# Conservative Versus Reversible Dynamical Systems

J.A.G. Roberts[1], T. Post[1], H.W. Capel[1] and G.R.W. Quispel[2]

[1]Institute for Theoretical Physics, University of Amsterdam,
Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands.
[2]Department of Mathematics, La Trobe University,
Bundoora, Victoria 3083, Australia.

Abstract. The study of mappings of the plane is one of the simplest, yet most usefull, ways of becoming acquainted with the behaviour of large classes of dynamical systems. The purpose of this paper is to study the relation between the two classes of planar mappings: *measure preserving* mappings and *reversible* mappings.

## 1. Introduction

Time-reversal symmetry has played, and still plays, an important role in physics and many of the differential equations of physics are time-reversible (for a readable account, see [1]). The classical concept of time-reversal symmetry refers to the invariance of equations under the transformation $t \to -t$. A dynamical system that is invariant under time-reversal will be called *reversible*.

A well-known example of a reversible dynamical system is a Hamiltonian system with Hamiltonian $H(\underset{\sim}{x}, \underset{\sim}{p})$, $\underset{\sim}{x}, \underset{\sim}{p} \in \mathbf{R}^n$, which is even in the momenta. In such a system, the reversibility manifests itself as invariance of the 2n equations of motion under $t \to -t$ together with a change of sign in the momenta i.e. $(\underset{\sim}{x}, \underset{\sim}{p}) \to (\underset{\sim}{x}, -\underset{\sim}{p})$. Devaney [2] generalised this invariance by defining a dynamical system as reversible if there is an involution G in phase space which reverses the direction of time. Recall that an involution is its own inverse i.e. $G \circ G = Id$, the identity mapping. In the Hamiltonian example, we have $G : (\underset{\sim}{x}, \underset{\sim}{p}) \to (\underset{\sim}{x}, -\underset{\sim}{p})$.

Significantly, the generalised time-reversal invariance is not restricted to Hamiltonian, or *conservative*, dynamical systems. Therefore there can exist reversible conservative systems, reversible non conservative systems as well as conservative non reversible systems [2-6]. In the present contribution, we illustrate this at the level of mappings of the plane, which are dynamical systems with a 2-dimensional (2D) phase space and with discrete time. The study of mappings of the plane has numerical and conceptual advantages over the study of differential equations with many of the dynamical features of the mappings being analogues of features of the continuous time flows [7-9].

The 2D mapping analogue of a Hamiltonian flow is a *measure preserving mapping*; the analogue of a reversible flow is a *reversible mapping*.

A mapping of the plane $L : \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$L: \quad x' = f(x, y) \qquad (1.1a)$$

$$\quad y' = g(x, y) \qquad (1.1b)$$

is called measure preserving if there is a function m(x, y) such that

$$\int_D m(x, y)\, dx\, dy = \int_{L(D)} m(x', y')\, dx'\, dy' \qquad (1.2)$$

for any region of the plane D. That is, the double integral of m(x, y) has the same value when extended over any region as it has when extended over the image of the region. The function m(x, y) is called a measure for the mapping L. The Jacobian determinant J = Det dL(x,y) of a measure preserving mapping can be written J = m(x,y) / m(x', y'), so that at a fixed point of the mapping J = 1. The most frequently-studied measure preserving mappings are *area preserving mappings* for which $m(x, y) \equiv 1$ so that J = 1 everywhere [7]. Measure preserving, or conservative, mappings cannot possess attractors.

A mapping of the plane (1.1) is called reversible [5] if there exists an involution $G : \mathbf{R}^2 \to \mathbf{R}^2$

$$G: \quad x' = u(x, y), \quad y' = v(x, y) \qquad (1.3)$$

satisfying

$$L \circ G \circ L = G. \qquad (1.4)$$

G is called a (reversing) *symmetry* of L. Usually G is also assumed to be orientation reversing i.e. Det dG < 0.

