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EQUATION OF MOTION FOR THE HEISENBERG SPIN CHAIN

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A systematic treatment is given of the equation of motion of the classical anisotropic Heisenberg spin chain, both in the discrete case and in the continuum limit in which the spins $\boldsymbol{S}_m(t)$ associated with the lattice sites m are replaced by a spin density $\boldsymbol{S}(x, t)$, which is a function of the time t and the position x on the chain. In the case of axial symmetry the equation of motion for the spins is shown to be equivalent to a new equation in terms of one real variable, i.e. $q_m(t)$ in the discrete case and $q(x, t)$ in the continuum limit. (From the treatment by A.E. Borovik it follows that the new equation of motion for $q(x, t)$ is completely integrable in the special case of quadratic anisotropy.) Explicit expressions are given for the Lagrangians, both in the ferromagnetic and in the antiferromagnetic case. The relation with the nonlinear Schrödinger equation on the one hand and the sine-Gordon equation on the other hand is discussed in some detail.

1. Introduction

The spin dynamics in one-dimensional spin systems has received a lot of attention. From an experimental point of view one-dimensional spin dynamics can be observed in systems consisting of weakly coupled chains. Due to the small interchain interactions the phase transition to 3-dimensional long-range ordering will occur at very low temperatures and a substantial amount of short-range ordering will show up well above the transition temperature.

From a theoretical point of view many exact and approximate results have been derived recently. Exact results have been obtained for the time-dependent correlations in the one-dimensional spin-$\frac{1}{2}$ XY-model, viz. the correlations between the z-components of spins at arbitrary temperatures[1,2]), and the correlations between x-components of spins at infinite temperature[3–5]). At zero temperature explicit results have been obtained for the correlations at large distances[6,7]). Nonlinear differential–difference equations which are valid at arbitrary temperatures[8,9]) have also been derived.

The spin dynamics in Heisenberg chains has not been solved exactly but a rather good insight has been obtained recently using an elegant description in terms of two-parameter spin-wave continua[10,11]), see also ref. 12 and references cited therein. A successful explanation has been given for the behaviour of the proton-spin lattice relaxation time in antiferromagnetic linear chain systems[13,14]). Recent progress in the quantum inverse scattering method may turn out to be important for the spin dynamics in Heisenberg chains, see ref. 15 and references cited therein.

Although quantum effects may be quite important, a number of interesting results have been obtained by investigating the equation of motion for a system of classical spins. Usually one applies a continuum approximation in which the spins $\boldsymbol{S}_m(t)$ at the lattice sites m are replaced by a spin density $\boldsymbol{S}(x, t)$ which is a function of time t and the position x on the chain, see ref. 16. A treatment of the equation of motion for the Heisenberg chain can be found e.g. in ref. 17 for the isotropic case, and in ref. 18 for the case of uniaxial anisotropy. The complete integrability of the Heisenberg chain was proved in the isotropic case[19]), cf. also ref. 20, as well as in the presence of a uniaxial[21]) or an orthorhombic[22]) crystalline field term. An exact relation between the isotropic Heisenberg chain and the nonlinear Schrödinger equation was given in ref. 23, see also ref. 24. A quantum mechanical derivation of the nonlinear Schrödinger equation was discussed in ref. 25.

Furthermore there is an approximate relation between the Heisenberg chain with uniaxial anisotropy of the Ising type and the nonlinear Schrödinger equation[22]), and another approximate relation between the Heisenberg chain with planar anisotropy and the sine-Gordon equation in the ferromagnetic[26]), as well as in the antiferromagnetic case[27]). Explicit solutions for the anisotropic Heisenberg chain can be found in refs. 28–33, and multisoliton solutions have been discussed in ref. 34. (For treatments of the equation of motion in two or more dimensions, see e.g. refs. 35–40.) The statistical mechanics of solitons in the Heisenberg chain with uniaxial anisotropy was treated in refs. 41, 42.

The description in terms of the sine-Gordon equation has been useful in the interpretation[26,27,43–45]) of the central peak in inelastic neutron scattering found in ferromagnetic $CsNiF_3$[46]), as well as in antiferromagnetic TMMC[47]). Further evidence for soliton motion has recently been obtained from Mössbauer linewidths in antiferromagnetic linear chains with Ising anisotropy[48]) and from (frequency-dependent) susceptibility measurements in quasi one-dimensional ferromagnetic TMNC[49]). A critical discussion of the mapping of an easy-plane ferromagnet onto the sine-Gordon equation has been given in ref. 50.

In the present paper we investigate in a systematic way the nonlinear differential equations for the anisotropic Heisenberg chain, both in a discrete and in a continuum description. (The treatment in this paper will be restricted mainly to the case of classical spins.) In a discrete description we have at each lattice site m a 3-dimensional (real) spin vector $\boldsymbol{S}_m$, subjected to a normalization condition $(\boldsymbol{S}_m \cdot \boldsymbol{S}_m)^{\frac{1}{2}} = s$, and the equation of motion is reviewed in section 2.

Using the Holstein–Primakoff transformation[51]) in the theory of spin waves, the $n = 3$ description in terms of spins $\boldsymbol{S}_m$ can be reduced to an $n = 2$ description with one complex field χ_m at every lattice site m. The equation of motion and the corresponding Lagrangian are given in section 3. In the case of axial symmetry we shall show that the spin dynamics can be described completely in terms of one *real* field q_m associated with every site m. In section 4 we give the equation of motion and the Lagrangian in an $n = 1$ description in terms of the variables q_m.

In section 5 we treat the continuum approximation for the Heisenberg chain using a continuum version of the Holstein–Primakoff transformation. Both in the ferromagnetic and in the antiferromagnetic case the spin dynamics is determined by one complex field $\chi(x, t)$ which is a function of time t and position x. The equation of motion and the Lagrangian in the continuum $n = 2$ description with one complex field $\chi(x, t)$ are given in section 5. In the case of axial symmetry one again has an $n = 1$ description with one real field $q(x, t)$. The equation of motion and the Lagrangian are given in sections 6 and 7 for the ferromagnetic and antiferromagnetic case, respectively. In section 8 we discuss the relations with the nonlinear Schrödinger equation and in section 9 an extension of the sine-Gordon equation is derived under the assumption that the spin density $\boldsymbol{S}(x, t)$ is nearly restricted to a plane for all x and t.* The equation of motion and the Lagrangian in an $n = 1$ description can be helpful in obtaining similarity solutions, which will be treated in a following paper[53]).

2. Discrete $n = 3$ description

Consider the Heisenberg spin chain described by the Hamiltonian

$$H = -\sum_{m=1}^{N-1} \boldsymbol{S}_m \cdot \boldsymbol{J}_m \cdot \boldsymbol{S}_{m+1} - \sum_{m=1}^{N} K_m(\boldsymbol{S}_m), \tag{2.1}$$

where $m = 1, \ldots, N$ labels the different sites on the chain. $\boldsymbol{J}_m$ describes a site-dependent exchange interaction between the components of spins at neighbouring sites m and $m + 1$ and $K_m(\boldsymbol{S}_m)$ is an anisotropy field acting on spin $\boldsymbol{S}_m$, arising e.g. from crystalline fields and external magnetic fields. Throughout this paper we shall assume that the tensor $\boldsymbol{J}_m$ is diagonal, i.e.

$$\boldsymbol{J}_m = J_m \begin{bmatrix} 1 + f_m & & \\ & 1 - f_m & \\ & & 1 + c_m \end{bmatrix}, \quad m = 1, \ldots, N - 1. \tag{2.2}$$

* It has been shown recently[52]) that this assumption is not always correct, even in cases when there is a substantial amount of planar anisotropy.

The components of the spins have the commutation relations

$$[\boldsymbol{S}_m, \boldsymbol{S}_{m'}] = \mathrm{i}\boldsymbol{\epsilon} \cdot \boldsymbol{S}_m \delta_{m',m}, \tag{2.3}$$

where $\boldsymbol{\epsilon}$ is the completely antisymmetric Levi–Civita tensor.

The time evolution of the operator $\boldsymbol{S}_m$ is described by the equation of motion

$$\dot{\boldsymbol{S}}_m = \mathrm{i}[H, \boldsymbol{S}_m]. \tag{2.4}$$

From (2.1), (2.3) and (2.4), taking into account that $\boldsymbol{J}_m$ is symmetric, we have

$$\dot{\boldsymbol{S}}_m = \boldsymbol{S}_m \times (\boldsymbol{J}_{m-1} \cdot \boldsymbol{S}_{m-1} + \boldsymbol{J}_m \cdot \boldsymbol{S}_{m+1}) + \mathscr{D}\boldsymbol{K}_m(\boldsymbol{S}_m), \tag{2.5}$$

where $\mathscr{D}K_m(\boldsymbol{S}_m)$ is the sum of all contributions which can be obtained replacing in the terms of $K_m(\boldsymbol{S}_m)$ a factor $\boldsymbol{S}_m$ by $(\boldsymbol{\epsilon} \cdot \boldsymbol{S}_m)$.

Assuming the spins in the right-hand side of (2.5) to be classical, i.e. ignoring the commutation relations between the spin components in $\mathscr{D}K_m(\boldsymbol{S}_m)$, eq. (2.5) reduces to

$$\dot{\boldsymbol{S}}_m = \boldsymbol{S}_m \times (\boldsymbol{J}_{m-1} \cdot \boldsymbol{S}_{m-1} + \boldsymbol{J}_m \cdot \boldsymbol{S}_{m+1}) + \boldsymbol{S}_m \times \boldsymbol{\nabla} K_m(\boldsymbol{S}_m), \tag{2.6}$$

where $\boldsymbol{\nabla} K_m \equiv \partial K_m / \partial \boldsymbol{S}_m$. Note that (2.6) is also valid exactly for quantum mechanical operators in the case that the K_m are linear functions of the components S_m^ξ, $\xi = x, y, z$, arising, e.g., from external magnetic fields. For spins with quantum number $S = \frac{1}{2}$, the K_m can always be chosen to be linear.

Eq. (2.6) has been derived using the quantum mechanical equation of motion (2.4). For classical spins (2.6) can equally well be derived using the equation of motion

$$\dot{\boldsymbol{S}}_m = \{H, \boldsymbol{S}_m\}, \tag{2.7}$$

together with the Poisson bracket

$$\{\boldsymbol{S}_m, \boldsymbol{S}_{m'}\} = -\boldsymbol{\epsilon} \cdot \boldsymbol{S}_m \delta_{m',m}. \tag{2.8}$$

3. Discrete $n = 2$ description

For classical spins with length $s = (\boldsymbol{S}_m \cdot \boldsymbol{S}_m)^{\frac{1}{2}}$, one can introduce a complex field χ_m for every site m, i.e.

$$\begin{aligned} S_m^+ &\equiv S_m^x + \mathrm{i} S_m^y = \chi_m s \sqrt{2}, \\ S_m^- &\equiv S_m^x - \mathrm{i} S_m^y = \chi_m^\dagger s \sqrt{2}, \\ S_m^z &= (1 - \chi_m^\dagger \chi_m - \chi_m \chi_m^\dagger)^{\frac{1}{2}} s, \end{aligned} \tag{3.1}$$

in which the χ_m satisfy the Poisson brackets

$$\{\chi_m, \chi_{m'}\} = \{\chi^\dagger_m, \chi^\dagger_{m'}\} = 0,$$
$$\{\chi_m, \chi^\dagger_{m'}\} = \mathrm{i}s^{-1}(1 - \chi^\dagger_m\chi_m - \chi_m\chi^\dagger_m)^{\frac{1}{2}}\delta_{m',m}. \tag{3.2}$$

For quantum mechanical spins one has the Holstein–Primakoff transformation[51])

$$S^+_m = \left(1 - \frac{a^\dagger_m a_m}{2S}\right)^{\frac{1}{2}} a_m\sqrt{2S},$$
$$S^-_m = a^\dagger_m\left(1 - \frac{a^\dagger_m a_m}{2S}\right)^{\frac{1}{2}}\sqrt{2S}, \tag{3.3}$$
$$S^z_m = S - a^\dagger_m a_m,$$

where S is the spin-quantum number and the operators a satisfy the boson commutation relations

$$[a_m, a_{m'}] = [a^\dagger_m, a^\dagger_{m'}] = 0, \quad [a_m, a^\dagger_{m'}] = \delta_{m',m}. \tag{3.4}$$

Introducing the operators

$$\chi_m = \left(1 - \frac{a^\dagger_m a_m}{2S}\right)^{\frac{1}{2}} \frac{a_m}{\sqrt{S+1}}, \tag{3.5}$$

with the commutation relations

$$[\chi_m, \chi_{m'}] = [\chi^\dagger_m, \chi^\dagger_{m'}] = 0,$$
$$[\chi_m, \chi^\dagger_{m'}] = (S(S+1))^{-\frac{1}{2}}(1 - \chi^\dagger_m\chi_m - \chi_m\chi^\dagger_m)^{\frac{1}{2}}\delta_{m',m}, \tag{3.6}$$

eq. (3.3) can be expressed in the form (3.1) with $s = (S(S+1))^{\frac{1}{2}}$ thereby implying that (3.1) holds also for quantum mechanical spins. In eqs. (3.1), (3.2) and (3.6) it is understood that the expression $(1 - \chi^\dagger_m\chi_m - \chi_m\chi^\dagger_m)^{\frac{1}{2}}$ can also have negative values corresponding to negative values of S^z_m.

Inserting (3.1) and (2.2) in (2.1) we obtain after a straightforward calculation

$$H = -s^2 \sum_{m=1}^{N-1} J_m[\chi^\dagger_m\chi_{m+1} + \chi^\dagger_{m+1}\chi_m + f_m(\chi^\dagger_m\chi^\dagger_{m+1} + \chi_{m+1}\chi_m)$$
$$+ (1 + c_m)(1 - 2|\chi^2_m|)^{\frac{1}{2}}(1 - 2|\chi^2_{m+1}|)^{\frac{1}{2}}] - \sum_{m=1}^{N} K_m(\boldsymbol{S}_m(\chi_m, \chi^\dagger_m)), \tag{3.7}$$

with $2|\chi^2_m| \equiv \chi_m\chi^\dagger_m + \chi^\dagger_m\chi_m$. The equation of motion for χ_m is given by

$$\dot{\chi}_m = \mathrm{i}s[(1 - 2|\chi^2_m|)^{\frac{1}{2}}\{J_{m-1}(\chi_{m-1} + f_{m-1}\chi^\dagger_{m-1}) + J_m(\chi_{m+1} + f_m\chi^\dagger_{m+1})\}$$
$$- \chi_m\{J_{m-1}(1 + c_{m-1})(1 - 2|\chi^2_{m-1}|)^{\frac{1}{2}} + J_m(1 + c_m)(1 - 2|\chi^2_{m+1}|)^{\frac{1}{2}}\}]$$
$$+ (\mathrm{i}/\sqrt{2})(1 - 2|\chi^2_m|)^{\frac{1}{2}}(\nabla_x + \mathrm{i}\nabla_y)K_m - \mathrm{i}\chi_m\nabla_z K_m. \tag{3.8}$$

Instead of using the fields χ_m one may also derive the equation of motion in terms of the operators a_m or the corresponding classical fields. There is, however, no need to use fields that satisfy boson commutation relations or the corresponding classical Poisson brackets. In the $n=2$ description in this paper we use the fields χ_m for convenience, but other choices are possible, see ref. 54.

