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CONSERVATIVE AND DISSIPATIVE BEHAVIOUR IN REVERSIBLE DYNAMICAL SYSTEMS

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Conservative and dissipative behaviour can coexist in reversible dynamical systems. We recognize the presence of both behaviours quantitatively by calculating critical exponents associated with period-doubling and the destruction of KAM curves. A "strange" attractor in a reversible system is also investigated.

1. Reversible dynamical systems [1] combine the properties of the two more commonly studied classes of dynamical system, i.e. conservative and dissipative systems. That is to say, generally the phase space of a reversible system contains both Kolmogorov–Arnol'd–Moser (KAM) tori (typical of conservative systems) and also (strange) attractors (typical of dissipative systems). For this reason reversible systems may also be of interest as perturbations of (reversible) conservative systems. Unlike dissipative perturbations, reversible perturbations of conservative systems can preserve the stability associated with KAM tori in part of the phase space, while introducing attractors and repellors in another part.

In this Letter we show the coexistence of conservative [2] and dissipative [3] behaviour in reversible dynamical systems by calculating critical exponents associated with period-doubling and the breaking of KAM tori. We restrict ourselves to the study of mappings of the plane. In particular we introduce a one-parameter reversible mapping of the plane which has an elliptic fixed point and invariant KAM curves in one part of the plane, and an attracting fixed point in another part. The critical exponents associated with the period-doubling of the elliptic fixed point and the breakup of the invariant curves are those previously found in conservative (i.e. area preserving) mappings (see also our previous Letter [4]). The critical exponents associated with the period-doubling of the attractive fixed point are those previously found in dissipative mappings. We present numerical evidence that a strange attractor may exist after the dissipative period-doubling cascade accumulates.

Area preserving mappings have a Jacobian determinant, J, equal to ± 1 everywhere in the plane, and have been well studied [5]. Like all conservative systems, area preserving mappings cannot have attractors. If the mapping is close to integrable or has an elliptic fixed point then the well-known KAM theorems [6,7] state that typically the plane contains infinitely many nested closed curves (KAM circles) that are left invariant by the mapping.

In dissipative mappings, $|J| < 1$ at any point. In these systems, attractors are common. An attractor may be a periodic orbit or it may be a strange attractor [2,8].

Quantitative differences in the behaviour of one-

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Table 1
Universal numbers of conservative and dissipative mappings of the plane.

	Conservative	Dissipative
δ_{PD}	8.721097200...	4.6692016091...
α_{PD}	−4.018076704...	2.5029078750...
β_{PD}	16.363896879...	–
R_{KAM}	0.2500888...	–
$\delta_{C,KAM}$	−2.6651429...	–
$\delta_{R,KAM}$	−1.6371161...	–

parameter families of conservative and dissipative mappings are found in the scaling exponents associated with the period-doubling route to chaos [9]. It is found in many typical cases that the parameter values C_n at which the period-doubling bifurcations occur, converge asymptotically geometrically to a value C_{PD} with rate δ_{PD}. Associated with the period-doubling is also an orbit scaling factor α_{PD}. The values of δ_{PD} and α_{PD} are found to be different for the class of area preserving mappings [10] and for the class of dissipative mappings [11], but are universal within each class [12,13] (note that we are restricting our discussion to the universality class of quadratic mappings). For area preserving mappings another universal orbit scaling factor β_{PD} also exists. These values are listed in table 1.

For one-parameter families of conservative mappings, universal scaling behaviour can also be associated with the breaking-up of a KAM curve with an irrational "noble" winding number as the map parameter is varied [14]. The parameter value C_{KAM} at which this occurs can be calculated by a method due to Greene [15]. Three numbers associated with the process, R_{KAM}, $\delta_{C,KAM}$ and $\delta_{R,KAM}$ (cf. ref. [4]), have been found to be universal. These numbers are also listed in table 1.