This definition implies that a reversible mapping can be written as the product (composition) of two involutions $H := L \circ G$ and G. Conversely, any mapping that decomposes as a product of involutions U and R i.e. $L = U \circ R$, is reversible with e.g. G = U or G = R, so that this is an equivalent way of defining a reversible mapping. The reversibility property ensures that a mapping is invertible. In fact, from (1.4) with G an involution, the inverse mapping $L^{-1}$ can be expressed as

$$L^{-1} = G \circ L \circ G, \qquad (1.5)$$

that is to say, the mapping and its inverse are conjugate. This conjugacy has the important consequence that the reflection by G of the forward (backward) orbit of a point $\underset{\sim}{x}_0$ gives the backward (forward) orbit of the point $G\underset{\sim}{x}_0$. This (discrete) time-reversal [1, 6] is illustrated in

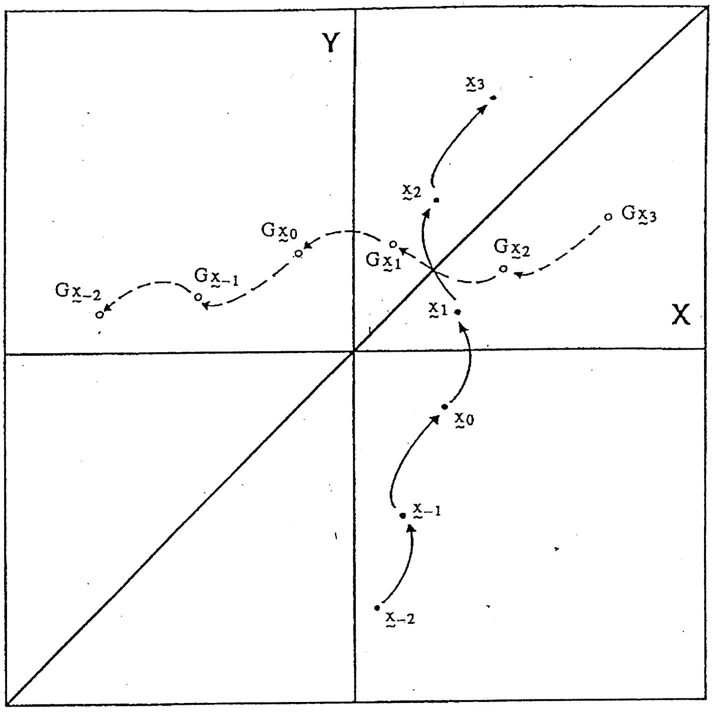


Figure 1. Illustration of the nature of the motion in a reversible mapping L of the plane with symmetry G: x' = y, y' = x. Shown is part of the trajectory of a point $x_0$ with unbroken arrows indicating the action of L on each point. Reversibility implies that the trajectory of $Gx_0$ is found by reflecting the trajectory of $x_0$ by G. The forward and backward trajectories are interchanged on reflection as the broken arrows indicate

Figure 1. Thus if the motion in one part of the plane is known, the motion in another part of the plane (i.e. in the region reflected by G) can be deduced.

The vast majority of reversible mappings studied to date have been conservative, and in particular area preserving. Conversely, almost without exception, all explicit area preserving mappings studied numerically in the literature have been reversible. For example, the mapping

$$M: \quad x' = x + f(y) \qquad (1.6a)$$

$$y' = y + g(x') \qquad (1.6b)$$

is readily verified to be area preserving. If f(y) or g(x') is an odd function, the mapping is also reversible. In the former case an involution is

$$G: \quad \begin{aligned} x' &= x + f(y), \quad f \text{ odd} \\ y' &= -y. \end{aligned} \qquad (1.7)$$

The choice $f(y) = K \sin y$, $g(x') = x'$ in (1.6) gives the well known Chirikov-Taylor or standard mapping. As far as we know, up to now no-one has been able to analytically show that an explicitly given conservative mapping of the plane is not reversible.

A possible relation between the two sets of conservative and reversible mappings is depicted in the Venn diagram in Figure 2. Region II comprising measure preserving reversible mappings has been thoroughly investigated. Here we show that such a Venn diagram is valid since there exist reversible non measure preserving mappings (Region I) and measure preserving non reversible mappings (Region III) cf. also [6].