The equation of motion (3.8) can be derived in various ways.

i) From (2.6) together with the transformation (3.1).

ii) From (3.7) and (3.6) with the quantum mechanical equation of motion $\dot{\chi}_m = \mathrm{i}[H, \chi_m]$, afterwards treating the spins in the right-hand side as classical spins. Ignoring the commutations in $\mathscr{D}K_m(\boldsymbol{S}_m)$, we have

$$\dot{\chi}_m = -\mathrm{i}\frac{\partial H}{\partial \chi^{\dagger}_m}[\chi_m, \chi^{\dagger}_m]. \tag{3.9}$$

iii) From (3.7) and (3.2) with the classical equation of motion

$$\dot{\chi}_m = \{H, \chi_m\} = -\frac{\partial H}{\partial \chi^{\dagger}_m}\{\chi_m, \chi^{\dagger}_m\}. \tag{3.10}$$

iv) From the classical Lagrangian

$$L = -H + \frac{s}{2\mathrm{i}}\sum_m \left(\frac{\partial}{\partial t}\ln\frac{\chi_m}{\chi^{\dagger}_m}\right)(1-2|\chi^2_m|)^{\frac{1}{2}}, \tag{3.11}$$

using the Lagrange equations

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\chi}^{\dagger}_m} - \frac{\partial L}{\partial \chi^{\dagger}_m} = 0, \tag{3.12}$$

as can be checked by explicit calculation using (3.10)–(3.12). Eq. (3.12) can also be derived using more general arguments. In fact, in view of the Poisson brackets

$$\left\{S^z_m, \arctan\frac{S^y_{m'}}{S^x_{m'}}\right\} = \left\{s(1-2|\chi^2_m|)^{\frac{1}{2}}, \frac{1}{2\mathrm{i}}\ln\frac{\chi_{m'}}{\chi^{\dagger}_{m'}}\right\} = \delta_{m',m}, \tag{3.13}$$

cf. (3.2) and also e.g. refs. 55, 56, in which this property has been derived previously, the Lagrangian (3.11) has the form

$$L = -H(\{X_n\}) + \sum_{m=1}^{N} P_m \dot{P}_{m+N}, \quad n = 1, \ldots, 2N, \tag{3.14}$$

where

$$X_m = \chi_m, \quad X_{m+N} = \chi^{\dagger}_m, \tag{3.15}$$

and

$$P_m = s(1-2|\chi_m^2|)^{\frac{1}{2}}, \quad P_{m+N} = \frac{1}{2\mathrm{i}} \ln \frac{\chi_m}{\chi_m^\dagger}, \quad m = 1, \ldots, N. \tag{3.16}$$

From (3.13) it is clear that the P's satisfy the fundamental Poisson brackets

$$\{P_m, P_{m'}\} = \{P_{m+N}, P_{m'+N}\} = 0,$$
$$\{P_m, P_{m'+N}\} = \delta_{m',m}, \quad m, m' = 1, \ldots, N, \tag{3.17}$$

and eq. (3.12) can be derived using the treatment of appendix A in the discrete case, cf. remark i) at the end of the appendix.

Remark. Instead of the Lagrangian (3.11) one may take any Lagrangian of the form (3.14) with arbitrary fields $P_m(\{\chi_{m'}\}, \{\chi_{m'}^\dagger\}), P_{m+N}(\{\chi_{m'}\}, \{\chi_{m'}^\dagger\})$, $(m, m' = 1, \ldots, N)$, satisfying the fundamental Poisson brackets (3.17). Lagrangians in an $n = 3$ description with constraints, i.e. in terms of spin vectors $\boldsymbol{S}_m$ with constant lengths $s = (\boldsymbol{S}_m \cdot \boldsymbol{S}_m)^{\frac{1}{2}}$ have been given in refs. 57, 58.

4. Axial symmetry. Discrete $n = 1$ description

In this section we investigate the special case of axial symmetry around the z axis, i.e. (2.1), (2.2) with

$$f_m = 0, \quad K_m(\boldsymbol{S}_m) \equiv \hat{K}_m(S_m^z), \tag{4.1}$$

and using (4.1), we shall show that the $n = 2$ description in terms of complex fields χ_m can be reduced to an $n = 1$ description in terms of real fields q_m.

Substituting

$$\chi_m = \kappa_m \, \mathrm{e}^{\mathrm{i}\phi_m}, \quad m = 1, \ldots, N, \tag{4.2}$$

eq. (3.8) for the time derivative $\dot{\chi}_m$ can be expressed as

$$\begin{aligned}\frac{\dot{\kappa}_m + \mathrm{i}\kappa_m\dot{\phi}_m}{(1-2\kappa_m^2)^{\frac{1}{2}}} &= \mathrm{i}s[J_{m-1}\kappa_{m-1}\,\mathrm{e}^{\mathrm{i}(\phi_{m-1}-\phi_m)} + J_m\kappa_{m+1}\,\mathrm{e}^{\mathrm{i}(\phi_{m+1}-\phi_m)} \\ &\quad - J_{m-1}(1+c_{m-1})\kappa_m(1-2\kappa_m^2)^{-\frac{1}{2}}(1-2\kappa_{m-1}^2)^{\frac{1}{2}} \\ &\quad - J_m(1+c_m)\kappa_m(1-2\kappa_m^2)^{-\frac{1}{2}}(1-2\kappa_{m+1}^2)^{\frac{1}{2}}] \\ &\quad - \mathrm{i}\kappa_m(1-2\kappa_m^2)^{-\frac{1}{2}}\nabla_z\hat{K}_m, \quad m = 1, \ldots, N,\end{aligned} \tag{4.3}$$

where we have introduced

$$J_0 = J_N = 0, \tag{4.4}$$

in addition to the J_m, $m = 1, \ldots, N-1$, in (2.2).

The real part of (4.3), i.e.

$$\frac{1}{2}\frac{\partial}{\partial t}(1-2\kappa_m^2)^{\frac{1}{2}} = s[-J_{m-1}\kappa_{m-1}\kappa_m \sin(\phi_m - \phi_{m-1}) + J_m\kappa_m\kappa_{m+1}\sin(\phi_{m+1}-\phi_m)], \tag{4.5}$$

can be solved formally, introducing a real variable r_m such that

$$(1-2\kappa_m^2)^{\frac{1}{2}} = r_m(t) - r_{m-1}(t), \quad m = 1, \dots, N. \tag{4.6}$$

From (4.5) and (4.6) we obtain the relation

$$\dot{r}_m = 2J_m s\kappa_m\kappa_{m+1}\sin(\phi_{m+1}-\phi_m) + g_m(t), \quad m = 0, \dots, N, \tag{4.7}$$

where

$$g_m(t) = g_{m+1}(t), \quad m = 0, \dots, N-1, \tag{4.8}$$

is an integration constant.

We can get rid of $g_m(t)$ by introducing the variables

$$q_m = r_m - r_0, \quad m = 0, \dots, N. \tag{4.9}$$

Eq. (4.9) implies in particular

$$q_0 = 0, \tag{4.10}$$

and from (4.9), (4.6), (4.2) and (3.1) we have

$$q_N = s^{-1}\sum_{m=1}^{N} S_m^z, \tag{4.11}$$

which is a constant of the motion, due to the axial symmetry. Eqs. (4.7)–(4.9) lead to the relations

$$\dot{q}_m = 2J_m s\kappa_m\kappa_{m+1}\sin(\phi_{m+1}-\phi_m), \quad m = 1, \dots, N-1, \tag{4.12}$$

and

$$(1-2\kappa_m^2)^{\frac{1}{2}} = q_m - q_{m-1}, \quad m = 1, \dots, N. \tag{4.13}$$

From (4.12) and (4.13) we have

$$\phi_{m+1} - \phi_m = \arcsin\left(\frac{\dot{q}_m(J_m s)^{-1}}{\{1-(q_m-q_{m-1})^2\}^{\frac{1}{2}}\{1-(q_{m+1}-q_m)^2\}^{\frac{1}{2}}}\right), m = 1, \dots, N-1. \tag{4.14}$$

Taking the time derivative of (4.14) and using the imaginary part of (4.3), i.e.

$$\begin{aligned}\dot{\phi}_m = {} & s(1-2\kappa_m^2)^{\frac{1}{2}}\{J_m(\kappa_{m+1}/\kappa_m)\cos(\phi_{m+1}-\phi_m) \\ & + J_{m-1}(\kappa_{m-1}/\kappa_m)\cos(\phi_m-\phi_{m-1})\} \\ & - s\{J_{m-1}(1+c_{m-1})(1-2\kappa_{m-1}^2)^{\frac{1}{2}} + J_m(1+c_m)(1-2\kappa_{m+1}^2)^{\frac{1}{2}}\} \\ & - \nabla_z\hat{K}_m(s(1-2\kappa_m^2)^{\frac{1}{2}}), \quad m = 1, \dots, N,\end{aligned} \tag{4.15}$$

we obtain

$$\frac{\partial}{\partial t}\arcsin\left(\frac{\dot{q}_m(J_m s)^{-1}}{\{1-(q_m-q_{m-1})^2\}^{\frac{1}{2}}\{1-(q_{m+1}-q_m)^2\}^{\frac{1}{2}}}\right)$$
$$=\frac{(q_{m+1}-q_m)(\zeta_{m+1}+\zeta_m)}{1-(q_{m+1}-q_m)^2}-\frac{(q_m-q_{m-1})(\zeta_m+\zeta_{m-1})}{1-(q_m-q_{m-1})^2}$$
$$-J_{m+1}s(1+c_{m+1})(q_{m+2}-q_{m+1})+J_m s(1+c_m)(q_{m+1}-q_m)$$
$$-J_m s(1+c_m)(q_m-q_{m-1})+J_{m-1}s(1+c_{m-1})(q_{m-1}-q_{m-2})$$
$$-\nabla_z\hat{K}_{m+1}(s(q_{m+1}-q_m))+\nabla_z\hat{K}_m(s(q_m-q_{m-1})),\quad m=1,\ldots,N-1, \tag{4.16}$$

where we have used the short-hand notation

$$\zeta_m=(J_m^2 s^2\{1-(q_{m+1}-q_m)^2\}\{1-(q_m-q_{m-1})^2\}-\dot{q}_m^2)^{\frac{1}{2}},\quad m=0,\ldots,N. \tag{4.17}$$

From a solution $q_m(t)$ of (4.16), the corresponding spin vector $\boldsymbol{S}_m(t)$ can be obtained using (3.1), (4.2) and the explicit expressions

$$\kappa_m(t)=(1/\sqrt{2})\{1-(q_m-q_{m-1})^2\}^{\frac{1}{2}}, \tag{4.18}$$

$$\phi_m(t)=\sum_{n=k}^{m-1}\arcsin(\dot{q}_n(J_n s)^{-1}\{1-(q_n-q_{n-1})^2\}^{-\frac{1}{2}}\{1-(q_{n+1}-q_n)^2\}^{-\frac{1}{2}})$$
$$+\int_0^t dt_1[(q_k-q_{k-1})(\zeta_k+\zeta_{k-1})\{1-(q_k-q_{k-1})^2\}^{-1}$$
$$-J_k s(1+c_k)(q_{k+1}-q_k)-J_{k-1}s(1+c_{k-1})(q_{k-1}-q_{k-2})$$
$$-\nabla_z\hat{K}_k(s(q_k-q_{k-1}))]+\phi_k(0), \tag{4.19}$$

for a constant value k, $k\in\{1,\ldots,m\}$, with the understanding that the sum over n in the first term vanishes for $k=m$. Eq. (4.19) can easily be derived from (4.14) and (4.15).

On the other hand, from every solution $\chi_m(t)$ of the Heisenberg spin chain equation (4.3), one obtains the corresponding $q_m(t)$ with

$$q_m(t)=\sum_{m'=1}^{m}\{1-2|\chi_{m'}^2|\}^{\frac{1}{2}},\quad m=1,\ldots,N. \tag{4.20}$$

Using (4.1), (4.2) and (4.13), the Hamiltonian (3.7) can be expressed as

$$H=-s^2\sum_{m=1}^{N-1}J_m[\{1-(q_m-q_{m-1})^2\}^{\frac{1}{2}}\{1-(q_{m+1}-q_m)^2\}^{\frac{1}{2}}\cos(\phi_{m+1}-\phi_m)$$
$$+(1+c_m)(q_m-q_{m-1})(q_{m+1}-q_m)]-\sum_{m=1}^{N}\hat{K}_m(s(q_m-q_{m-1})). \tag{4.21}$$

From (3.13), (4.2) and (4.13) we have the Poisson brackets

$$\{s(q_m - q_{m-1}), \phi_{m'}\} = \delta_{m',m}, \quad m, m' = 1, \ldots, N, \tag{4.22}$$

implying that, cf. (4.10),

$$\begin{aligned} \{s(\phi_{m+1} - \phi_m), q_{m'}\} &= \left\{ s(\phi_{m+1} - \phi_m), \sum_{k=1}^{m'} (q_k - q_{k-1}) \right\} \\ &= \sum_{k=1}^{m} (-\delta_{k,m+1} + \delta_{k,m}) = \delta_{m',m}, \quad m, m' = 1, \ldots, N, \end{aligned} \tag{4.23}$$

where we have put $\phi_{N+1} \equiv 0$.

