). More general situations with $a_{2j+2P} = a_{2j} + C_0$, $a_{2j+1+2P} = a_{2j+1} + C_1$ can be treated as well using a slightly different Lax representation [18].

4 Integrals

To evaluate the integrals of (3.5) we compare the basis functions u_k, v_k of the *Lax representation* (3.4) at the points $(0,0)$ and (P,P) at the beginning and at the end of the staircase. We have

$$(p-k)^P(q-k)^P \begin{pmatrix} u_k(P,P) \\ v_k(P,P) \end{pmatrix} = \mathcal{T}_k \begin{pmatrix} u_k(0,0) \\ v_k(0,0) \end{pmatrix} \tag{4.1}$$

in which the *monodromy matrix* is the product of all Lax matrices along the staircase, i.e.

$$\mathcal{T}_k\left(a_0, a_1, \cdots, a_{2P-1}\right) = \overset{\longleftarrow}{\prod_{j=0}^{2P-1}} L_k\left(a_j, a_{j+1}, p_j\right) \tag{4.2}$$

$$L_k\left(a_j, a_{j+1}, p_j\right) = \begin{pmatrix} p_j - a_{j+1} & 1 \\ k^2 - p_j^2 + * & p_j + a_j \end{pmatrix} \tag{4.3}$$

In eq. (4.3) $p_j = p$ for $j =$ even, $p_j = q$ for $j =$ odd and $*$ denotes the product of the diagonal elements, $\longleftarrow$ in eq. (4.2) means that the matrices in the product are ordered from the right to the left.

But there is another way to connect the basis functions at $(0,0)$ and (P,P) as shown in Fig.1. From a_0 we go down to Va_0, next we go through the shifted staircase until we reach Va_{2P} and finally we go to a_{2P}. Thus $\mathcal{T}_k$ is also given by

$$\mathcal{T}_k = L_k^{-1}\left(a_{2P}, Va_{2P}, q\right) \left(\overset{\longleftarrow}{\prod_{j=0}^{2P-1}} L_k\left(Va_j, Va_{j+1}, p_j\right) \right) L_k\left(a_0, Va_0, q\right) \tag{4.4}$$

Because of the periodicity the Lax matrices involving a_0, Va_0 and a_{2P}, Va_{2P} are the same. Hence, the trace of the product of Lax matrices along the staircase is invariant under the mapping [17]

$$\operatorname{tr} \mathcal{T}_k = \operatorname{tr} \overset{\longleftarrow}{\prod_{j=0}^{2P-1}} L_k\left(a_j, a_{j+1}, p_j\right) = \sum_{j=0}^{P} k^{2j} I_j = \text{invariant} \tag{4.5}$$

Eq. (4.5) holds for arbitrary values of the spectral parameter k and all coefficients $I_j, j = 0, 1, \cdots, P$, of the expansion in powers of k^2 must be invariant. The coefficients I_P, I_{P-1} associated with k^{2P}, k^{2P-2} turn out to be trivial, but the coefficients $I_0, I_1, \cdots, I_{P-2}$ give $P-1$ nontrivial integrals for the $2P-2$ dimensional mapping in terms of the v_j (3.5).

5 Poisson bracket structure

For the lattice KdV we have the action

$$S = \sum_{n,m \in \mathbf{Z}} V^n H^m \mathcal{L} \tag{5.1}$$

with the Lagrangian $\mathcal{L}$ given by

$$\mathcal{L} = (Vu)(HVu - u) + \varepsilon\delta \log(\varepsilon + u - HVu) \tag{5.2}$$

for fields u at the sites of the $2D$ lattice. Assuming S to be invariant under infinitesimal variations of the fields u at the different sites of the $2D$ lattice, we have the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial u} + V^{-1}\frac{\partial \mathcal{L}}{\partial Vu} + (HV)^{-1}\frac{\partial \mathcal{L}}{\partial HVu} = 0 \tag{5.3}$$

Eq. (5.3) is automatically satisfied for any solution u of the lattice KdV (3.3).

For the $2P$ dimensional KdV-type of mapping we introduce the action [18]

$$S = \sum_{n \in \mathbf{Z}} V^n L(\{a_{2j}\}, \{Va_{2j}\}) \tag{5.4}$$

in which the Lagrangian L is a function of the even a's $a_0, a_2, \cdots, a_{2P-2}$ on the staircase and the shifted values $Va_0, Va_2, \cdots, Va_{2P-2}$ given by

$$L(\{a_{2j}\}, \{Va_{2j}\}) = \sum_{j=0}^{P-1} \left[(Va_{2j})(a_{2j+2} - a_{2j}) + \varepsilon\delta \log(\varepsilon + a_{2j} - a_{2j+2}) + \frac{1}{2}(Va_{2j})^2 - \frac{1}{2}a_{2j}^2 \right] , \tag{5.5}$$

where we have added 2 terms which do not affect the action but which are convenient for the introduction of canonical momenta. The Euler-Lagrange equations

$$\frac{\partial L}{\partial a_{2j}} + V^{-1}\frac{\partial L}{\partial Va_{2j}} = 0 \quad , \quad j = 0, \cdots, P-1 \tag{5.6}$$

with (5.5) yield the mapping (3.5).