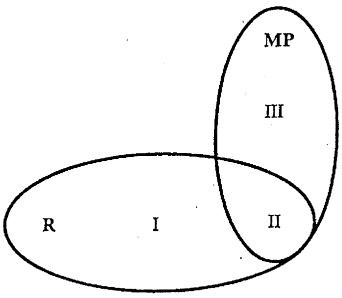


Figure 2. Venn diagram illustrating the relation between measure preserving mappings (denoted by MP) and reversible mappings (denoted by R)

## 2. Reversible Mappings need not be Measure Preserving

We have been able to construct reversible non conservative mappings and study their properties. We do this by creating a non area preserving involution and composing it with an area preserving involution. We illustrate the method with a simple example. We create the non area preserving involution $H_1$

$$H_1 : \begin{aligned} x' &= y\,(1+(y'-1)^2) \\ y' &= \frac{x}{1+(y-1)^2}, \end{aligned} \tag{2.1}$$

which we compose with the simple one parameter area preserving involution

$$G_1 : \begin{aligned} x' &= x \\ y' &= C - y. \end{aligned} \tag{2.2}$$

The result $H_1 \circ G_1$ is the reversible mapping :

$$L_1 : \begin{aligned} x' &= (C-y)(1+(y'-1)^2) \\ y' &= \frac{x}{1+(C-y-1)^2}. \end{aligned} \tag{2.3}$$

Note that $H_1$ and $G_1$ are orientation reversing and that their fixed points form lines in the plane e.g. $y = C/2$ for $G_1$. These lines are called symmetry lines (actually, once given one decomposition of L into symmetries H and G, one can find an infinite family of other symmetries with their own symmetry lines). A fixed point (periodic orbit) of a reversible mapping is called symmetric if it is also invariant under the symmetries. Otherwise, it is asymmetric and its reflection under the symmetries is another fixed point (periodic orbit) of the mapping.

One finds that for a substantial range of the parameter C the mapping $L_1$ above has a symmetric fixed point and a pair of asymmetric fixed points, e.g., at $C = 3$ these are readily calculated: the symmetric fixed point is at (15/8, 3/2) and the asymmetric pair at (2, 1) and (2, 2). The Jacobian determinant of (2.3) is:

$$J_1(x, y) = \frac{1 + (y'-1)^2}{1 + (C-y-1)^2}. \tag{2.4}$$

So at the asymmetric fixed points when $C = 3$, we find $J_1 = 0.5$ and $J_1 = 2$ respectively. Further calculation of the linearisation at these points reveals that they are in fact attracting, respectively repelling, fixed points. Therefore $L_1$ is not measure preserving. In fact, the time-reversal property of a reversible mapping implies that the presence of an attractor implies the presence of a repellor. The symmetric fixed point of $L_1$ has $J = 1$ and this approximate area preservation associated with symmetric periodic orbits is also a general property of reversible mappings.

The fact that reversible mappings of the plane can have both conservative and dissipative behaviour is illustrated pictorially in the phase portrait of another of our mappings in Figure 3. The origin in the picture is an elliptic symmetric fixed point and the curves and island chains around it are reminiscent of an area preserving mapping (in fact these are KAM curves because a KAM theorem exists for reversible mappings cf. refs. [3-5]). In the bottom right hand corner, there is a spirally attracting fixed point at A surrounded by an attracting 5 cycle. The reflection of these features in the symmetry line $y = x$ gives a repelling fixed point R and a repelling 5 cycle.

Numerical investigations of our mappings (cf. [6], [10-11]) have revealed that :

* Symmetric fixed points period double symmetrically with the scalings previously found in area preserving mappings [7] e.g. $\delta_{PD,SYM} = 8.721\ldots$.
* Asymmetric fixed points period double with the scalings previously found in dissipative ($|J| < 1$) mappings [8] e.g. $\delta_{PD,ASYM} = 4.669\ldots$.
* As a parameter is varied, the KAM curves with 'noble' winding number surrounding symmetric fixed points break up with the scalings previously found in area preserving mappings [7].