Introducing the variables

$$p_m = s(\phi_{m+1} - \phi_m), \quad m = 1, \ldots, N, \tag{4.24}$$

we have the fundamental Poisson brackets

$$\begin{aligned} &\{p_m, p_{m'}\} = \{q_m, q_{m'}\} = 0, \\ &\{p_m, q_{m'}\} = \delta_{m',m}, \quad m, m' = 1, \ldots, N. \end{aligned} \tag{4.25}$$

From the treatment of appendix A for the discrete case (with $P_m = p_m$, $P_{m+N} = q_m$, $m = 1, \ldots, N$) it is clear that the equation of motion (4.16) can be derived from the Lagrangian

$$\begin{aligned} L(\{q_m\}, \{\dot{q}_m\}, \{p_m\}) &= -H + \sum_{m=1}^{N} p_m \dot{q}_m \\ &= s^2 \sum_{m=1}^{N-1} J_m [\{1 - (q_m - q_{m-1})^2\}^{\frac{1}{2}} \{1 - (q_{m+1} - q_m)^2\}^{\frac{1}{2}} \\ &\quad \times \cos(p_m s^{-1}) + (1 + c_m)(q_m - q_{m-1})(q_{m+1} - q_m)] \\ &\quad + \sum_{m=1}^{N} \hat{K}_m(s(q_m - q_{m-1})) + \sum_{m=1}^{N} p_m \dot{q}_m, \end{aligned} \tag{4.26}$$

using the Lagrange equations

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{p}_m} - \frac{\partial L}{\partial p_m} = -\frac{\partial L}{\partial p_m} = 0, \tag{4.27}$$

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_m} - \frac{\partial L}{\partial q_m} = 0, \quad m = 1, \ldots, N. \tag{4.28}$$

Eq. (4.27) is equivalent to (4.14); and (4.28) with (4.24) and (4.14) leads to the equation of motion (4.16). From (4.26), (4.14) and (4.24) we obtain the expression

$$L(\{q_m\},\{\dot q_m\}) = s\sum_m [J_m^2 s^2\{1-(q_m-q_{m-1})^2\}\{1-(q_{m+1}-q_m)^2\}-\dot q_m^2]^{\frac{1}{2}}$$
$$+\sum_m [J_m s^2(1+c_m)(q_m-q_{m-1})(q_{m+1}-q_m)+\hat K_m(s(q_m-q_{m-1}))]$$
$$+ s\sum_m \dot q_m \arcsin(\dot q_m (J_m s)^{-1}\{1-(q_m-q_{m-1})^2\}^{-\frac{1}{2}}\{1-(q_{m+1}-q_m)^2\}^{-\frac{1}{2}}). \quad (4.29)$$

In view of (4.27) it is clear that the equation of motion (4.16) can also be obtained from

$$\frac{\partial}{\partial t}\frac{\partial L(\{q_m\},\{\dot q_m\})}{\partial \dot q_m} - \frac{\partial L(\{q_m\},\{\dot q_m\})}{\partial q_m} = 0, \quad m = 1,\dots,N, \quad (4.30)$$

in which the derivatives, in contrast to (4.28), are not taken at constant p_m. Therefore $L(\{q_m\},\{\dot q_m\})$ is an appropriate Lagrangian for the discrete Heisenberg spin chain in the case of axial symmetry.

Remark. A discrete $n = 1$ description can also be given in the special case that

$$\boldsymbol{J}_m = \begin{bmatrix} J_m^x & & \\ & -J_m^x & \\ & & J_m^z \end{bmatrix}, \quad K_m(\boldsymbol{S}_m) \equiv \hat K_m(S_m^z), \quad (4.31)$$

in eq. (2.1). This can be seen directly from the treatment in the case of axial symmetry using the canonical transformation $S_m^x \to S_m^x$, $S_m^y \to (-1)^m S_m^y$, $S_m^z \to (-1)^m S_m^z$.

5. Continuum $n = 2$ description

In order to give a continuum description it is necessary to formulate the problem in terms of variables which vary slowly over one lattice distance. If the spin components satisfy $|S_{m-1}^\xi - S_m^\xi| \ll 1$, then $\boldsymbol{S}_m$ can be replaced by a function $\boldsymbol{S}(x, t)$ which varies continuously as a function of the position x on the chain, and differences between neighbouring spins can be taken into account using a few terms in a Taylor-series expansion. This is all right in the case of small excitations from a ferromagnetic classical ground state, if one is interested in excitations with small wave numbers q; it is not valid when there is a substantial amount of antiferromagnetic short-range ordering. This short-range ordering can exist, e.g., in systems of weakly coupled chains well above the transition temperature T_N[48]). (The weak interchain interaction can also give rise to antiferromagnetic long-range ordering (LRO) at low temperatures $T < T_N$, but in a state with 3-dimensional LRO the domains of excited spins are not expected to move rapidly due to strong damping forces[48]).) In the Hamiltonian (2.1) only the strong intrachain interactions have been taken into account.

In order to have slowly varying fields also in the antiferromagnetic case, it is convenient to apply a slightly modified version[59]) of the Holstein–Primakoff transformation that we applied in section 3, cf. (3.1), (3.3), (3.5),

$$\begin{aligned}
S_m^+ &= \tilde{\chi}_m s\sqrt{2},\\
S_m^- &= \tilde{\chi}_m^\dagger s\sqrt{2},\\
S_m^z &= (1-\tilde{\chi}_m^\dagger\tilde{\chi}_m - \tilde{\chi}_m\tilde{\chi}_m^\dagger)^{\frac{1}{2}} s, \quad m = \text{even},\\
S_m^+ &= (\mu\tilde{\chi}_m + \nu\tilde{\chi}_m^\dagger)s\sqrt{2},\\
S_m^- &= (\mu^*\tilde{\chi}_m^\dagger + \nu^*\tilde{\chi}_m)s\sqrt{2},\\
S_m^z &= (|\mu|^2-|\nu|^2)(1-\tilde{\chi}_m^\dagger\tilde{\chi}_m - \tilde{\chi}_m\tilde{\chi}_m^\dagger)^{\frac{1}{2}} s, \quad m = \text{odd},
\end{aligned} \tag{5.1}$$

where the $\tilde{\chi}_m$ satisfy the Poisson brackets (3.2), or the commutation relations (3.6) in the quantum mechanical case.

In view of the relation $s = (\boldsymbol{S}_m \cdot \boldsymbol{S}_m)^{\frac{1}{2}}$ also for odd values of m, we have $\mu\nu = 0$ and $|\mu|^2 + |\nu|^2 = 1$. Restricting ourselves to real values of μ and ν, we have four possibilities μ, $\nu = 0$, ± 1 and for slowly varying fields, i.e. $|\tilde{\chi}_{m+1} - \tilde{\chi}_m| \ll 1$, one can easily write down approximate equalities ($\approx$) between spin components at neighbouring sites

case	*spin components*		
a) $\mu = 1, \nu = 0,$	$S_m^x \approx S_{m+1}^x,$	$S_m^y \approx S_{m+1}^y,$	$S_m^z \approx S_{m+1}^z,$
b) $\mu = -1, \nu = 0,$	$S_m^x \approx -S_{m+1}^x,$	$S_m^y \approx -S_{m+1}^y,$	$S_m^z \approx S_{m+1}^z,$
c) $\mu = 0, \nu = 1,$	$S_m^x \approx S_{m+1}^x,$	$S_m^y \approx -S_{m+1}^y,$	$S_m^z \approx -S_{m+1}^z,$
d) $\mu = 0, \nu = -1,$	$S_m^x \approx -S_{m+1}^x,$	$S_m^y \approx S_{m+1}^y,$	$S_m^z \approx -S_{m+1}^z.$

(5.2)

Eq. (5.2a) is suitable to describe small excitations from a ferromagnetic state. For $\nu = 0$ we have a ferromagnetic classical ground state with $S_m^z = s$ for all m and the excitations $\tilde{\chi}_m^\dagger$ are contained in S_m^- for all m.

Eq. (5.2b) may be of some use in a description of small excitations from the spin-flop state in antiferromagnets with preference for the z direction close to the critical field at which the antiferromagnetic ordering in the xy plane tends to zero. (An appropriate spin-wave treatment, however, should start from the classical spin-flop ground state in which neighbouring spins have nonvanishing antiparallel components in the xy plane[59–61]).) Eqs. (5.2c) and (5.2d) describe excitations from the classical antiferromagnetic ground state with $S_m^z = s$ for $m =$ even, and $S_m^z = -s$ for $m =$ odd. (Note that most spin-wave treatments, cf. ref. 62, introduce two different sets of boson operators a_k, $a_k^\dagger$ and b_k, $b_k^\dagger$ for both antiferromagnetic sublattices, but (5.1) gives an equivalent description[59]).) Usually low lying spin-wave excitations in antiferromagnets occur for wave numbers $q \approx 0$, or $q \approx \pi$, and the modes $q \approx \pi$ can be taken into account changing the sign of ν.

Therefore the low lying excitations are included in (5.2) together with the condition

$$|\tilde{\chi}_{m+1} - \tilde{\chi}_m| \ll 1, \tag{5.3}$$

which is necessary for the continuum approximation

$$\begin{aligned}&\tilde{\chi}_m(t) \to \chi(x, t),\\ &\tilde{\chi}_{m\pm 1}(t) \to \chi(x, t) \pm d\chi'(x, t) + \tfrac{1}{2}d^2\chi''(x, t) + \mathcal{O}(d^3),\end{aligned} \tag{5.4}$$

where d is the lattice distance and the primes denote differentiations with respect to x.

We now apply the transformation (5.1) together with the continuum approximation (5.4) to the Hamiltonian (3.7) in the special case that

$$\begin{aligned}&\boldsymbol{J}_m = \boldsymbol{J}, \quad \text{independent of } m,\\ &K_m(\boldsymbol{S}_m) = \begin{cases} K(\boldsymbol{S}_m) + K_s(\boldsymbol{S}_m), & m = \text{even},\\ K(\boldsymbol{S}_m) - K_s(\boldsymbol{S}_m), & m = \text{odd},\end{cases}\end{aligned} \tag{5.5}$$

corresponding to a homogeneous exchange interaction and to a staggered anisotropy field $K_s(\boldsymbol{S}_m)$ in addition to the homogeneous term $K(\boldsymbol{S}_m)$. Replacing the summation Σ_m in (3.7) by an integration $(1/d) \int \mathrm{d}x$ we have

$$H = \int \mathrm{d}x\, \mathcal{H}(\chi, \chi^\dagger), \tag{5.6}$$

with the Hamilton density

$$\begin{aligned}\mathcal{H}(\chi, \chi^\dagger) = -\tfrac{1}{2}Js^2 d[&(\mu + f\nu)(\chi^\dagger\chi'' + \chi\chi''^\dagger) + (\nu + f\mu)(\chi^\dagger\chi''^\dagger + \chi\chi'')\\ &+ (\mu^2 - \nu^2(1 + c)(1 - 2|\chi^2|)^{\frac{1}{2}}(1 - 2|\chi|^2)^{\frac{1}{2}''}]\\ &- F(\chi, \chi^\dagger)/d + \mathcal{O}(d^3),\end{aligned} \tag{5.7a}$$

or, using an integration by parts and $\chi' \to 0$ for $x \to \pm\infty$,

$$\begin{aligned}\mathcal{H}(\chi, \chi^\dagger) = \tfrac{1}{2}Js^2 d[&(\mu + f\nu)(\chi'^\dagger\chi' + \chi'\chi'^\dagger) + (\nu + f\mu)(\chi'^\dagger\chi'^\dagger + \chi'\chi')\\ &+ (\mu^2 - \nu^2)(1 + c)(1 - 2|\chi^2|)^{\frac{1}{2}'^2}] - F(\chi, \chi^\dagger)/d + \mathcal{O}(d^3),\end{aligned} \tag{5.7b}$$

where $2|\chi^2| \equiv \chi^\dagger\chi + \chi\chi^\dagger$. Furthermore,

$$F(\chi, \chi^\dagger) = Js^2[2\{\mu + f\nu - (\mu^2 - \nu^2)(1+c)\}|\chi^2| \\ + (\nu + f\mu)(\chi^\dagger\chi^\dagger + \chi\chi)] + k(\chi, \chi^\dagger), \tag{5.8}$$

with

$$k(\chi, \chi^\dagger) = \tfrac{1}{2}(K + K_s)\Big(\frac{(\chi + \chi^\dagger)s}{\sqrt{2}}, \frac{(\chi - \chi^\dagger)s}{i\sqrt{2}}, (1 - 2|\chi^2|)^{\frac{1}{2}}s\Big) \\ + \tfrac{1}{2}(K - K_s)\Big(\frac{(\mu + \nu)(\chi + \chi^\dagger)s}{\sqrt{2}}, \frac{(\mu - \nu)(\chi - \chi^\dagger)s}{i\sqrt{2}}, \\ + (\mu^2 - \nu^2)(1 - 2|\chi^2|)^{\frac{1}{2}}s\Big). \tag{5.9}$$

For example, when

$$K(\boldsymbol{S}) = \tfrac{1}{2}u_1 S^{x^2} + \tfrac{1}{2}u_2 S^{y^2} + \tfrac{1}{2}u_3 S^{z^2} + \boldsymbol{b}\cdot\boldsymbol{S}, \quad K_s(\boldsymbol{S}) = \boldsymbol{b}_s\cdot\boldsymbol{S}, \tag{5.10}$$

we have

$$k(\chi, \chi^\dagger) = \tfrac{1}{4}[(u_1 - u_2)(\chi^\dagger\chi^\dagger + \chi\chi) + 2(u_1 + u_2 - 2u_3)|\chi^2| + 2u_3]s^2 \\ + \Big[\frac{\beta^x(\chi + \chi^\dagger)}{\sqrt{2}} + \frac{\beta^y(\chi - \chi^\dagger)}{i\sqrt{2}} + \beta^z(1 - 2|\chi^2|)^{\frac{1}{2}}\Big]s, \tag{5.11}$$

with

$$\beta^x = \tfrac{1}{2}(b^x(1 + \mu + \nu) + b_s^x(1 - \mu - \nu)), \\ \beta^y = \tfrac{1}{2}(b^y(1 + \mu - \nu) + b_s^y(1 - \mu + \nu)), \\ \beta^z = \tfrac{1}{2}(b^z(1 + \mu^2 - \nu^2) + b_s^z(1 - \mu^2 + \nu^2)). \tag{5.12}$$

For the fields $\chi(x)$ we have the Poisson brackets (cf. (2.8) and also ref. 63 for the Poisson brackets between the components of the spin density)

$$\{\chi(x), \chi(y)\} = \{\chi^\dagger(x), \chi^\dagger(y)\} = 0, \\ \{\chi(x), \chi^\dagger(y)\} = i d s^{-1}(1 - 2|\chi^2|)^{\frac{1}{2}}\delta(x - y), \tag{5.13}$$

using (3.2) and the continuum limit

$$\sum_{m,m'}\{\chi_m, \chi^\dagger_{m'}\} \to (1/d^2)\int dx \int dy\{\chi(y), \chi^\dagger(x)\}. \tag{5.14}$$

The equation of motion for the field χ is given by

$$\dot{\chi}(y) = \Big\{\int dx\, \mathscr{H}(\chi, \chi^\dagger), \chi(y)\Big\}, \tag{5.15}$$

and with the property

$$\left\{\int \mathrm{d}x \ g(\chi,\chi^\dagger)\frac{\partial^2}{\partial x^2}h(\chi,\chi^\dagger),\chi(y)\right\}$$
$$= -\left\{\int \mathrm{d}x \frac{\partial g}{\partial x}(\chi,\chi^\dagger)\frac{\partial h}{\partial x}(\chi,\chi^\dagger),\chi(y)\right\}$$
$$= -\mathrm{i}ds^{-1}(1-2|\chi^2|)^{\frac{1}{2}}\left(\frac{\partial g}{\partial \chi^\dagger}\frac{\partial^2 h}{\partial y^2}+\frac{\partial h}{\partial \chi^\dagger}\frac{\partial^2 g}{\partial y^2}\right), \tag{5.16}$$

we find, either from (5.7a), or from (5.7b),

$$\dot{\chi} = \mathrm{i}Jsd^2[(1-2|\chi^2|)^{\frac{1}{2}}\{(\mu+f\nu)\chi''+(\nu+f\mu)\chi^{\dagger\prime\prime}\}$$
$$-(\mu^2-\nu^2)(1+c)\chi(1-2|\chi^2|)^{\frac{1}{2}\prime\prime}]+\mathrm{i}s^{-1}(1-2|\chi^2|)^{\frac{1}{2}}\frac{\partial F}{\partial \chi^\dagger}. \tag{5.17}$$