2. A mapping L is called reversible [16] if there exists an involution G such that

$$L \circ G \circ L = G . \quad (1)$$

This is equivalent to saying that L can be written as the composition of two involutions

$$L = H \circ G , \quad H \circ H = G \circ G = \mathrm{Id} . \quad (2)$$

This definition is a generalization of the classical definition of a reversible mechanical system [1]. It is invariant under arbitrary (autonomous) coordinate transformations.

A mapping $L: (x, y) \to (x', y')$ is called measure-preserving (MP) if there is a function $f(x, y)$ such that

$$\int_D f(x, y)\, dx\, dy = \pm \int_{L(D)} f(x', y')\, dx'\, dy' \quad (3)$$

for any region D. An equivalent requirement is

$$J_L = \pm \frac{f(x, y)}{f(x', y')} . \quad (4)$$

Area preserving mappings are MP with $f(x, y) \equiv 1$. All periodic orbits in MP systems have return Jacobian determinant ±1. Measure preservation is invariant under arbitrary (autonomous) coordinate transformations (where under the transformation the measure f is multiplied by the Jacobian determinant of the transformation) [17]. It is not difficult to show that each differential involution is MP. However a reversible mapping need not be MP: a necessary condition for it not to be MP is that its composing involutions be incommensurate (i.e. do not preserve the same measure).

Until recently most area preserving systems studied have been reversible and many reversible systems studied have been conservative [14,18–21]. For flows, however, Moser [22] has shown that reversibility alone is sufficient for KAM theorems to hold. Furthermore in ref. [23], an example was presented of a reversible flow that has "dissipativelike structures" (cf. also ref. [24]). For mappings, Arnol'd and Sevryuk [16,25,26] have very recently proved KAM theorems for reversible (non-area-preserving) diffeomorphisms.

An involution G satisfying (1) is called a symmetry of L. The fixed points of G often form a line in the plane, called a symmetry line. If G is a symmetry then so are $L^n G$ for n an integer. The fixed lines of these other symmetries are obtained by taking images of the symmetry lines of G and LG. The points of intersection of different symmetry lines can be shown to be periodic points of L.

A trajectory that is left invariant by a symmetry is called a symmetric trajectory. It is readily shown that

any symmetric trajectory intersects a symmetry line and, conversely, that any trajectory that cuts a symmetry line is symmetric. If a mapping has an attractor A then it also has a repellor GA. An attractor cannot intersect a symmetry line because then $GA = A$ and it is also a repellor, which is impossible.

A symmetric periodic orbit has a return Jacobian determinant equal to ±1. When it equals +1, the linearization around the orbit is typically elliptic or hyperbolic. In the former case, the theorems of refs. [16,25,26] can be applied to show that generically the orbit is surrounded by KAM circles which are symmetric with respect to the symmetry line involved. When a symmetric periodic orbit turns hyperbolic and period doubles it does so along and across a symmetry line, maintaining two points of the new elliptic orbit on the line itself.

3. An example of a one-parameter family of reversible mappings that we have investigated numerically is

$$L_1: \quad x' = -y + y^3 + x , \quad (5a)$$

$$y'^3 - y^3 - y'^2 - y^2 + y' - y + 2Cx' + 2x'^2 - 2x' = 0 . \quad (5b)$$

Eq. (5b) has only one real solution for y' as a function of x' and y and this can be explicitly written out in terms of cube roots. The mapping L_1 is also invertible. It can be written as the composition of the following two involutions,

$$H_1: \quad x' = x ,$$

$$y'^3 + y^3 - y'^2 - y^2 + y' + y + 2Cx + 2x^2 - 2x = 0 , \quad (6)$$

$$G_1: \quad x' = -y + y^3 + x , \quad y' = -y . \quad (7)$$

Note that the symmetry line of G_1 is simply $y = 0$.

The mapping L_1 with the quadratic and cubic terms in y and y' removed reduces to the area preserving Hénon mapping [27], apart from a simple coordinate transformation. However when these terms are present the mapping is not area preserving. Instead its Jacobian determinant varies throughout the plane.