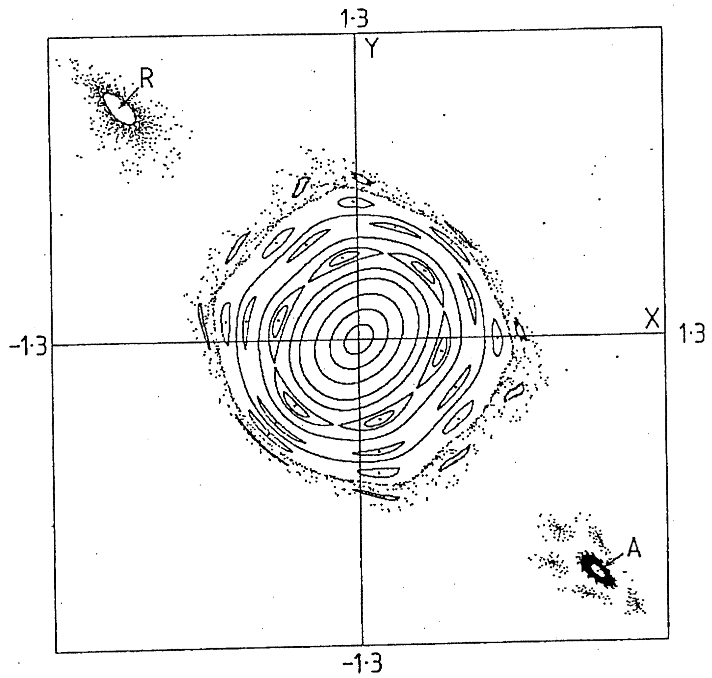


Figure 3 : This picture illustrates some of the important properties of a non measure preserving reversible mapping L. Here the mapping has a symmetry G : x' = y, y' = x. The origin is an elliptic symmetric fixed point and is surrounded by invariant curves. These curves intersect the symmetry line y = x and their symmetry with respect to reflection about this line is obvious. Also shown are elliptic symmetric 6, 7 and 8-cycles at the centres of the concentric 'rings' of 6, 7 and 8 islands. External to these islands is a hyperbolic 10-cycle which nearby points move around before escaping to infinity (creating the 10 fuzzy islands). Away from the origin, the fixed point A at (1, –1) is attracting and its reflection GA = R at (–1, 1) is repelling. A trajectory that spirals in towards A is shown; its relection by G spirals outwards from R. An attracting 5-cycle is found near A and a trajectory that spirals towards it is also shown. The trajectories of many of the points not enclosed by curves around the origin or near A escape to infinity

These results extend the universality classes previously associated with these exponents and show that a reversible mapping can have two period doubling cascades in different parts of the plane occurring with different rates, one conservative and one dissipative. Other observations include the appearance of strange attractors (and therefore, because of reversibility, strange repellors) beyond the point of accumulation of the asymmetric period doubling cf. [6, 11].

## 3. Attractors and Repellors born from Conservative Features

In Figure 3, we see that the attractor and repellor in our reversible mapping lie on the outside of the KAM curves. It is also possible to have the attractor and repellor encircled by these curves. This can happen when an elliptic symmetric fixed point surrounded by KAM curves (Figure 4a) turns unstable with both eigenvalues equal to +1. In this case an asymmetric attracting fixed point and a repelling fixed point are born on either side of the symmetry line on which the symmetric fixed point lies (cf. Figure 4b - the symmetry line is the dashed horizontal line shown). The KAM curves present before the bifurcation appear to remain for some parameter interval afterwards, enclosing the unstable symmetric fixed point and two asymmetric fixed points. Figures 4a and 4b are actually phase portraits of (2.3) for C = 2.827 and C = 2.83, respectively. The bifurcation occurs at $C = 2\sqrt{2} = 2.8284..$ . For more details of this bifurcation cf. [12], and see [13] for a similar bifurcation in area preserving mappings.

One situation in which we observe such a bifurcation is when the period doubling of a symmetric periodic orbit terminates because of Trace Turnaround [6]. By this we mean that the Trace of the Jacobian matrix of the symmetric daughter orbit of a symmetric period doubling does not proceed in the usual way from +2 (its point of creation) to –2 (its point of period doubling), but rather reaches a minimum between the two values and then climbs back towards +2. On reaching +2, the symmetric orbit turns unstable and gives birth to an attractor and repellor as described above. It is obviously important to know when and why this anomalous behaviour occurs because it introduces expansion and dissipation in the vicinity of the symmetric periodic orbit.

## 4. Measure Preserving Mappings need not be Reversible

We have been able to construct measure preserving and area preserving mappings which we can show are not reversible.