The same result can be derived from the equation of motion (3.8) inserting (5.1) and (5.4), cf. also (5.8). The equation of motion (5.17) can also be derived from the Lagrangian density

$$\mathscr{L}(\chi,\chi^\dagger) = -\mathscr{H}(\chi,\chi^\dagger)+\frac{s}{2\mathrm{i}d}\left(\frac{\partial}{\partial t}\ln\frac{\chi}{\chi^\dagger}\right)(1-2|\chi^2|)^{\frac{1}{2}}$$
$$= -\tfrac{1}{2}Js^2d[(\mu+f\nu)(\chi^{\dagger\prime}\chi'+\chi'\chi^{\dagger\prime})+(\nu+f\mu)(\chi^{\dagger\prime}\chi^{\dagger\prime}+\chi'\chi')$$
$$+(\mu^2-\nu^2)(1+c)(1-2|\chi^2|)^{\frac{1}{2}\prime 2}]$$
$$+F(\chi,\chi^\dagger)/d+\frac{s}{2\mathrm{i}d}\left(\frac{\partial}{\partial t}\ln\frac{\chi}{\chi^\dagger}\right)(1-2|\chi^2|)^{\frac{1}{2}}, \tag{5.18}$$

using the Lagrange equation

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{\chi}^\dagger}-\frac{\partial \mathscr{L}}{\partial \chi^\dagger}+\frac{\partial}{\partial x}\frac{\partial \mathscr{L}}{\partial \chi^{\dagger\prime}}=0. \tag{5.19}$$

Eq. (5.19) can be verified by a straightforward calculation, cf. (5.17); it also follows taking the continuum limit (5.1), (5.4) of the Lagrangian (3.11), (3.7). Furthermore it is a direct consequence of the treatment in appendix A, in the special case $D = N = 1$. This is clear, since the Lagrangian (5.18) can be expressed in the form

$$\mathscr{L}(\chi,\chi^\dagger) = -\mathscr{H}(X_1,X_2)+P_1\dot{P}_2, \tag{5.20}$$

where

$$X_1=\chi,\quad X_2=\chi^\dagger,$$
$$P_1 = s(1-2|\chi^2|)^{\frac{1}{2}},\quad P_2=\frac{1}{2\mathrm{i}d}\ln\frac{\chi}{\chi^\dagger}. \tag{5.21}$$

From (5.13) it follows that the fields P_1 and P_2 satisfy the fundamental Poisson brackets

$$\{P_1(x), P_1(y)\} = \{P_2(x), P_2(y)\} = 0,$$
$$\{P_1(x), P_2(y)\} = \delta(x-y), \tag{5.22}$$

implying that the equation of motion (5.17) is given by (5.19).

Remarks

i) Instead of (5.18) one may also take any Lagrangian density of the form (5.20) with fields $P_k(\chi, \chi', \chi'', \ldots, \chi^\dagger, \chi^{\dagger\prime}, \chi^{\dagger\prime\prime}, \ldots)$, $k = 1, 2$, satisfying the fundamental Poisson brackets (5.22). Lagrangian densities in an $n = 3$ description with constraints, i.e. in terms of a spin density $\boldsymbol{S}(x, t)$ with constant length, have been given in ref. 57.

ii) If one inserts $\chi = (1/\sqrt{2}) \sin\theta\, e^{i\phi}$ in (5.7), (5.15) and (5.17), one obtains the $n = 2$ description in terms of the polar angles θ and ϕ. Special cases of this description for ferromagnets have been treated, e.g., in refs. 17, 18 and 26. In ref. 27 a description has been given for the antiferromagnetic case leading to four coupled equations containing four angles, but only two are necessary to account for low-lying excitations.

6. Axial symmetry. Continuum $n = 1$ description for ferromagnets

In this section we consider the continuum description of the Heisenberg spin chain (5.7)–(5.9), (5.17), (5.18), in the special case of axial symmetry around the z axis, for excitations from a classical ferromagnetic ground state with $\mu = \pm 1$, $\nu = 0$. (The antiferromagnetic case will be treated in section 7.)

We assume that

$$f = 0,\; K(\boldsymbol{S}) \equiv \hat{K}(S^z) = \hat{K}((1 - 2|\chi^2|)^{\frac{1}{2}} s), \quad K_s = 0, \tag{6.1}$$

so that, cf. (5.8), (5.9),

$$F(\chi, \chi^\dagger) \equiv \hat{F}((1-2|\chi^2|)^{\frac{1}{2}} s) = 2Js^2(\mu - 1 - c)|\chi^2| + \hat{K}((1-2|\chi^2|)^{\frac{1}{2}} s). \tag{6.2}$$

We shall derive the continuum $n = 1$ description applying the continuum analogue of the treatment in section 4. The spin chain is supposed to be finite, with the x-coordinate ranging from x_L to x_R, i.e. $x \in [x_L, x_R]$. (In the thermodynamic limit, for a infinite chain, $x_L \to -\infty$, $x_R \to \infty$.)

In analogy with (4.2) and (4.6) we introduce

$$\chi = \kappa\, e^{i\phi}, \tag{6.3}$$

$$(1 - 2\kappa^2)^{\frac{1}{2}} = r'(x, t). \tag{6.4}$$

Note that we do not include a factor d in front of r', in the right-hand side of (6.4), as $r_m - r_{m-1}$ in (4.6) does not tend to 0 in the limit $d \to 0$.

From (5.21) and (5.22) we have the Poisson brackets

$$\{sr'(x), sr'(y)\} = \left\{\frac{\phi(x)}{d}, \frac{\phi(y)}{d}\right\} = 0,$$

$$\left\{sr'(x), \frac{\phi(y)}{d}\right\} = \delta(x-y), \tag{6.5}$$

for positions x and y on the chain, i.e. $x_{\mathrm{L}} \leqslant x \leqslant x_{\mathrm{R}}$, $x_{\mathrm{L}} \leqslant y \leqslant x_{\mathrm{R}}$.

Choosing a point $x_0 \uparrow x_{\mathrm{L}}$, at the left of the chain, we have

$$\left\{s\,\frac{\phi'(y)}{d}, r(x) - r(x_0)\right\} = \left\{s\,\frac{\phi'(y)}{d}, \int_{x_0}^{x} r'(x_1)\,\mathrm{d}x_1\right\}$$

$$= -\int_{x_0}^{x} \mathrm{d}x_1 \frac{\partial}{\partial y}\,\delta(x_1 - y) = \delta(x-y). \tag{6.6}$$

For the variables

$$q(x) = r(x) - r(x_0), \quad p(x) = \phi'(x) s d^{-1}, \tag{6.7}$$

we have the fundamental Poisson brackets

$$\{p(x), p(y)\} = \{q(x), q(y)\} = 0, \quad \{p(x), q(y)\} = \delta(x-y). \tag{6.8}$$

From the treatment in appendix A with $D = N = 1$ and

$$X_1(x) = P_1(x) = p(x), \quad X_2(x) = P_2(x) = q(x), \tag{6.9}$$

it is clear that the equation of motion can be derived from the Lagrangian density

$$\mathscr{L} = -\mathscr{H} + p(x)\dot{q}(x), \tag{6.10}$$

which in view of (5.7), (6.1), (6.3), (6.4) and (6.7) can be expressed as, $(\nu = 0)$,

$$\mathscr{L} = -Js^2 d\mu\{\kappa'^2 + \kappa^2\phi'^2 + \tfrac{1}{2}\mu(1+c)(1-2\kappa^2)^{\frac{1}{2}\prime 2}\} + d^{-1}\hat{F}((1-2\kappa^2)^{\frac{1}{2}}s) + p\dot{q}$$

$$= -Js^2 d\mu \left\{\tfrac{1}{2}\frac{q''^2 q'^2}{1-q'^2} + \tfrac{1}{2}\frac{d^2p^2}{s^2}(1-q'^2) + \tfrac{1}{2}\mu(1+c)q''^2\right\} + d^{-1}\hat{F}(sq') + p\dot{q}. \tag{6.11}$$

The equation of motion is determined by the Lagrange equations

$$\frac{\partial \mathscr{L}}{\partial p} = 0, \tag{6.12}$$

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{q}} + \frac{\partial}{\partial x}\frac{\partial \mathscr{L}}{\partial q'} - \frac{\partial^2}{\partial x^2}\frac{\partial \mathscr{L}}{\partial q''} = 0. \tag{6.13}$$

Eq. (6.12) gives

$$p = \frac{\dot{q}}{Jd^3\mu(1-q'^2)} = \frac{sq_\tau}{d(1-q'^2)}, \tag{6.14}$$

where we have introduced a scaled time variable

$$\tau = \mu J d^2 st, \quad q_\tau \equiv \frac{\partial q}{\partial \tau} = \frac{\dot{q}}{\mu J d^2 s}. \tag{6.15}$$

Eq. (6.13) leads to the equation of motion

$$\frac{\partial}{\partial\tau}\left(\frac{q_\tau}{1-q'^2}\right) + \frac{\partial}{\partial x}\left\{\frac{q'''}{1-q'^2} + \frac{(q''^2+q_\tau^2)}{(1-q'^2)^2}\,q' + (\mu(1+c)-1)q''' + (\mu J d^2 s^2)^{-1}\frac{\partial \hat{F}(sq')}{\partial q'}\right\} = 0. \tag{6.16}$$

In view of (6.12), eq. (6.16) is also equivalent to the Lagrange equation

$$\frac{\partial}{\partial\tau}\frac{\partial\mathscr{L}(q_\tau, q', q'')}{\partial q_\tau} + \frac{\partial}{\partial x}\frac{\partial\mathscr{L}(q_\tau, q', q'')}{\partial q'} - \frac{\partial^2}{\partial x^2}\frac{\partial\mathscr{L}(q_\tau, q', q'')}{\partial q''} = 0. \tag{6.17}$$

associated with the Lagrangian density $\mathscr{L}(q_\tau, q', q'')$ that can be obtained inserting (6.14) in (6.11), i.e.

$$\mathscr{L}(q_\tau, q', q'') = Js^2 d\mu\left\{\tfrac{1}{2}\frac{q_\tau^2 - q''^2}{1-q'^2} + \tfrac{1}{2}(1-\mu(1+c))q''^2 + \frac{\hat{F}(sq')}{\mu J d^2 s^2}\right\}. \tag{6.18}$$

The solutions of eq. (6.16) are in one to one correspondence with the solutions for the spin density $\boldsymbol{S}(x,t)$ in the (ferromagnetic) Heisenberg chain in the special case (6.1) of axial symmetry.

In fact, from an arbitrary solution $q(x,\tau)$ of the $n=1$ equation of motion (6.16), the spin density $\boldsymbol{S}(x,\tau)$ can be obtained using (5.1) with $\nu = 0$, (5.4), (6.3) and the expressions of κ and ϕ in terms of q:

$$\kappa = (1/\sqrt{2})(1-q'^2)^{\frac{1}{2}}, \tag{6.19}$$

and

$$\phi(x,\tau) = \int_C \{\mathrm{d}l_x\,\phi'(x_1,\tau_1) + \mathrm{d}l_\tau\,\phi_\tau(x_1,\tau_1)\} + \phi(0,0). \tag{6.20}$$

Here (x_1, τ_1) denotes a point on an arbitrary curve C in the $x\tau$-plane going from $(0,0)$ to (x,τ) and $(\mathrm{d}l_x, \mathrm{d}l_\tau)$ is an infinitesimal two-dimensional vector tangent to C at (x_1, τ_1).

The explicit expressions for ϕ' and ϕ_τ in terms of q are

$$\phi' = q_\tau/(1-q'^2), \tag{6.21}$$

$$\phi_\tau = \frac{-q'''}{1-q'^2} - \frac{(q''^2+q_\tau^2)}{(1-q'^2)^2}\,q' - (\mu(1+c)-1)q''' - \frac{\partial\hat{F}/\partial q'}{\mu J d^2 s^2}, \tag{6.22}$$

and $\phi(0,0)$ is an arbitrary constant.

Eq. (6.21) is obvious from (6.7) and (6.14). Eq. (6.12) is equivalent to the Lagrange equation

$$\frac{s}{d}\,\phi_\tau + \frac{\partial\mathcal{L}}{\partial z} - \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial z'} = 0, \quad z \equiv (1-2\kappa^2)^{\frac{1}{2}}, \tag{6.23}$$

which follows from the Lagrangian density $\mathcal{L} = -\mathcal{H} + (s/d)\dot{\phi}z$, as given by (5.18), in the special case (6.1), (6.2), considering ϕ and z as independent variables. Note that (6.23) is the integrated form of (6.17) with respect to x, in which the integration constant is fixed uniquely.

The result given in (6.20) is independent of the curve C, which is an immediate consequence of Stokes' theorem, together with the compatibility condition $(\phi')_\tau = (\phi_\tau)'$ which is equivalent to the equation of motion (6.16).

On the other hand, from every solution $\chi(x,\tau)$ of the Heisenberg spin-chain equation (5.17) in the case of uniaxial symmetry (6.1) with $\nu = 0$, one obtains the corresponding $q(x,\tau)$ by

$$q(x,\tau) = \int_C [\mathrm{d}l_x\{1-2\kappa^2(x_1,\tau_1)\}^{\frac{1}{2}} + \mathrm{d}l_\tau(2\kappa^2\phi')(x_1,\tau_1)] + q(0,0), \tag{6.24}$$

which is independent of the curve C, in view of Stokes' theorem and the relation $\partial(1-2\kappa^2)^{\frac{1}{2}}/\partial\tau = (2\kappa^2\phi')'$, cf. (5.17), (6.1), (6.3) and (6.15).

Remarks

i) In the special case

$$c = 0, \quad \hat{F}(sq') = bsq' + \tfrac{1}{2}us^2q'^2, \tag{6.25}$$

it is clear from the treatment in refs. 21, 22 that the fourth-order partial differential equation (6.16) is completely integrable.

ii) Eqs. (6.17) and (6.18) provide the continuum $n = 1$ description in the case of uniaxial anisotropy (6.1), for a ferromagnetic classical ground state. The same results can be obtained taking the continuum limit of the corresponding expressions (4.29) and (4.30) of the discrete $n = 1$ description. Some details of the derivation are given in appendix B.