The mapping L_1 has two symmetric fixed points at (0, 0) and $(1-C, 0)$. These points lie at the intersection of the symmetry lines of H_1 and G_1, and it is readily checked that at these points $J = 1$. The points are elliptically stable for $-1 < C < 1$ and $1 < C < 3$ respectively. The mapping also possesses four asymmetric fixed points which come in the pairs $(x_+, \pm 1)$ and $(x_-, \pm 1)$ where

$$x_\pm = [1 - C \pm \sqrt{(1-C)^2 + 4}\,]/2 . \quad (8)$$

Each point in a pair is the reflection of the other point in the pair by the involutions H_1 and G_1. We have concentrated on the pair $(x_+, -1)$ and $(x_-, +1)$ which have $J = 1/3$ and $J = 3$, respectively, independent of C. When $(1-C)^2 < 12$, they form an attractor/repellor pair. In fig. 1 we have shown part of the phase portrait of (5) when $C = 0.5$ (note the trajectories of two points just outside the KAM circles and island chain which, for many iterations, seem to "stick" to a KAM curve but eventually converge to the attracting fixed point).

We have analyzed the period-doubling of the origin as the parameter C is decreased below $C = -1$, where it bifurcates to a symmetric two-cycle. We have also looked at the period-doubling of the attractor at

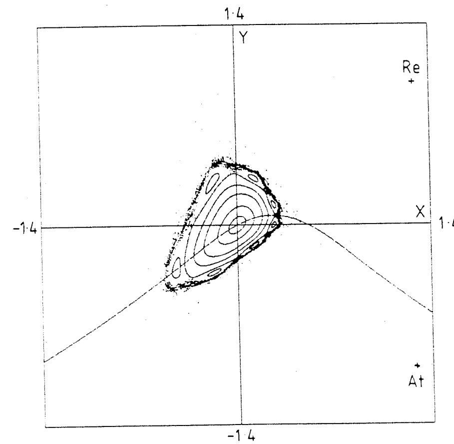

Fig. 1. Phase portrait of the reversible mapping (5) for $C = 0.5$. Two symmetry lines are the x axis and the humped curve shown. The symmetric fixed point at (0, 0) is surrounded by KAM circles. The attracting (At) and repelling (Re) fixed points are marked with crosses. External to the KAM circles and island chain are shown two trajectories that eventually converge to the attracting fixed point.

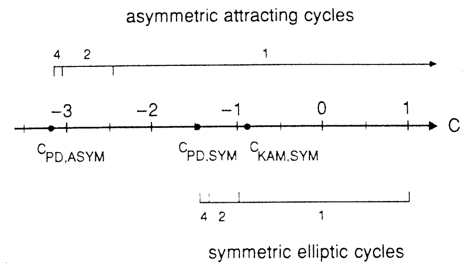

Fig. 2. Intervals in the parameter C for the existence of (asymmetric) attracting cycles and (symmetric) elliptic cycles in the mapping (5). The numbers of 1, 2, 4 label the windows of existence of cycles with these periods in the symmetric and asymmetric period-doubling.

$(x_+, -1)$ below $C=-2.46...$, where it bifurcates to an attracting two-cycle. In fig. 2 we display the parameter ranges in which these two period-doubling processes occur. The symmetric period-doubling accumulates at $C_{PD,SYM}=-1.473315...$ and the asymmetric period-doubling at $C_{PD,ASYM}=-3.179095...$.