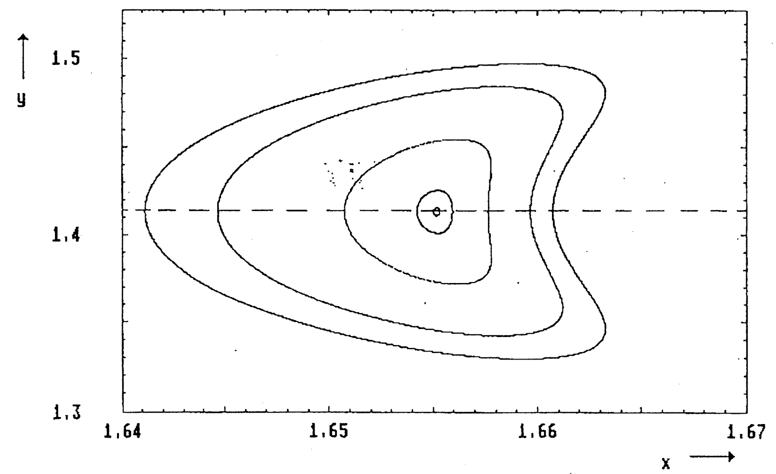


Figure 4a. Before the bifurcation of an attractor and a repellor from a symmetric fixed point

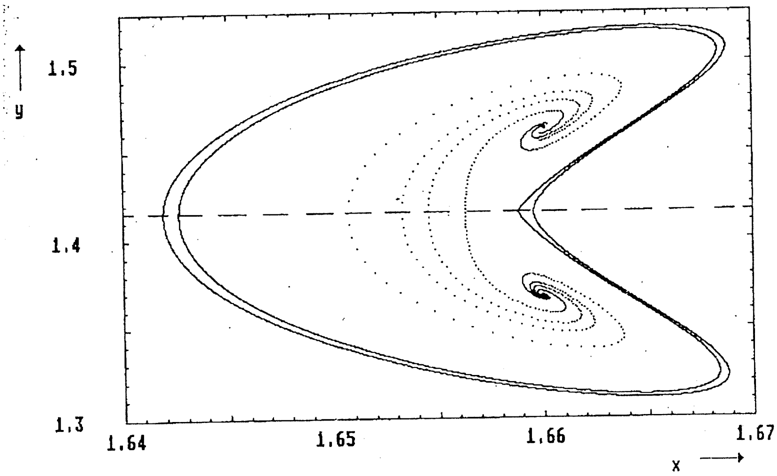


Figure 4b. After the bifurcation of an attractor and a repellor from a symmetric fixed point

To do this we use the idea of *local reversibility* (cf. [14]). The idea is that if a mapping is reversible, then the relation (1.4) holds throughout the plane. In particular, if we have a symmetric fixed point, which by definition is fixed under G *and* L, then (1.4) holds in a neighbourhood of the point. For a given measure preserving mapping which is not known *a priori* to be reversible, we can identify a fixed point as necessarily symmetric if the mapping is reversible by comparing its eigenvalues with those of the other fixed points of the mapping. This uses the fact that asymmetric fixed points of a reversible mapping necessarily have inverse eigenvalues. Consequently if the fixed point has no 'partner' with inverse eigenvalues it is symmetric if the mapping is reversible. By now taking the expansions of L and a possible G around the fixed point and substituting in (1.4), we can determine at the nth order of the expansion necessary conditions on the coefficients of L in order that there exist some mapping satisfying (1.4) which is an involution to order n. If these conditions are not satisfied then L is not reversible because not locally reversible around the fixed point implies not globally reversible.

Application of this method to third order around hyperbolic and elliptic fixed points of measure preserving mappings and fixed points with linearisation (1, b, 0, 1) with $b \neq 0$ revealed that local measure preservation always implies local reversibility [14].

We have applied the method to expansions of measure preserving mappings around fixed points which have linear part given by the identity matrix (1, 0, 0, 1) and no second order terms [15]. At fourth order we find an additional necessary condition for reversibility that is not implied by measure preservation.

As a result, the following area preserving mapping is generically not reversible [15]:

$$P: \quad x' = x + ay^3 + by^4 \qquad (4.1a)$$

$$y' = y + cx'^3 + dx'^4 \qquad (4.1b)$$

when $ac \neq 0$ and $bd \neq 0$. Note that this is of the form of (1.6) except that here neither of the functions f(y) nor g(x') are odd. It raises the question as to whether (1.6) with both even and odd terms in f and g is more generally not reversible. We hope to report on this and other questions concerning the similarities and differences between conservative and reversible mappings in the near future.

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