7. Axial symmetry. Continuum $n = 1$ description for antiferromagnets

The Lagrangian density in the antiferromagnetic case will be derived taking the continuum limit of the discrete Lagrangian (4.29). In addition to (6.1) it will be assumed in this section that $\hat{K}$ is an even function of S^z, i.e.

$$K(\boldsymbol{S}) \equiv \hat{K}(S^z) = \hat{K}(-S^z). \tag{7.1}$$

The assumption (7.1) does not include a homogeneous magnetic field in the z direction. This is not a serious restriction, since a homogeneous field term $-b\,\Sigma_m S^z_m$ gives a contribution $ib\chi$ (in the right-hand side of (5.17)) which in the case of uniaxial symmetry can be eliminated using the substitution $\chi \to \chi\, e^{ibt}$.

We first consider the case $\mu = 0$, $\nu = 1$. From (5.2c) and (5.1) it is clear that

$$(-1)^m \cos\theta_m = (-1)^m(1 - 2\kappa^2_m)^{\frac{1}{2}}, \quad \text{and } \tilde{\phi}_m \equiv (-1)^m \phi_m, \tag{7.2}$$

are slowly varying functions of m. (Here it is understood that $(1 - 2\kappa^2_m)^{\frac{1}{2}}$ can also have negative values corresponding to negative values of $\cos\theta_m$.)

Using (7.2), eq. (4.5) with $J_m = J$ can be rewritten as

$$\frac{\partial}{\partial t}(-1)^m(1 - 2\kappa^2_m)^{\frac{1}{2}} = sJ[\kappa_{m-1}\kappa_m \sin(\tilde{\phi}_{m-1} + \tilde{\phi}_m) + \kappa_m\kappa_{m+1}\sin(\tilde{\phi}_m + \tilde{\phi}_{m+1})]. \tag{7.3}$$

In (7.3), the left-hand side, as well as both terms in the right-hand side are slowly varying functions of m. Eq. (7.3) can be solved formally introducing a slowly varying function $\tilde{q}_m$ such that

$$(-1)^m(1 - 2\kappa^2_m)^{\frac{1}{2}} = \tilde{q}_m + \tilde{q}_{m-1}. \tag{7.4}$$

Comparing (7.4) with (4.13), we find that

$$\tilde{q}_m = (-1)^m q_m, \tag{7.5}$$

is a slowly varying function of m.

From (4.24), (4.14) and (7.2), (7.5) we have

$$\begin{aligned} p_m s^{-1} &= -(-1)^m(\tilde{\phi}_m + \tilde{\phi}_{m+1}) \\ &= \varepsilon \arcsin\left(\frac{\dot{\tilde{q}}_m (Js)^{-1}}{\{1 - (\tilde{q}_m + \tilde{q}_{m-1})^2\}^{\frac{1}{2}}\{1 - (\tilde{q}_{m+1} + \tilde{q}_m)^2\}^{\frac{1}{2}}}\right) + \tfrac{1}{2}(\epsilon - 1)\pi + 2\pi n, \end{aligned} \tag{7.6}$$

where $\varepsilon(= \pm 1)$ and the integer n have been introduced to restrict the arcsin function to the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

Using (7.6) and the relation

$$\cos(p_m s^{-1}) = \varepsilon\{1 - \dot{\tilde{q}}^2_m (Js)^{-2}\{1 - (\tilde{q}_m + \tilde{q}_{m-1})^2\}^{-1}\{1 - (\tilde{q}_{m+1} + \tilde{q}_m)^2\}^{-1}\}^{\frac{1}{2}}, \tag{7.7}$$

the discrete Lagrangian (4.26), (4.29) can be expressed in terms of the slowly varying functions $\tilde{q}_m$, cf. also (4.17) for the definition of ζ_m. The result is, using (6.1), (7.1), and taking $J_m = J$, $c_m = c$ independent of m:

$$L(\{\tilde{q}_m\}, \{\dot{\tilde{q}}_m\}) = Js^2\varepsilon \sum_m \{\{1-(\tilde{q}_m+\tilde{q}_{m-1})^2\}\{1-(\tilde{q}_{m+1}+\tilde{q}_m)^2\} - \dot{\tilde{q}}_m^2(Js)^{-2}\}^{\frac{1}{2}}$$
$$-Js^2(1+c)\sum_m(\tilde{q}_m+\tilde{q}_{m-1})(\tilde{q}_{m+1}+\tilde{q}_m) + \sum_m \hat{K}(s(\tilde{q}_m+\tilde{q}_{m-1}))$$
$$+\varepsilon s\sum_m \dot{\tilde{q}}_m \arcsin(\dot{\tilde{q}}_m(Js)^{-1}\{1-(\tilde{q}_m+\tilde{q}_{m-1})^2\}^{-\frac{1}{2}}\{1-(\tilde{q}_{m+1}+\tilde{q}_m)^2\}^{-\frac{1}{2}}). \tag{7.8}$$

In the continuum approximation we insert

$$\tilde{q}_m + \tilde{q}_{m-1} \to q,$$
$$\tilde{q}_{m+1} + \tilde{q}_m \to q + dq' + \tfrac{1}{2}d^2q'' + \mathcal{O}(d^3), \tag{7.9}$$

and from this it follows that

$$\dot{\tilde{q}}_m \to \tfrac{1}{2}\dot{q} + \tfrac{1}{4}d\dot{q}' + \mathcal{O}(d^3) \tag{7.10}$$

In fact, when we introduce the replacement $\tilde{q}_m \to \frac{1}{2}\tilde{q}$, we have the relation $\tilde{q} - \frac{1}{2}d\tilde{q}' + \frac{1}{4}d^2\tilde{q}'' + \mathcal{O}(d^3) = q$, which can be solved by iteration to give $\tilde{q} = q + \frac{1}{2}dq' + \mathcal{O}(d^3)$.

The Lagrangian density $\mathcal{L}$ corresponding to (7.8) is obtained replacing the summation Σ_m by an integration $(1/d)\int dx$, so that $L \to \int dx\,\mathcal{L}$. Then $\mathcal{L}$ is given by

$$\mathcal{L} = Js^2\varepsilon d^{-1}((1-q^2)\{1-(q+dq'+\tfrac{1}{2}d^2q'')^2\} - (\dot{q}+\tfrac{1}{2}d\dot{q}')^2(2Js)^{-2})^{\frac{1}{2}}$$
$$-Js^2d^{-1}(1+c)q(q+dq'+\tfrac{1}{2}d^2q'') + d^{-1}\hat{K}(sq) + Js^2\varepsilon d^{-1}\left(\frac{\dot{q}+\frac{1}{2}d\dot{q}'}{2Js}\right)$$
$$\times \arcsin\left(\frac{(\dot{q}+\frac{1}{2}d\dot{q}')(2Js)^{-1}}{(1-q^2)^{\frac{1}{2}}\{1-(q+dq'+\frac{1}{2}d^2q'')^2\}^{\frac{1}{2}}}\right) + \mathcal{O}(d^2). \tag{7.11}$$

We insert the relation

$$(1-q^2)(1-\{q+dq'+\tfrac{1}{2}d^2q''\}^2) = (1-\{q+\tfrac{1}{2}dq'\}^2)^2$$
$$-d^2(1-\{q+\tfrac{1}{2}dq'\}^2)(qq''+\tfrac{1}{2}q'^2) - d^2q^2q'^2 + \mathcal{O}(d^3), \tag{7.12}$$

and a straightforward expansion in powers of d yields the Lagrangian density

$$\mathcal{L} = Js^2\varepsilon d^{-1}\left(\frac{\dot{q}+\frac{1}{2}d\dot{q}'}{2Js}\right)\arcsin\left(\frac{(\dot{q}+\frac{1}{2}d\dot{q}')(2Js)^{-1}}{\{1-(q+\frac{1}{2}dq')^2\}}\right) + d^{-1}\hat{K}(sq)$$
$$+ Js^2\varepsilon d^{-1}(\{1-(q+\tfrac{1}{2}dq')^2\}^2 - (\dot{q}+\tfrac{1}{2}d\dot{q}')^2(2Js)^{-2})^{\frac{1}{2}}$$
$$\times(1-\tfrac{1}{2}d^2(qq''-\tfrac{1}{2}q'^2)(1-q^2)^{-1} - \tfrac{1}{2}d^2q'^2(1-q^2)^{-2})$$
$$- Js^2(1+c)d^{-1}q(q+dq'+\tfrac{1}{2}d^2q'') + \mathcal{O}(d^2). \tag{7.13}$$

The Lagrange equation

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{q}}+\frac{\partial}{\partial x}\frac{\partial \mathscr{L}}{\partial q'}-\frac{\partial^2}{\partial x\,\partial t}\frac{\partial \mathscr{L}}{\partial \dot{q}'}-\frac{\partial^2}{\partial x^2}\frac{\partial \mathscr{L}}{\partial q''}=0, \tag{7.14}$$

gives the correct equation of motion in a continuum $n=1$ description in the case of axial symmetry for the antiferromagnetic classical ground state. We shall not write down the explicit result which is more complicated than the corresponding equation (6.16) in the ferromagnetic case.

Remarks

i) The result (7.13) has been derived for $\nu=1$. The result for $\nu=-1$ can be obtained replacing ε by $\varepsilon\nu$, as can be seen by applying a rotation over π around the z axis at the odd lattice sites leading to the transformation

$$J\to -J,\quad J(1+c)\to J(1+c),\quad \hat{K}\to\hat{K}, \tag{7.15}$$

in the Hamiltonian (2.1) and the replacement $\varepsilon\to\varepsilon\nu$ in (7.13).

ii) An important simplification can be obtained, if $\dot{\tilde{q}}_m$ is small. Assuming that $\dot{\tilde{q}}_m$ is of order d, it can be of use to introduce a scaled time τ such that

$$\tau=2Jdst,\quad 2\dot{\tilde{q}}_m\to\dot{q}+\tfrac{1}{2}d\dot{q}'=2Jdsq_\tau+\mathscr{O}(d^2). \tag{7.16}$$

Eq. (7.16) is similar to eq. (B.3) derived in appendix B in the case of a ferromagnetic ground state. There is, however, an important difference. In the ferromagnetic case eq. (4.12) ensures that $\dot{q}_m$ is of order d in the continuum limit; in the antiferromagnetic case the time derivative of the slowly varying $\tilde{q}_m$ is in general of order 1. The assumption that $\dot{\tilde{q}}_m$ is of order d involves a restriction to solutions with small values of $\sin 2\tilde{\phi}_m$.
For solutions with $\tilde{\phi}_m\sim\mathscr{O}(d)$, use can be made of the relation

$$\begin{aligned}&(\{1-(q+\tfrac{1}{2}dq')^2\}^2-(\dot{q}+\tfrac{1}{2}d\dot{q}')^2(2Js)^{-2})^{\frac{1}{2}}\\&\quad=(1-q^2)-dqq'-\tfrac{1}{4}d^2q'^2-\tfrac{1}{2}d^2\frac{q_\tau^2}{1-q^2}+\mathscr{O}(d^3).\end{aligned} \tag{7.17}$$

Inserting (7.17) in (7.13) we obtain the Lagrangian density

$$\begin{aligned}\mathscr{L}=&\,Js^2\varepsilon d\tfrac{1}{2}\frac{q_\tau^2-q'^2}{1-q^2}+d^{-1}\hat{K}(sq)\\&-Js^2\varepsilon d^{-1}(1+\varepsilon(1+c))(q^2+dqq')-\tfrac{1}{2}Js^2\varepsilon d(1+\varepsilon(1+c))qq''+\mathscr{O}(d^2).\end{aligned} \tag{7.18}$$

Omitting a total differential which does not contribute to the equation of motion, $\mathscr{L}$ can be expressed as

$$\mathscr{L} = Js^2\varepsilon d\left[\frac{1}{2}\frac{q_\tau^2 - q'^2}{1-q^2} + \tfrac{1}{2}(1+\varepsilon(1+c))q'^2 + \frac{(1+\varepsilon(1+c))}{d^2}(1-q^2) + \frac{\hat{K}(sq)}{Js^2\varepsilon d^2}\right]. \tag{7.19}$$

Eq. (7.19) is identical to (6.18), cf. also (B.7), derived in the ferromagnetic case, apart from the replacements $\mu J \to \varepsilon J$, $\mu(1+c)J \to -\varepsilon(1+c)J$, $q \to q'$, $q' \to q''$. Yet these replacements cause (6.18) and (7.19) to lead to quite different equations of motion. Eq. (6.18) is in special cases equivalent to the nonlinear Schrödinger equation as will be discussed in section 8. Eq. (7.19), on the other hand, is related to generalizations of the sine-Gordon equation. This will be discussed in section 9, where it will also be shown that the restriction to solutions with small values of ϕ will also lead to sine-Gordon type equations in the absence of axial symmetry, cf. also ref. 27 where some special cases have been treated.

8. Relation with the nonlinear Schrödinger equation

Consider a special case of axial symmetry for a ferromagnetic classical ground state, i.e.

$$\nu = 0, \quad f = 0, \quad K = bS^z + \tfrac{1}{2}uS^{z2}, \quad K_s = 0. \tag{8.1}$$

Using (5.10), (8.1) and (6.15), and taking $\tau = t$ for convenience, eq. (5.17) can be expressed as

$$-\mathrm{i}\dot{\chi} = \chi''(1-2|\chi^2|)^{\frac{1}{2}} - (1+c)\mu\chi(1-2|\chi^2|)^{\frac{1}{2}''} - \beta\chi - \alpha\chi(1-2|\chi^2|)^{\frac{1}{2}}, \tag{8.2}$$

with

$$\alpha = d^{-2}\{2\mu(1+c) - 2 + u(J\mu)^{-1}\}, \quad \beta = b(\mu J d^2 s)^{-1}. \tag{8.3}$$

Neglecting in the right-hand side of (8.2) 5th and higher order terms in χ, as well as 3rd order terms containing derivatives χ' or χ'', we have

$$-\mathrm{i}\dot{\chi} = \chi'' - (\alpha+\beta)\chi + \alpha\chi^\dagger\chi\chi. \tag{8.4}$$

Substituting

$$\psi = \mathrm{e}^{-\mathrm{i}(\alpha+\beta)t}\chi, \tag{8.5}$$

we have

$$-\mathrm{i}\dot{\psi} = \psi'' + \alpha\psi^\dagger\psi\psi, \tag{8.6}$$

which is the nonlinear Schrödinger equation (NLS). The relation between the

Heisenberg chain with uniaxial anisotropy and the NLS was noted previously in ref. 22. It suggests that the NLS, like eq. (5.17) in the case of axial symmetry, is in one to one correspondence with an ($n = 1$) equation with one real variable $y(x, t)$.