In table 2 (example 1) we present our calculations for the parameter and orbit scalings of the two sequences of period-doublings in L_1, based on periods up to 1024. It is seen that the critical exponents for period-doubling of the origin are those found in conservative mappings and that those for period-doubling of the attractor are those associated with dissipative mappings. (Note that for some other mappings, which differ from (5) only in the constant coefficients of the quadratic and cubic terms in

Table 2
Universal numbers of reversible mappings of the plane. Example 1 is the mapping L_1 given by eq. (5). Example 2 is the mapping L_2 given by eq. (9)

	Example 1	Example 2
$\delta_{PD,SYM}$	8.7210...	8.7210972...
$\alpha_{PD,SYM}$	−4.018...	−4.0180767...
$\beta_{PD,SYM}$	16.36...	16.3639...
$\delta_{PD,ASYM}$	4.66920...	4.6692
$\alpha_{PD,ASYM}$	2.5029...	2.502...
$R_{KAM,SYM}$	0.250...	0.25008...
$\delta_{C,KAM,SYM}$	−2.6...	−2.665...
$\delta_{R,KAM,SYM}$	−1.6...	−1.63...

y and y', we have observed that the normal symmetric period-doubling terminates at a four-cycle. We observe that the symmetric four-cycle becomes hyperbolic and bifurcates asymmetrically to produce an attracting four-cycle and a repelling four-cycle which are initially enclosed by apparently invariant curves that still encircle the hyperbolic symmetric four-cycle, cf. also refs. [23,24].) An analysis of the possible bifurcations in reversible non-area-preserving mappings is given in ref. [28] (and cf. refs. [19,20] for the reversible area preserving case).

In table 2 we have also listed critical exponents calculated for the breaking in the mapping L_1 of the KAM circle around the origin which has winding number $(3-\sqrt{5})/2$. Applying Greene's method, we find that the circle breaks at a value $C_{KAM,SYM}=-0.889335...$.

Under the heading example 2 of table 2, we have included numerical results for another reversible non-area-preserving mapping which we partially investigated in ref. [4]:

$$L_2:\quad x'=\frac{-y/(1+y^2/2500)+x}{1-xy^3/(1+2500y^2)},$$

$$y'=2x'-2Cx'-2x'^2+y. \qquad (9)$$

Unlike the mapping L_1, this example has singularity lines, i.e. lines in the plane where the mapping is not defined. The numerical results however are in good agreement. Furthermore, the critical exponents found for symmetric period-doubling and KAM curve destruction in the mappings L_1 and L_2 agree well with other examples studied in ref. [4]

For both L_1 and L_2 we have found that for many values of C in some range beyond $C_{PD,ASYM}$ an apparently aperiodic attractor appears. For L_1, this attractor consists of two pieces for $-3.23\lesssim C\lesssim -3.20$ and one piece for $-3.301\leqslant C\leqslant -3.23$. It does not seem to exist for $C<-3.301$. This attractor arises in a similar way to the one found in Hénon's dissipative mapping [29]. Note that Hénon's dissipative mapping is not reversible though because, for example, it can be shown that whilst it has an attractive fixed point in some parameter range, its other fixed point is not a repellor but rather hyperbolic. Like Hénon's attractor, we find self-similarity and an apparently endless series of folds in our attractor – see fig. 3. It would seem therefore that this attractor is