Having established the Lagrangian property of the mapping we can follow refs. [14,15] to introduce canonical momenta and a discrete-time hamiltonian. We have the relations

$$Vp_{2j} \equiv \frac{\partial L}{\partial Va_{2j}} \qquad , \qquad p_{2j} = a_{2j} + a_{2j+1} - a_{2j-1} \tag{5.7}$$

$$\begin{aligned} \mathcal{H}(\{Vp_{2j}\}, a_{2j}) &= \sum_{j=0}^{P-1} (Vp_{2j})(Va_{2j} - a_{2j}) - L \\ &= \sum_{j=0}^{P-1} \left[\frac{1}{2}(Vp_{2j} - a_{2j+2})^2 + \frac{1}{2}(a_{2j+2} - a_{2j})^2 - \varepsilon\delta \log(\varepsilon + a_{2j} - a_{2j+2}) \right] \end{aligned} \tag{5.8}$$

The Hamiltonian acts as the generating functional of the mapping, i.e. one has the discrete hamiltonian equations

$$Va_{2j} - a_{2j} = \frac{\partial \mathcal{H}}{\partial Vp_{2j}} \quad , \quad Vp_{2j} - p_{2j} = -\frac{\partial \mathcal{H}}{\partial a_j} \tag{5.9}$$

The hamiltonian is not invariant under the mapping, but is the generating function of the canonical transformation associated with the mapping. In fact, the standard Poisson brackets

$$\{p_{2j}, a_{2j'}\} = \delta_{jj'} \quad , \quad \{a_{2j}, a_{2j'}\} = \{p_{2j}, p_{2j'}\} = 0 \tag{5.10}$$

are invariant under the mapping.

6 Involution

For the trace of the monodromy matrix given by (4.3) and (4.5) one can derive the explicit expression

$$\operatorname{tr} \mathcal{T}_k = \left(\prod_{j=0}^{2P-1} v_j \right) \left\{ 1 + \sum_{\substack{J_{\nu+1} - J_\nu \geq 2 \\ J_f \leq 2P-1 \\ J_1 - J_f + 2P \geq 2}} \prod_{\nu=1}^{f} \frac{k^2 - p_{J_\nu}^2}{v_{J_\nu} v_{J_\nu + 1}} \right\} \tag{6.1}$$

with v_j and p_j as in (3.5) and (5.7). From (5.10) we have the Poisson brackets

$$\{v_j, v_{j'}\} = \delta_{j',j-1} - \delta_{j',j+1} \tag{6.2}$$

To prove the involution property use can be made of the determinant formula

$$\operatorname{tr} \mathcal{T}_k = \det Y_k + 1 + \left(p^2 - k^2\right)^P \left(q^2 - k^2\right)^P \tag{6.3}$$

with the $2P \times 2P$ matrix Y_k given by

$$(Y_k)_{jj'} = \left(p_j^2 - k^2\right) \delta_{j',j+1}(\mathrm{mod} 2P) + \delta_{j,j-1}(\mathrm{mod} 2P) + v_j \delta_{j'j} \tag{6.4}$$

$j, j' = 0, 1, \cdots, 2P-1$, where $\delta_{j,k}(\mathrm{mod} 2P) = 1$, if $j - k$ is a multiple of $2P$, and 0 otherwise. From (6.2) and (6.4) we find

$$\{\operatorname{tr} \mathcal{T}_k, \operatorname{tr} \mathcal{T}_{k'}\} = \sum_{J=0}^{2P-1} \det \left(Y_k^{(J)} \cdot Y_{k'}^{(J-1)} \right) - (k \leftrightarrow k') \tag{6.5}$$

with

$$\left(Y_k^{(J)}\right)_{jj'} = \left(p_{j+J}^2 - k^2\right) \delta_{j',j+1} + \delta_{j',j-1} + v_{J+j} \delta_{j'j} \tag{6.6}$$

$j, j' = 1, 2, \cdots, 2P-1, p_{j+2P} = p_j \cdot (k \leftrightarrow k')$ in (6.5) denotes the previous term with k and k' interchanged. The matrix product $Y_k^{(J)} \cdot Y_{k'}^{J-1}$ is symmetric in k and k', apart from the $(1,1)$ and $(2P-1, 2P-1)$ elements, but it can be shown that the sum of the contributions from these elements to the first term in (6.5) is symmetric in k and k' as well. Therefore,

$$\{\operatorname{tr} \mathcal{T}_k, \operatorname{tr} \mathcal{T}_{k'}\} = 0 \tag{6.7}$$

implying that the integrals of the mapping are in involution.