This is indeed the case*. In fact, using the substitution

$$\psi = |\psi|\, e^{i\theta}, \tag{8.7}$$

eq. (8.6) can be expressed as

$$-i\frac{\partial}{\partial t}|\psi| + |\psi|\dot{\theta} = |\psi|'' + 2i|\psi|'\theta' + i|\psi|\theta'' - |\psi|\theta'^2 + \alpha|\psi|^3. \tag{8.8}$$

Using the decomposition into a real and an imaginary part, we have

$$\frac{\partial}{\partial t}\tfrac{1}{2}|\psi|^2 = -\frac{\partial}{\partial x}(|\psi|^2\theta'), \tag{8.9}$$

$$\dot{\theta} = \frac{|\psi|''}{|\psi|} - \theta'^2 + \alpha|\psi|^2. \tag{8.10}$$

Eq. (8.9) can be solved formally, introducing $\eta(x, t)$ through

$$|\psi|^2 = 2\eta'. \tag{8.11}$$

From (8.9) we have

$$\dot{\eta} = -2\eta'\theta' + g(t). \tag{8.12}$$

Introducing the variable

$$y = \eta - \int_0^t dt_1 g(t_1), \tag{8.13}$$

we have

$$y' = \tfrac{1}{2}|\psi|^2, \tag{8.14}$$

$$\dot{y} = \dot{\eta} - g(t) = -2\eta'\theta' = -2y'\theta'. \tag{8.15}$$

Inserting (8.14), (8.15) into the derivative of (8.10) with respect to x, we obtain

$$\frac{\partial}{\partial t}\left(\frac{-\dot{y}}{2y'}\right) = \frac{\partial}{\partial x}\left(\frac{y'''}{2y'} - \frac{(y''^2 + \dot{y}^2)}{4y'^2} + 2\alpha y'\right) = 0, \tag{8.16}$$

which is the equivalent real form of the NLS[64]).

* Very recently this equation has been derived (independently) by Boiti, Laddomada and Pempinelli using the same type of reasoning[64]).

From an arbitrary solution of (8.16) we obtain a solution of the NLS, viz.

$$\psi(x,t) = [2y'(x,t)]^{\frac{1}{2}} \exp\left[\mathrm{i} \int_C \{ \mathrm{d}l_x\, \theta'(x_1,t_1) + \mathrm{d}l_t\, \dot{\theta}(x_1,t_1) \} + \mathrm{i}\theta(0,0) \right], \tag{8.17}$$

where (x_1, t_1) is a point on an arbitrary curve C in the xt-plane connecting $(0,0)$ with (x,t), and $(\mathrm{d}l_x, \mathrm{d}l_t)$ is an infinitesimal two-dimensional vector tangent to C at (x_1, t_1). The explicit expressions for θ' and $\dot{\theta}$ in terms of y are given by, cf. (8.14) and (8.10),

$$\theta' = \frac{-\dot{y}}{2y'}, \tag{8.18}$$

$$\dot{\theta} = \frac{y'''}{2y'} - \frac{(y''^2 + \dot{y}^2)}{4y'^2} + 2\alpha y', \tag{8.19}$$

and $\theta(0,0)$ is an arbitrary constant. The integral in (8.17) is independent of the choice of the curve C, as follows from Stokes' theorem and the compatibility relation $\partial\theta'/\partial t = \partial\dot{\theta}/\partial x$ which is equivalent to (8.16).

On the other hand, from every solution of the NLS the corresponding solution $y(x,t)$ of (8.16) can be obtained using

$$y(x,t) = \int_C \{ \mathrm{d}l_x \tfrac{1}{2} |\psi|^2(x_1,t_1) - \mathrm{d}l_t (|\psi|^2\theta')(x_1,t_1) \} + y(0,0), \tag{8.20}$$

so that there is a one to one correspondence between the solutions of (8.16) and (8.6).

Eq. (8.16) can also be derived from the Lagrangian density

$$\mathscr{L} = -\frac{(\dot{y}^2 - y''^2)}{4y'} - \alpha y'^2, \tag{8.21}$$

using the Lagrange equation

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{y}} + \frac{\partial}{\partial x}\frac{\partial \mathscr{L}}{\partial y'} - \frac{\partial^2}{\partial x^2}\frac{\partial \mathscr{L}}{\partial y''} = 0. \tag{8.22}$$

The Lagrangian density (8.21) can also be derived from the Lagrangian density in terms of ψ for the NLS[65]), analogously to the derivation in section 6 for the Heisenberg spin chain.

A direct connection between the $n = 1$ equations (6.16) and (8.16) may be obtained as follows. Substituting:

$$q = x - w, \quad q' = 1 - w', \quad q_\tau = -\dot{w}, \quad 1 - q'^2 = 2w' - w'^2, \tag{8.23}$$

and expanding

$$(\mu J d^2 s^2)^{-1} \frac{\partial F}{\partial w'} = \sum_l f_l w'^l, \tag{8.24}$$

eq. (6.16), with $\tau = t$, can be expressed as

$$\frac{\partial}{\partial t}\left(\frac{-\dot{w}}{2w' - w'^2}\right) = \frac{\partial}{\partial x}\left\{\frac{1}{2w''}\frac{\partial}{\partial x}\left(\frac{w''^2 + \dot{w}^2}{2w' - w'^2}\right) - \frac{\dot{w}\dot{w}'}{(2w' - w'^2)w''} + (\mu(1+c) - 1)w''' + \sum_l f_l w'^l\right\}. \tag{8.25}$$

Using the scale transformation

$$w \to w\lambda^3, \quad \frac{\partial}{\partial t} \to \lambda^{-4}\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x} \to \lambda^{-2}\frac{\partial}{\partial x}, \tag{8.26}$$

and multiplying both sides by λ^6, eq. (8.25) changes into

$$\frac{\partial}{\partial t}\left(\frac{-\dot{w}}{2w' - \lambda w'^2}\right) = \frac{\partial}{\partial x}\left\{\frac{1}{2w''}\frac{\partial}{\partial x}\left(\frac{w''^2 + \dot{w}^2}{2w' - \lambda w'^2}\right) - \frac{\dot{w}\dot{w}'}{(2w' - \lambda w'^2)w''} + (\mu(1+c) - 1)\lambda w''' + \sum_l f_l \lambda^{l+4} w'^l\right\}. \tag{8.27}$$

In the special case (8.1) we have, cf. (5.8), (5.9), (8.3), 6.2), (6.3) and (6.7) (omitting a constant),

$$F = \tfrac{1}{2}\mu J d^2 s^2 \{\alpha(1 - w')^2 + 2\beta(1 - w')\}, \tag{8.28}$$

in which the term with β does not contribute to the equation of motion. If we take the limit $\lambda w'^2 \to 0$, and assume $\alpha\lambda^5 w'$ to be finite, eq. (8.27) reduces to

$$\frac{\partial}{\partial t}\left(\frac{-\dot{w}}{2w'}\right) = \frac{\partial}{\partial x}\left(\frac{1}{2w''}\frac{\partial}{\partial x}\left(\frac{w''^2 + \dot{w}^2}{2w'}\right) - \frac{\dot{w}\dot{w}'}{2w'w''} + 2\tilde{\alpha}w'\right), \quad \tilde{\alpha} \equiv \tfrac{1}{2}\alpha\lambda^5, \tag{8.29}$$

which is equivalent to the NLS (8.16). The equivalence holds in the limit of large α, i.e. large anisotropy, and for small values of w', i.e. small κ or $S^z \sim s$, cf. ref. 22.

Remarks

i) From the considerations above one may infer that the term $-\{\mu(1+c) - 1\}q'''$ in the equation of motion (6.16) is not really important. This can also be seen directly from the Lagrangian density (6.18). In fact using (6.2), (8.1), (8.3) and the substitution

$$q(x, \tau) \to q(x, \tau) - \tfrac{1}{4}\left[\frac{\mu(1+c) - 1}{\mu(1+c) - 1 + u/(2J\mu)}\right] d^2 q''(x, \tau), \tag{8.30}$$

we find, neglecting terms of order d^3 and higher,

$$\mathscr{L} = Js^2 d\mu \left[\tfrac{1}{2} \frac{q_\tau^2 - q''^2}{1 - q'^2} + \tfrac{1}{2} \alpha q'^2 \right], \tag{8.31}$$

apart from a total differential.

ii) For the isotropic Heisenberg spin chain, i.e.

$$f = c = 0, \quad \mu = 1, \quad K = K_s = F = 0, \tag{8.32}$$

there is another relation[23,24]) with the NLS. Lakshmanan[23]) has shown, that if $\boldsymbol{S}(x, t)$ is a solution of the equation of motion

$$s\dot{\boldsymbol{S}} = \boldsymbol{S} \times \boldsymbol{S}'', \tag{8.33}$$

of the isotropic Heisenberg spin chain with $t = \tau$, then

$$\psi = \frac{(\boldsymbol{S}' \cdot \boldsymbol{S}')^{\frac{1}{2}}}{s} \exp \mathrm{i} \int^x \mathrm{d}x_1 \left(- \frac{\boldsymbol{S}' \cdot \dot{\boldsymbol{S}}}{\boldsymbol{S}' \cdot \boldsymbol{S}'} \right), \tag{8.34}$$

is a solution of the coupled set of equations formed by (8.9) and the spatial derivative of (8.10) with $\alpha = \frac{1}{2}$. A similar relation exists between (6.16) with $\mu = 1$, $c = 0$, $F = 0$ for the Heisenberg spin chain and eq. (8.16) giving the $n = 1$ description of the NLS.

In fact, let $q(x, t)$ be an arbitrary solution of (6.16) with $\mu = 1$, $c = F = 0$ and $\tau = t$, then the function

$$y(x, t) = \int_C \{ \mathrm{d}l_x\, y'(x_1, t_1) + \mathrm{d}l_t\, \dot{y}(x_1, t_1) \} + y(0, 0), \tag{8.35}$$

with

$$y' = \frac{1}{4\alpha} \frac{q''^2 + \dot{q}^2}{1 - q'^2}, \tag{8.36}$$

$$\dot{y} = \frac{1}{2\alpha} \left\{ \frac{q''\dot{q}' - q'''\dot{q}}{1 - q'^2} - \frac{\dot{q}q'(q''^2 + \dot{q}^2)}{(1 - q'^2)^2} \right\}, \tag{8.37}$$

and where (x_1, t_1) denotes a point on an arbitrary curve C in the xt-plane going from $(0, 0)$ to (x, t) and $(\mathrm{d}l_x, \mathrm{d}l_t)$ is an infinitesimal two-dimensional vector tangent to C at (x_1, t_1), is a solution of eq. (8.16), for arbitrary $\alpha (\neq 0)$.

Note that $y(x, t)$ as defined by (8.35)–(8.37) is independent of the curve C, as follows from Stokes' theorem together with the compatibility relation $\partial y'/\partial t - \partial \dot{y}/\partial x = 0$, or

$$\frac{\partial}{\partial t} \tfrac{1}{2} \frac{q''^2 + \dot{q}^2}{1 - q'^2} - \frac{\partial}{\partial x} \left(\frac{q''\dot{q}' - q'''\dot{q}}{1 - q'^2} - \frac{\dot{q}q'(q''^2 + \dot{q}^2)}{(1 - q'^2)^2} \right) = 0, \tag{8.38}$$

which can be rewritten as

$$\dot{q}\left[\frac{\partial}{\partial t}\frac{\dot{q}}{1-q'^2}+\frac{\partial}{\partial x}\left(\frac{q'''}{1-q'^2}+\frac{q'(q''^2+\dot{q}^2)}{(1-q'^2)^2}\right)\right]=0. \tag{8.39}$$

Eq. (8.39) is satisfied if $q(x, t)$ is a solution of (6.16) with $\mu = 1$, $c = F = 0$ and $\tau = t$. The proof that $y(x, t)$, as defined by (8.35)–(8.37) is indeed a solution of (8.16) is more cumbersome and is presented in appendix C. The corresponding solution $\psi(x, t)$ of the NLS (8.6), or equivalently (8.9), (8.10), can be obtained substituting (8.35) into (8.17)–(8.19).

9. Relation with the sine-Gordon equation

In section 7 we derived the Lagrangian density (7.13) for the Heisenberg chain with uniaxial anisotropy (6.1), (7.1) in a continuum ($n = 1$) description with one real field $q(x, t)$ for excitations from a classical antiferromagnetic ground state. Eq. (7.13) and the corresponding equation of motion, which we have not written down explicitly, are rather complicated. Substantial simplifications occur under the assumption that the spin density $\boldsymbol{S}(x, t)$ is nearly restricted to the xz-plane, or equivalently that the polar angle $\phi(x, t)$ is small. Under this assumption we have derived the simplified Lagrangian density (7.19) which may be considered to be an extension of the Lagrangian density for the sine-Gordon equation. However, starting from the assumption that ϕ is small, see also ref. 52, one can derive a Lagrangian density of the sine-Gordon type under much more general conditions, also when there are large deviations from axial symmetry.

To see this, we start from the general Hamiltonian density (5.7b), (5.8), which under the substitution $\chi = \kappa\, \mathrm{e}^{\mathrm{i}\phi}$, cf. (6.3), can be rewritten in the form

$$\begin{aligned}\mathcal{H} = {} & Js^2d[(1+f)(\nu+\mu)\kappa'^2+(\nu+f\mu)\kappa'^2(-1+\cos 2\phi)\\ & +\kappa^2\phi'^2\{\mu+f\nu-(\nu+f\mu)\cos 2\phi\}\\ & -2\kappa\kappa'\phi'(\nu+f\mu)\sin 2\phi+\tfrac{1}{2}(\mu^2-\nu^2)(1+c)(1-2\kappa^2)^{\frac{1}{2}\prime 2}]\\ & -2Js^2d^{-1}[\{\mu+f\nu-(\mu^2-\nu^2)(1+c)\}\kappa^2+(\nu+f\mu)\kappa^2\cos 2\phi]\\ & -d^{-1}k(\chi,\chi^\dagger),\end{aligned} \tag{9.1}$$

where $k(\chi, \chi^\dagger)$ in the special case (5.10) is given by, cf. (5.11) and (6.3),

$$\begin{aligned}k(\chi,\chi^\dagger) = {} & \tfrac{1}{2}s^2(u_1+u_2-2u_3)\kappa^2+\tfrac{1}{2}s^2(u_1-u_2)\kappa^2\cos 2\phi\\ & +s\sqrt{2}(\beta^x\kappa\cos\phi+\beta^y\kappa\sin\phi)+s\beta^z(1-2\kappa^2)^{\frac{1}{2}}+\tfrac{1}{2}u_3s^2.\end{aligned} \tag{9.2}$$

From the Poisson bracket

$$\{p(x), -q(y)\} = \delta(x-y), \tag{9.3}$$

for the quantities

$$p(x) = d^{-1}s\phi(x), \quad q(y) = (1-2\kappa^2)^{\frac{1}{2}}(y), \tag{9.4}$$

cf. (5.22), we obtain upon applying the treatment of appendix A with $N = D = 1$, the Lagrangian density

$$\begin{aligned}\mathscr{L} &= -\mathscr{H} - p\dot{q}\\ &= -\tfrac{1}{2}Js^2d\left[\frac{q^2q'^2}{1-q^2}(1+f)(\nu+\mu) + (\mu^2-\nu^2)(1+c)q'^2 + g(p,p',q,q')\right]\\ &\quad + d^{-1}\tilde{F}(q,p) - p\dot{q}.\end{aligned} \tag{9.5}$$