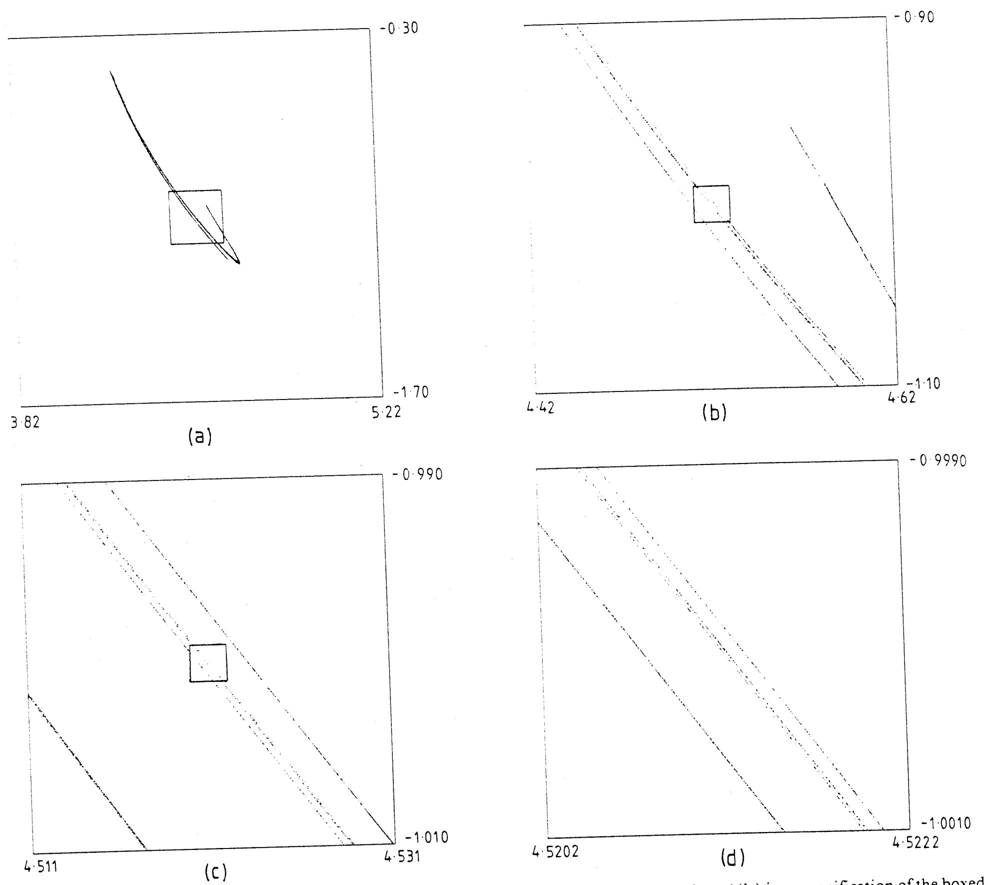

Fig. 3. (a)–(d) Strange attractor in the mapping (5) for $C=-3.300$, and successive magnifications ((b) is a magnification of the boxed region in (a), etc.). At the centre of each picture is the hyperbolic fixed point $(x_+, -1)$ which was originally attracting.

strange. We note too that in our example, reversibility implies that coincident with the period-doubling cascade of attractors and an eventual strange attractor there exists a period-doubling cascade of repellors and a strange repellor.

It is interesting to see how the strange attractor manages to avoid cutting the symmetry lines of the reversible mapping. In fig. 4, we show portions of the symmetry lines of $L_1^5G_1$ and $L_1^7G_1$ in the vicinity of the strange attractor of L_1. We find that the higher the power of L_1 in the symmetry $L_1^{2i+1}G_1$, the closer a part of its symmetry line comes to the attractor and the more loops it has. These observations are consistent with the fact that the symmetry lines of these symmetries are just images (under L_1^i) of that of $L_1G_1=H_1$. The increasing proximity to the attractor with increasing i indicates that the symmetry line of H_1 must intersect its basin of attraction. As a result, all symmetries of the form $L_1^{2i+1}G_1$ have part of their symmetry lines in the basin of attraction, and the folding of these lines is a consequence of the nature of the attractor. Symmetry lines cannot intersect in

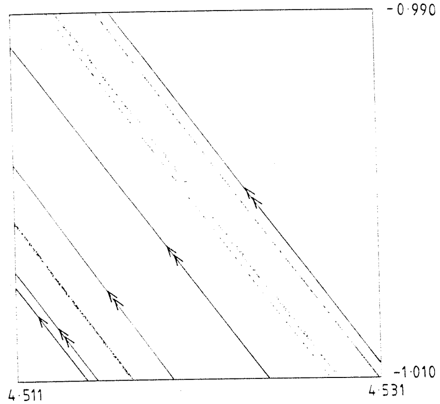

Fig. 4. Fig. 3c with the addition of some parts of the symmetry lines of $L_1^5 G_1$ (single arrow) and $L_1^7 G_1$ (double arrows).

the basin of attraction since their intersection yields a periodic orbit which is not attracted by the attractor. Consequently, symmetry lines are nested parallel to one another and to the attractor in this region as fig. 4 indicates. Note however that all the symmetry lines are "anchored" outside the basin of attraction due to the fact that they pass through the two symmetric fixed points.