7 Concluding remarks

i) We have established complete integrability in the Liouville-Arnol'd sense for a family of $2P$ dimensional mappings associated with a vertical shift V for periodic solutions of the lattice KdV (3.3). One can also derive the complete integrability for the mappings associated with a diagonal shift $D = H^{-1}V$.

ii) The involution property has been proved directly on the basis of a determinant formula [18], but a more fundamental justification is obtained via an r-matrix structure of a rather unusual non-ultralocal structure [19]. In ref. [21] the complete integrability for a class of discrete-time Toda lattices has been obtained using the usual r-matrix formalism.

iii) The lattice KdV has been treated as an example, but completely integrable mappings can also be found starting from other integrable 2D lattice equations, cf. [17,18] for some results concerning a (mixed) lattice version of the modified Korteweg-de Vries (MKdV) and Toda equation. More complicated mappings not included in the considerations of section 2 arise from lattice versions of the Gel'fand-Dikii hierarchy [19].

iv) On the 2D lattice with sites (l,m) one can investigate similarity solutions satisfying $u_{l,m} = u_n$ with $n = z_1 l - z_2 m$, z_1 and z_2 being relatively prime. The initial data can be chosen on a so called standard staircase [22] with z_2 horizontal and z_1 vertical steps. The similarity reduction amounts to a $z_1 + z_2$ dimensional mapping. A sufficient number of integrals has been found for the mappings associated with the lattice versions of the KdV, the MKdV and the sine-Gordon (SG) equations [22]. We expect that an invariant Poisson structure for these mappings can be found.

v) After a continuum limit the 2D lattice equations yield hierarchies of partial difference equations with time-dependent fields at the sites of a 1D chain, together with an infinite number of conserved quantities in involution [7]. Taking stationary solutions (or a slightly different simple time-dependence) one can obtain a variety of mappings. The simple two-dimensional examples belong to the 18-parameter family of ref. [16]. For some of these mappings complete integrability has been established on the basis of a Poisson structure with a sufficient number of integrals in involution [14].

vi) Starting from a 3D lattice version of the Kadomtsev-Petviashvili (KP) equation one can derive a variety of 2D difference equations as well as mappings more general than the ones arising from the 2D lattice equations. In simple cases the mappings have been identified with known integrable cases. Although one may anticipate to obtain a larger class of integrable mappings, none of the underlying ideas (like the staircase of initial data, the evaluation of integrals and the Poisson structure) has been established with a satisfactory amount of generality.

References

[1] F.W. Nijhoff, G.R.W. Quispel and H.W. Capel, Phys.Lett. **97A**(1983)125.

[2] G.R.W. Quispel, F.W. Nijhoff, H.W. Capel and J. van der Linden, Physica **125A**(1984)344.

[3] F.W. Nijhoff, H.W. Capel, G.L. Wiersma and G.R.W. Quispel, Phys.Lett. **103A**(1984)293; **105A**(1984)267.

[4] H.W. Capel, G.L. Wiersma and F.W. Nijhoff, Physica **138A**(1986)76.

[5] R.Hirota, J.Phys.Soc. Japan **43**(1977)1424,2074,2079; **50**(1981)3785.

[6] E. Date, M. Jimbo and T. Miwa, J.Phys.Soc. Japan **51**(1982)4125; **52**(1983)388,766.

[7] G.L. Wiersma and H.W. Capel, Physica **142A**(1987)199.

[8] G.L. Wiersma and H.W. Capel, Physica **147A**(1988)49,75.

[9] F. Kako and M. Mugibayashi, Progr.Theor.Phys. **60**(1978)975; **61**(1979)778.

[10] E.M. McMillan, in: *Topics in Physics*, eds. W.E. Britten and H. Odabasi, (Colorado Associated Univ. Press, Boulder, 1971) p.219.

[11] A.P. Veselov, Funct.Anal.Appl. **22**(1988)83; Theor.Math.Phys. **71**(1987)446.

[12] J. Moser and A.P. Veselov, Preprint ETH (Zürich), 1989.

[13] P.A. Deift and L.C. Li, Commun.Pure Appl.Math. **42**(1989)963.

[14] S. Maeda, Proc.Japan Acad. **63A**(1987)198; Math. Japonica **25**(1988)405.

[15] M. Bruschi, O. Ragnisco, P.M. Santini and G.-Z. Tu, *Integrable Symplectic Maps*, Preprint Università di Roma I, June 1990.

[16] G.R.W. Quispel, J.A.G. Roberts, and C.J. Thompson, Phys.Lett. **A128** (1988)419; Physica **D34**(1989)183.

[17] V.G. Papageorgiou, F.W. Nijhoff and H.W. Capel, Phys.Lett. **A147** (1990)106.

[18] H.W. Capel, F.W. Nijhoff and V.G. Papageorgiou, *Complete Integrability and Lattices of the KdV type*, Preprint INS # 165/90.

[19] F.W. Nijhoff, V.G. Papageorgiou and H.W. Capel, *Integrable time-discrete systems: Lattices and Mappings*, Preprint INS # 166/90. Proceedings Second International Workshop on Quantum Groups, Leningrad, November 1990.

[20] A.S. Fokas and M.J. Ablowitz, Phys.Rev.Lett. **47**(1981)1096.

[21] Yu.B. Suris, Phys.Lett. **A145**(1990)113.

[22] G.R.W. Quispel, H.W. Capel, V.G. Papageorgiou and F.W. Nijhoff, *Integrable Mappings derived from Soliton Equations*, Preprint La Trobe University, October 1990, to be published in Physica A.