Here

$$\begin{aligned}g(p,p',q,q') = {}& d^2s^{-2}p'^2(1-q^2)\{\mu + f\nu - (\nu+f\mu)\cos(2ds^{-1}p)\}\\ &+ 2ds^{-1}p'qq'(\nu+f\mu)\sin(2ds^{-1}p)\\ &- q^2q'^2(1-q^2)^{-1}(\nu+f\mu)(1-\cos(2ds^{-1}p)),\end{aligned} \tag{9.6}$$

and using (9.1), (9.2), cf. (5.8),

$$\begin{aligned}\tilde{F}(q,p)(\equiv F(\chi,\chi^\dagger)) = {}& Js^2(1-q^2)(\alpha_1 + \alpha_2\cos(2ds^{-1}p))\\ &+ s(1-q^2)^{\frac{1}{2}}\{\beta^x\cos(ds^{-1}p) + \beta^y\sin(ds^{-1}p)\} + s\beta^z q + \tfrac{1}{2}u_3s^2,\end{aligned} \tag{9.7}$$

with

$$\begin{aligned}\alpha_1 &= (\mu + f\nu) - (\mu^2-\nu^2)(1+c) + \tfrac{1}{4}J^{-1}(u_1+u_2-2u_3),\\ \alpha_2 &= (\nu + f\mu) + \tfrac{1}{4}J^{-1}(u_1-u_2).\end{aligned} \tag{9.8}$$

The Lagrange equation for p, i.e.

$$\frac{\partial\mathscr{L}}{\partial p} - \frac{\partial}{\partial x}\frac{\partial\mathscr{L}}{\partial p'} = 0, \tag{9.9}$$

can be expressed as

$$-\dot{q} + d^{-1}\frac{\partial}{\partial p}\tilde{F}(q,p) - \tfrac{1}{2}Js^2d\left(\frac{\partial g}{\partial p} - \frac{\partial}{\partial x}\frac{\partial g}{\partial p'}\right) = 0. \tag{9.10}$$

We now investigate solutions with ϕ small, i.e. $\phi = \mathcal{O}(d)$, or equivalently $p = \mathcal{O}(1)$. Then we can expand

$$\tilde{F}(q,p) = \tilde{F}_0(q) - \tilde{F}_1(q)pd - \tfrac{1}{2}\tilde{F}_2(q)p^2d^2 + \mathcal{O}(d^3), \tag{9.11}$$

where

$$\begin{aligned}\tilde{F}_0(q) &= Js^2(\alpha_1+\alpha_2)(1-q^2) + s\beta^x(1-q^2)^{\frac{1}{2}} + s\beta^z q + \tfrac{1}{2}u_3s^2,\\ \tilde{F}_1(q) &= -\beta^y(1-q^2)^{\frac{1}{2}},\\ \tilde{F}_2(q) &= 4J\alpha_2(1-q^2) + s^{-1}\beta^x(1-q^2)^{\frac{1}{2}}.\end{aligned} \tag{9.12}$$

For large $\tilde{F}_2$ we indeed have solutions with small pd, and from (9.10) and (9.11) we have

$$pd = -(\dot{q} + \tilde{F}_1)\tilde{F}_2^{-1} + \mathcal{O}(d^3). \tag{9.13}$$

Inserting (9.12) in (9.5) we obtain the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}Js^2 d\left[-(1+f)(\nu+\mu)\frac{q'^2}{1-q^2} + \{(1+f)(\nu+\mu) - (\mu^2-\nu^2)(1+c)\}q'^2\right] + d^{-1}[\tilde{F}_0(q) + (\dot{q}+\tilde{F}_1)^2(2\tilde{F}_2)^{-1}]. \tag{9.14}$$

Introducing a scaled time $\tau = 2Jdst$, cf. (7.16), and assuming the magnetic field to be small, i.e. $\beta^x \sim \mathcal{O}(d^{\eta_1})$, $\beta^y \sim \mathcal{O}(d^{1+\eta_2})$, $\beta^z \sim \mathcal{O}(d^{\eta_3})$, with η_1, η_2, $\eta_3 > 0$, eq. (9.14) can be simplified to

$$\mathcal{L} = Js^2 d\left[\tfrac{1}{2}\frac{\alpha_2^{-1}q_\tau^2 - (1+f)(\nu+\mu)q'^2}{1-q^2} + \tfrac{1}{2}\{(1+f)(\nu+\mu) - (\mu^2-\nu^2)(1+c)\}q'^2 + \frac{\tilde{F}_0(q)}{Js^2d^2}\right], \tag{9.15}$$

apart from terms of higher order in d. Eq. (9.15) is a straightforward generalization of the Lagrangian density (7.19) with $\hat{K}(sq) = \frac{1}{2}s^2(u_1 + (u_3 - u_1)q^2)$. In fact, (7.19) can be obtained from (9.15) inserting $f = \mu = 0$, $u_1 = u_2$, $\nu = \varepsilon$, $\beta^x = \beta^z = 0$.

The term with q'^2 can again be eliminated using the substitution

$$q(x,\tau) \to q(x,\tau) + \tfrac{1}{4}\frac{\{(\mu^2-\nu^2)(1+c) - (1+f)(\nu+\mu)\}}{\alpha_1+\alpha_2}d^2q'', \tag{9.16}$$

and with the substitution $q = \cos\theta$, we obtain the Lagrangian density

$$\mathcal{L} = Js^2 d\left[\tfrac{1}{2}\frac{\theta_\tau^2}{\alpha_2} - \tfrac{1}{2}(1+f)(\nu+\mu)\theta'^2 - \tfrac{1}{2}\frac{(\alpha_1+\alpha_2)}{d^2}\cos 2\theta + \frac{\beta^x}{Jsd^2}\sin\theta + \frac{\beta^z}{Jsd^2}\cos\theta\right], \tag{9.17}$$

omitting a constant which does not contribute to the equation of motion and terms of higher order $\mathcal{O}(d^{1+\eta})$, with $\eta > 0$.

Choosing the magnetic field in the z or x direction, i.e. $\beta^x = 0$ or $\beta^z = 0$, we obtain from (9.17) the Lagrangian for the double sine-Gordon equation. For special cases, i.e. a vanishing magnetic field ($\beta^x = \beta^z = 0$) and $\alpha_1 + \alpha_2 \neq 0$, cf. ref. 22, or $\alpha_1 + \alpha_2 = 0$, $(1+f)(\nu+\mu) - (\mu^2-\nu^2)(1+c) = 0$, and $\beta^z = 0$, $\beta^x \neq 0$, cf. refs. 26, 27, eq. (9.17) reduces to the Lagrangian density for the sine-Gordon equation. Note that the double sine-Gordon equation is not completely integrable[66]), in contrast to the sine-Gordon equation.

Lagrangians of the type (9.17) may be suitable in cases with large α_2, and therefore in general for systems with a substantial antiferromagnetic short-range ordering ($\nu = 1$). For excitations from a ferromagnetic classical ground state ($\nu = 0$) we can only obtain (9.17) in the presence of a large orthorhombic anisotropy. Furthermore, for θ values close to zero, or $q \sim 1$, some difficulties may be expected, if $\dot{q}$ is not small compared to $\tilde{F}_2(q)$, cf. ref. 52, For ferromagnetic systems with small deviations from axial symmetry, eq. (6.16) which can be related to the NLS if χ is small ($q' \sim 1$), will give a better description.

10. Concluding remark

In this paper we have investigated the equation of motion for the classical anisotropic Heisenberg spin chain, both in the discrete case, as well as in the continuum limit. In the case of axial symmetry the equation of motion in the discrete case has been shown to be equivalent to an equation in terms of one real variable $q_m(t)$ associated with every lattice site m. A Lagrangian from which this new equation may be derived has also been obtained.

In the continuum limit we have derived the equation of motion and the Lagrangian density in terms of one real field $q(x, t)$ which is a function of the time t and the position x on the chain. The results obtained in the continuum limit depend on the type of (short-range) ordering which is present in the chain. The equation in the ferromagnetic case with quadratic anisotropy is completely integrable, and the relations with an equivalent real form of the nonlinear Schrödinger equation[64]) have been discussed in some detail. The equation in the antiferromagnetic case is much more complicated. Simplifications can be derived assuming that the spin density vector $\boldsymbol{S}(x, t)$ is nearly restricted to a plane. Under this assumption one can derive various extensions of the sine-Gordon equation, also when there are (large) deviations from axial symmetry.

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Appendix A

In this appendix we discuss a way of obtaining Lagrangian densities associated with a fairly general class of Hamiltonian systems.

Suppose we have a classical Hamiltonian

$$H = \int \mathrm{d}\boldsymbol{r}\, \mathcal{H}(\{X_n\}, \{\partial_k X_n\}, \{\partial_k \partial_l X_n\}, \ldots), \tag{A.1}$$

where $\boldsymbol{r} = (x_1, \ldots, x_D)$ is a D-dimensional vector, $\{X_n(\boldsymbol{r}, t)\}$ is a set of $2N$ (complex) fields depending on $\boldsymbol{r}$ as well as on t and the ∂_k for $k = 1, \ldots, D$ denote the differentiations with respect to x_k.

We also suppose that the Poisson brackets for the X_n are arbitrary given functions, i.e.

$$\{X_n(\boldsymbol{r}), X_{n'}(\boldsymbol{r}')\} \equiv a_{nn'}(\boldsymbol{r}, \boldsymbol{r}'), \quad n, n' = 1, \ldots, 2N, \tag{A.2}$$

and for two arbitrary functions $A \equiv A(\{X_n\})$, $B = B(\{X_n\})$ we have the Poisson bracket

$$\{A, B\} = \sum_{n,n'} \left(\frac{\partial A}{\partial X_n} \frac{\partial B}{\partial X_{n'}} - \frac{\partial A}{\partial X_{n'}} \frac{\partial B}{\partial X_n} \right) a_{nn'}(\boldsymbol{r}, \boldsymbol{r}'). \tag{A.3}$$

We also assume that the equation of motion for the X_n is given by the Poisson bracket

$$\dot{X}_n = \{H, X_n\}, \quad n = 1, \ldots, 2N. \tag{A.4}$$

In this appendix we will prove that the equation of motion is the Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial X_n} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{X}_n} - \sum_k \partial_k \frac{\partial \mathcal{L}}{\partial \partial_k X_n} + \sum_{k \leq l} \partial_k \partial_l \frac{\partial \mathcal{L}}{\partial \partial_k \partial_l X_n} + \cdots, \quad n = 1, \ldots, 2N, \tag{A.5}$$

for any Lagrangian density of the type

$$\mathcal{L}(\{X_n\}, \{\partial_k X_n\}, \{\partial_k \partial_l X_n\}, \ldots) = -\mathcal{H} + \sum_{m=1}^{N} P_m \dot{P}_{m+N}, \tag{A.6}$$

in which the

$$P_n = P_n(\{X_{n'}\}, \{\partial_k X_{n'}\}, \{\partial_k \partial_l X_{n'}\}, \ldots), \tag{A.7}$$

satisfy the fundamental Poisson brackets

$$\{P_m(\boldsymbol{r}), P_{m'}(\boldsymbol{r}')\} = \{P_{m+N}(\boldsymbol{r}), P_{m'+N}(\boldsymbol{r}')\} = 0,$$
$$\{P_m(\boldsymbol{r}), P_{m'+N}(\boldsymbol{r}')\} = \delta_{m',m} \delta_D(\boldsymbol{r} - \boldsymbol{r}'), \quad m, m' = 1, \ldots, N, \tag{A.8}$$

$\delta_D(\boldsymbol{r} - \boldsymbol{r}')$ being the delta function in D dimensions, assuming also that (A.7) can be inverted to give

$$X_{n'} = X_{n'}(\{P_n\}, \{\partial_k P_n\}, \{\partial_k \partial_l P_n\}, \ldots). \tag{A.9}$$

Proof. Starting from the equation of motion (A.4), we have using (A.7)

$$\dot{P}_n = \{H, P_n\}. \tag{A.10}$$

Expressing the Hamiltonian (A.1) in terms of the P_n, i.e.

$$H = \int \mathrm{d}\boldsymbol{r}\, \tilde{\mathscr{H}}(\{P_n\}, \{\partial_k P_n\}, \{\partial_k \partial_l P_n\}, \ldots), \tag{A.11}$$

cf. (A.9), and using (A.10) and (A.8), it follows after integration by parts that

$$\dot{P}_n + \frac{\partial \tilde{\mathscr{H}}}{\partial P_{n+N}} - \sum_k \partial_k \frac{\partial \tilde{\mathscr{H}}}{\partial \partial_k P_{n+N}} + \sum_{k \leq l} \partial_k \partial_l \frac{\partial \tilde{H}}{\partial \partial_k \partial_l P_{n+N}} + \cdots = 0, \tag{A.12}$$

for $n = 1, \ldots, 2N$ with the notation $P_{m+2N} \equiv -P_m$ for $m = 1, \ldots, N$. It is straightforward to show that (A.12) can be obtained as the Lagrange equation

$$\frac{\partial \tilde{\mathscr{L}}}{\partial P_n} - \frac{\partial}{\partial t} \frac{\partial \tilde{\mathscr{L}}}{\partial \dot{P}_n} - \sum_k \partial_k \frac{\partial \tilde{\mathscr{L}}}{\partial \partial_k P_n} + \sum_{k \leq l} \partial_k \partial_l \frac{\partial \tilde{\mathscr{L}}}{\partial \partial_k \partial_l P_n} + \cdots = 0, \; n = 1, \ldots, 2N, \tag{A.13}$$

for the Lagrangian density

$$\tilde{\mathscr{L}}(\{P_n\}, \{\partial_k P_n\}, \{\partial_k \partial_l P_n\}, \ldots) = -\tilde{\mathscr{H}} + \sum_{m=1}^{N} P_m \dot{P}_{m+N}. \tag{A.14}$$

Eq. (A.13) can also be derived from the variational principle

$$\delta \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}\boldsymbol{r}\, \tilde{\mathscr{L}} = 0, \tag{A.15}$$

under variations

$$P_n(\boldsymbol{r}, t) \to P_n(\boldsymbol{r}, t) + \delta P_n(\boldsymbol{r}, t), \tag{A.16}$$

with

$$\begin{aligned} &\delta P_n(\boldsymbol{r}, t_1) = \delta P_n(\boldsymbol{r}, t_2) = 0, \\ &\delta P_n(\boldsymbol{r}, t) = 0, \quad \text{for } \boldsymbol{r} \text{ outside a finite volume.} \end{aligned} \tag{A.17}$$

From (A.15) and (A.9) we then have

$$\delta \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}\boldsymbol{r}\, \mathscr{L} = 0, \tag{A.18}$$

under variations

$$X_n(\boldsymbol{r}, t) \to X_n(\boldsymbol{r}, t) + \delta X_n(\boldsymbol{r}, t), \tag{A.19}$$

with

$$\delta X_n(\boldsymbol{r}, t_1) = \delta X_n(\boldsymbol{r}, t_2) = 0,$$
$$\delta X_n(\boldsymbol{r}, t) = 0, \quad \text{for } \boldsymbol{r} \text{ outside a finite volume.} \tag{A.20}$$

Eq. (A.18) together with (A.19) is equivalent to (A.5), and this completes the proof that (A.4) is the Lagrange equation for (A.6).