4. In closing we would like to point out that the mapping L_1 is not an isolated example. Rather it is typical of reversible non-area-preserving mappings that we have been able to construct by combining area preserving involutions (like G_1) with non-area-preserving involutions (like H_1). The methods we use will be expanded on elsewhere [30]. It seems to us that our results should have some physical relevance – they indicate, for example, that a perturbation of a reversible conservative system need not itself be dissipative (but just reversible) in order to introduce dissipative features into the dynamics. In some cases the regions of conservative or dissipative behaviour in the phase space may be so small that they are only seen for special initial conditions. It would be interesting to look experimentally for the presence of both behaviours [23].

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THEORY OF FIRST ORDER PHASE TRANSITIONS FOR CHAOTIC ATTRACTORS OF NONLINEAR DYNAMICAL SYSTEMS

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A theory is presented for first order phase transitions of multifractal chaotic attractors of nonhyperbolic two-dimensional maps. (These phase transitions manifest themselves as a discontinuity in the derivative with respect to q (analogous to temperature) of the fractal dimension q-spectrum, D_q (analogous to free energy).) A complete picture of the behavior associated with the phase transition is obtained.

Recently the multifractal properties of chaotic attractors have been a subject of intense study. In particular, the spectrum of dimensions D_q characterizing the multifractal properties of attractors [1] has been shown to follow from a thermodynamic formalism [2]. In this formalism, a first order phase transition [3–5] is manifested in the q-dependence of the dimension spectrum as a discontinuity in the derivative dD_q/dq. (D_q is analogous to the free energy while q plays the role of temperature.) In this note we present a theory which yields a complete picture of the behavior of D_q and the singularity dimension $f(\alpha)$ associated with these first order phase transitions for nonhyperbolic two-dimensional maps with constant Jacobian. (These results are contained in our eqs. (18)–(24) at the end of the paper.) Our results for q_T and D_T (eqs. (18) and (19)) have been obtained previously in ref. [5] using a different method.

As a guide to how we will proceed in the two-dimensional case, consider the one-dimensional quadratic map at criticality [4], $x_{n+1}=rx_n(1-x_n)$, with $r=4$. Almost every initial condition in [0, 1] produces an invariant probability density, $\rho(x)=[\pi^2 x(1-x)]^{-1/2}$ for $0\leqslant x\leqslant 1$. The definition of D_q for an attractor lying in an M-dimensional phase space is

$$D_q=\lim_{\epsilon\to 0}(q-1)^{-1}\ln I_q(\epsilon)/\ln\epsilon\,, \qquad (1)$$

where $I_q(\epsilon)=\sum_i p_i^q$ and p_i is the fraction of time a typical orbit falls in the ith M-dimensional cube of an M-dimensional cubic grid of unit grid size ϵ. That is, p_i is the "natural measure" in cube i. (For $M=1$ the "cubes" are intervals of length ϵ.) Applying eq. (1) to the quadratic map at criticality, we express I_q as

$$I_q(\epsilon)=I_q^{(h)}(\epsilon)+I_q^{(nh)}(\epsilon)\,. \qquad (2)$$

In eq. (2), $I_q^{(h)}$ represents a "hyperbolic contribution" to I_q which arises from the bulk of the intervals away from the special points, $x=0$, 1, where $\rho(x)$ goes to infinity. For these intervals $p_i\sim\epsilon$, and there are $\sim 1/\epsilon$ such intervals, yielding $I_q^{(h)}\sim\epsilon^{q-1}$. The term $I_q^{(nh)}(\epsilon)$ is the "nonhyperbolic contribution" arising from the inverse square root singularities in $\rho(x)$ at $x=0$, 1. From the contributions to $\sum_i p_i^q$ from