Remarks

i) The considerations in this appendix apply in particular also to the discrete case in which the Hamiltonian is a function of $2N$ discrete fields $X_n(t)$ which do not depend on $\boldsymbol{r}$. In fact, in this special case where the dimension $D = 0$, we have $\int \mathrm{d}\boldsymbol{r} = 1$, $\partial_k X_n = 0$ and $\delta_0(\boldsymbol{r} - \boldsymbol{r}') = 1$, in eq. (A.8). As an application of (A.5) one can derive (3.12) from (3.14).

ii) In section 5, the treatment of this appendix can be applied in the special case $D = 1$, $N = 1$ to derive (5.19).

iii) To obtain explicit expressions for the Lagrangian density (A.6) the $P_n = P_n(\{X_{n'}\}, \{\partial_k X_{n'}\}, \{\partial_k \partial_l X_{n'}\}, \ldots)$ must be solved from a set of partial differential equations which can be found on the basis of (A.3) and (A.8). In the special case that $P_n = P_n(\{X_{n'}\})$ this set of differential equations is the following:

$$\sum_{n_3,n_4} \left\{ \frac{\partial P n_1}{\partial X_{n_3}} \frac{\partial P_{n_2}}{\partial X_{n_4}} - \frac{\partial P_{n_1}}{\partial X_{n_4}} \frac{\partial P_{n_2}}{\partial X_{n_3}} \right\} a_{n_3 n_4}(\boldsymbol{r}, \boldsymbol{r}') =$$
$$(\delta_{n_2 - n_1, N} - \delta_{n_1 - n_2, N}) \delta_D(\boldsymbol{r} - \boldsymbol{r}'), \quad n_1, \ldots, n_4 \in \{1, \ldots, 2N\}. \tag{A.21}$$

Appendix B

In this appendix we discuss the derivation of (6.18) taking the continuum limit of (4.29) in the case of a ferromagnetic classical ground state. We first consider the case $\mu = 1$, $\nu = 0$. In that case $q_m - q_{m-1}$ and ϕ_m, cf. (4.6) and (5.2a) may be considered to be slowly varying functions of m. We therefore apply the continuum approximation $q_m - q_{m-1} \to q'$, where we do not include a factor d in front of q', as $q_m - q_{m-1}$ is not small in the limit $d \to 0$, cf. (6.4) and (6.7).

From eq. (4.12) we then have

$$\dot{q}_m \to 2Js\kappa^2 \phi' d + \mathcal{O}(d^2), \tag{B.1}$$

implying that

$$\dot{q}' \to 2Js(\kappa^2\phi')'d^2 + \mathcal{O}(d^3). \tag{B.2}$$

From (B.1) and (B.2) we have

$$\dot{q}_m \to \dot{q}d^{-1} + \mathcal{O}(d^2) = Jsq_\tau d + \mathcal{O}(d^2), \tag{B.3}$$

where we have used the time scaling (6.15). The right-hand side of (B.3) does not contain an integration constant of order d, as a consequence of the condition $q_0 = q(x_0) = 0$, cf. (4.10) and (6.7).

The Lagrangian density $\mathcal{L}$ corresponding to (4.29) may be obtained replacing the summation Σ_m in (4.29) by an integration $(1/d)\int dx$, so that $L \to \int dx\, \mathcal{L}$. To find $\mathcal{L}$ correctly up to order d, we insert the following expressions in (4.29)

$$\begin{aligned} q_m - q_{m-1} &\to q', \\ q_{m+1} - q_m &\to q' + q''d + \tfrac{1}{2}q'''d^2 + \mathcal{O}(d^3), \\ \dot{q}_m &\to Jsq_\tau d + \mathcal{O}(d^2). \end{aligned} \tag{B.4}$$

Then

$$\begin{aligned} \mathcal{L} = Js^2[\{(1-q'^2)(1-(q'+q''d+\tfrac{1}{2}q'''d^2)^2) - q_\tau^2 d^2\}^{\frac{1}{2}} \\ + (1+c)q'(q'+q''d+\tfrac{1}{2}q'''d^2)]d^{-1} + \hat{K}(sq')d^{-1} \\ + Js^2\frac{q_\tau^2}{1-q'^2}d + \mathcal{O}(d^2). \end{aligned} \tag{B.5}$$

Expanding the terms in the right-hand side of (B.5) in powers of d, the Lagrangian density $\mathcal{L}$ can be rewritten as

$$\begin{aligned} \mathcal{L} = Js^2\{(1-q'^2) + (1+c)q'^2\}d^{-1} + \hat{K}(sq')d^{-1} + Js^2\{(1+c)-1\}q'q'' \\ + Js^2\left\{\frac{1}{2}\frac{q_\tau^2 - q''^2}{1-q'^2} + \tfrac{1}{2}((1+c)-1)q'q'''\right\}d + \mathcal{O}(d^2). \end{aligned} \tag{B.6}$$

Eq. (B.6) has been derived for the case $\mu = 1$. The Lagrangian density for $\mu = -1$ can be derived from a discrete Lagrangian using slowly varying functions which can be obtained from the ones for $\mu = 1$ applying a rotation over π around the z axis at the odd lattice sites, leading to the transformation

$$J \to -J, \quad J(1+c) \to J(1+c), \quad \hat{K} \to \hat{K}, \tag{B.7}$$

in the Hamiltonian (2.1). Apart from a total differential which does not contribute to the equation of motion, the Lagrangian density for $\mu = \pm 1$ is given by

$$\mathcal{L} = Js^2 d\mu \left\{ \tfrac{1}{2} \frac{(q_\tau^2 - q''^2)}{1 - q'^2} + \tfrac{1}{2}(1 - \mu(1 + c))q''^2 + \frac{(1 - \mu(1 + c))}{d^2}(1 - q'^2) + \frac{\hat{K}(sq')}{\mu J d^2 s^2} \right\}, \tag{B.8}$$

which is equal to (6.18) in view of (6.2).

Appendix C

In this appendix we prove that the function $y(x, t)$, defined by (8.35)–(8.37), i.e.

$$y' = \frac{1}{2\alpha} \frac{E}{(1 - q'^2)},$$
$$\dot{y} = \frac{1}{2\alpha} \left(\frac{J_1}{1 - q'^2} - \frac{J_2}{(1 - q'^2)^2} \right), \tag{C.1}$$

with

$$E = \tfrac{1}{2}(q''^2 + \dot{q}^2),$$
$$J_1 = q''\dot{q}' - q'''\dot{q}, \quad J_2 = 2\dot{q}q'E, \tag{C.2}$$

is a solution of the equation

$$\ddot{y} + y'''' - \frac{\partial}{\partial x}\left(\frac{y''^2 + \dot{y}^2}{y'} \right) + 4\alpha y' y'' = 0, \tag{C.3}$$

(which is equivalent to (8.16) for $y' \neq 0$), provided that $q(x, t)$ is a solution of (6.16) with $\tau = t$, $\mu = 1$, $c = F = 0$.

To show this, we first evaluate the derivatives of y, using (C.1). A straightforward calculation yields

$$2\alpha y'' = \frac{E'}{1 - q'^2} + \frac{2q'q''E}{(1 - q'^2)^2},$$
$$2\alpha y''' = \frac{E''}{1 - q'^2} + \frac{4q'q''E' + 2q''^2E + 2q'q'''E}{(1 - q'^2)^2} + \frac{8q'^2q''^2E}{(1 - q'^2)^3},$$
$$2\alpha y'''' = \frac{E'''}{1 - q'^2} + \frac{6q'q''E'' + 6q''^2E' + 6q'q'''E' + 6q''q'''E + 2q'q''''E}{(1 - q'^2)^2} + \frac{24q'^2q''^2E' + 24q'q''^3E + 24q'^2q''q'''E}{(1 - q'^2)^3} + \frac{48q'^3q''^3E}{(1 - q'^2)^4}, \tag{C.4}$$

and

$$2\alpha\ddot{y} = \frac{\dot{J}_1}{1-q'^2} + \frac{2J_1 q'\dot{q}' - \dot{J}_2}{(1-q'^2)^2} - \frac{4q'\dot{q}'J_2}{(1-q'^2)^3} = \frac{\dot{q}''\dot{q}' + \ddot{q}'q'' - \ddot{q}q''' - \dot{q}\dot{q}'''}{1-q'^2}$$
$$+\frac{2q'\dot{q}'(q''\dot{q}' - \dot{q}q''') - 2\dot{q}q'\dot{E} - 2\ddot{q}q'E - 2\dot{q}\dot{q}'E}{(1-q'^2)^2} - \frac{8\dot{q}\dot{q}'q'^2E}{(1-q'^2)^3}. \quad \text{(C.5)}$$

We furthermore have, evaluating $(\dot{y}^2 + y''^2)/y'$ and its derivative:

$$2\alpha\frac{\partial}{\partial x}\left(\frac{y''^2 + \dot{y}^2}{y'}\right) = \frac{4q'''q'''' + 4\dot{q}'\dot{q}''}{1-q'^2}$$
$$+\frac{8q'q'''E' + 8q'q''''E + 8q''q'''E + 4q'q''(q'''^2 + \dot{q}'^2)}{(1-q'^2)^2}$$
$$+\frac{16q'^2EE' + 16q'q''E^2 + 32q'^2q''q'''E}{(1-q'^2)^3} + \frac{48q'^3q''E^2}{(1-q'^2)^4}, \quad \text{(C.6)}$$

and

$$8\alpha^2 y'y'' = \frac{2EE'}{(1-q'^2)^2} + \frac{4q'q''E^2}{(1-q'^2)^3} \quad \text{(C.7)}$$

The derivatives of E are given by

$$E' = q''q''' + \dot{q}\dot{q}', \quad E'' = q'''^2 + q''q'''' + \dot{q}'^2 + \dot{q}\dot{q}'',$$
$$E''' = 3q'''q'''' + q''q''''' + 3\dot{q}'\dot{q}'' + \dot{q}\dot{q}'''. \quad \text{(C.8)}$$

From (C.2) and (C.4)–(C.8) we see that

$$2\alpha\left[\ddot{y} + y'''' - \frac{\partial}{\partial x}\left(\frac{y''^2 + \dot{y}^2}{y'}\right) + 4\alpha y'y''\right]$$
$$= \frac{\gamma_1}{1-q'^2} + \frac{\gamma_2}{(1-q'^2)^2} + \frac{\gamma_3}{(1-q'^2)^3} + \frac{\gamma_4}{(1-q'^2)^4}, \quad \text{(C.9)}$$

with

$$\gamma_1 = q''(\ddot{q}' + q''''') - q'''(\ddot{q} + q''''),$$
$$\gamma_2 = 4q'q''\dot{q}'^2 - 4q'q'''\dot{q}\dot{q}' + 4q'q''\dot{q}\dot{q}'' + 6q''^2\dot{q}\dot{q}' - 3q'q''''\dot{q}^2 - q''^2q'\ddot{q}$$
$$-3q'\dot{q}^2\ddot{q} + 6q''^3q''' + 3q'q''^2q'''', \quad \text{(C.10)}$$
$$\gamma_3 = 24q'^2q''^2E' + E(-24\dot{q}\dot{q}'q'^2 + 18q'q''^3 - 24q'^2q''q''' - 6q'q''\dot{q}^2),$$
$$\gamma_4 = 24Eq'^3q''(q''^2 - \dot{q}^2).$$

Let us now consider an expression of the form

$$\left(\frac{K_1}{(1-q'^2)} + K_2 + K_3\frac{\partial}{\partial x}\right)\left[\frac{\partial}{\partial t}\frac{\dot q}{1-q'^2} + \frac{\partial}{\partial x}\left(\frac{q'''}{1-q'^2} + \frac{2E}{(1-q'^2)^2}\right)\right]$$
$$\equiv \frac{\tilde\gamma_1}{1-q'^2} + \frac{\tilde\gamma_2}{(1-q'^2)^2} + \frac{\tilde\gamma_3}{(1-q'^2)^3} + \frac{\tilde\gamma_4}{(1-q'^2)^4}, \tag{C.11}$$

where K_1, K_2, K_3 are as yet unspecified, and the second factor in the left-hand side of (C.11) is equal to $\dot q^{-1}\times$ left-hand side of (8.39). Evaluating the derivatives in the left-hand side of (C.11) and using (C.8) for E' and E'', we have

$$\tilde\gamma_1 = K_3(\ddot q' + q''''') + K_2(\ddot q + q''''),$$
$$\tilde\gamma_2 = \ddot q(K_1 + 2q'q''K_3) + q''''(K_1 + 6q'q''K_3) + q'''q'(4q''K_2 + 4q'''K_3)$$
$$+ 4\dot q\dot q'q'K_2 + q''^2(q''K_2 + 7q'''K_3) + \dot q^2(q''K_2 + q'''K_3)$$
$$+ 6\dot q\dot q'q''K_3 + 4q'\dot q'^2K_3 + 4\dot q q'\dot q''K_3, \tag{C.12}$$
$$\tilde\gamma_3 = E'(4q'K_1 + 24q'^2q''K_3) + 2Eq''K_1 + 8E(q'^2q''K_2 + 3q'q''^2K_3 + q'^2q'''K_3),$$
$$\tilde\gamma_4 = 8Eq'^2q''(K_1 + 6q'q''K_3).$$

Choosing

$$K_1 = -3q'(q''^2 + \dot q^2) = -6q'E, \quad K_2 = -q''', \quad K_3 = q'', \tag{C.13}$$

it is straightforward to check that

$$\tilde\gamma_1 = \gamma_1, \quad \tilde\gamma_2 = \gamma_2, \quad \tilde\gamma_3 = \gamma_3, \quad \tilde\gamma_4 = \gamma_4, \tag{C.14}$$

implying that

$$2\alpha\left[\ddot y + y'''' - \frac{\partial}{\partial x}\left(\frac{y''^2 + \dot y^2}{y'}\right) + 4\alpha y'y''\right]$$
$$= \left(\frac{-3q'(q''^2 + \dot q^2)}{1-q'^2} - q''' + q''\frac{\partial}{\partial x}\right)$$
$$\times\left[\frac{\partial}{\partial t}\frac{\dot q}{1-q'^2} + \frac{\partial}{\partial x}\left(\frac{q'''}{1-q'^2} + \frac{q'(q''^2 + \dot q^2)}{(1-q'^2)^2}\right)\right] = 0. \tag{C.15}$$

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