UNIVERSAL FUNCTIONAL EQUATION FOR PERIOD DOUBLING IN CONSTANT-JACOBIAN MAPS

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The period-doubling behavior of one-parameter families of maps of constant jacobian is related to a fixed-point equation of the form $g_{B_e^2}=A_{B_e}\circ g_{B_e}\circ g_{B_e}\circ A_{B_e}^{-1}$ and $\det Dg_{B_e}=B_e$, $g_{B_e}: \mathbb{R}^2\to\mathbb{R}^2$. (For $B_e=0$ the Feigenbaum–Cvitanovic equation is recovered.) The scaling transformation A_{B_e}: $\mathbb{R}^2\to\mathbb{R}^2$ is "conjugate" to the transformation $A^0_{B_e}:(x, y)\to(\alpha_{B_e}x, \beta_{B_e}y)$. Expanding the fixed-point equation above in inverse powers of α_{B_e}, a low-order polynomial approximate fixed-point solution g_{B_e} is found together with an approximation to the crossover scaling functions α_{B_e} and β_{B_e}.

1. Introduction

The universality of period doubling in dissipative systems was discovered by Feigenbaum [1–5]. Subsequently it was found that another universality class exists for period doubling in conservative mappings in two dimensions [6–8] [‡1]. A more rigorous mathematical foundation of this universality was later given in terms of the approach in a function space to the fixed points of certain universal functional equations [10–17]. In the 1D case Sullivan has proven that the quadratic map approaches the well-known fixed point [18], and Groeneveld has constructed the general unimodal solution [19].

On the other hand the crossover from conservative to dissipative behavior has been studied for 2D maps of constant jacobian by several authors [20–29]. In this paper I derive and study the functional equation describing this universal crossover behavior for 2D maps of constant jacobian.

2. Functional equation

Consider a 2D map of constant jacobian B:

$$\boldsymbol{x}'=f_B(\boldsymbol{x})\ , \quad f_B: \mathbb{R}^2\to\mathbb{R}^2\ , \quad \boldsymbol{x}\in\mathbb{R}^2\ , \quad B\in[0, 1]\ . \tag{1}$$

The conservative and dissipative limits of eq. (1) correspond to $B=1$ and $B=0$ respectively. For $B^2=B$ it is found numerically [1] that

$$g_r := \lim_{n\to\infty} A^n\circ f^{2^n}_{C_{n+r}}\circ A^{-n}\ , \tag{2}$$

where g_r is a universal function, independent of the specific function f we started from, C_n is the value of the bifurcation parameter at the nth bifurcation, and A is a scaling transformation. For $B^2\neq B$, numerical evidence indicates that eq. (2) generalizes to

$$g_{r,B^{2^n}} := \lim_{\substack{n\to\infty,\ B\to 1\\ B^{2^n}=:B_e}} \prod_{m=n-1}^{0} A_{B^{2^m}}\circ f^{2^n}_{C_{n+r},B}\circ\prod_{l=0}^{n-1} A^{-1}_{B^{2^l}}\ , \tag{3}$$

in taking this limit the effective jacobian $B_e := B^{2^n}$ is kept fixed. Taking the composition of two functions g we get

$$g_{r,B^{2^n}}\circ g_{r,B^{2^n}} \approx A^{-1}_{B^{2^n}}\circ\prod_{m=n}^{0} A_{B^{2^m}}\circ f^{2^{n+1}}_{C_{n+1+r-1},B}$$

$$\circ\prod_{l=0}^{n} A^{-1}_{B^{2^l}}\circ A_{B^{2^n}} \approx A^{-1}_{B^{2^n}}\circ g_{r-1,B^{2^{n+1}}}\circ A_{B^{2^n}}\ . \tag{4}$$

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[‡1] Very recently new universality classes were discovered for 4D conservative mappings [9].

Hence the functions g_r obey the universal equation:

$$g_{r-1,B_e^2}=A_{B_e}\circ g_{r,B_e}\circ g_{r,B_e}\circ A_{B_e}^{-1}\ . \tag{5}$$

Taking the limit $r\to\infty$ we obtain the equation satisfied by $g_{B_e}:=\lim\limits_{r\to\infty} g_{r,B_e}$:

$$g_{B_e^2}=A_{B_e}\circ g_{B_e}\circ g_{B_e}\circ A_{B_e}^{-1} \tag{6a}$$

and

$$\det Dg_{B_e}=B_e \tag{6b}$$

(cf. the independent work in ref. [27]). At $B_e=0$ the Feigenbaum–Cvitanovic equation is recorded [1–5,30]. At $B_e=1$ we recover the area-preserving functional equation studied by Collet et al. [13,14] and Greene et al. [8]. Eq. (6) is invariant under

$$g_{B_e}\to S_{B_e}\circ g_{B_e}\circ S_{B_e}^{-1}\ ,\quad A_{B_e}\to S_{B_e^2}\circ A_{B_e}\circ S_{B_e}^{-1}\ , \tag{7}$$

where $S_{B_e}: \mathbb{R}^2\to\mathbb{R}^2$ has a constant jacobian that may depend on B_e. (Note that eq. (6) is also invariant under $g_{B_e}\to g^j_{B_e^{1/j}}$, where $j\in\mathbb{Z}\setminus\{0\}$ and $g^j_{B_e^{1/j}}$ denotes the jth functional power of $g_{B_e^2}$.) We will use this invariance to restrict ourselves to

$$A^0_{B_e}: (x,y)\to(\alpha_{B_e}x, \beta_{B_e}y)\ . \tag{8}$$

3. Solution

For $B_e^2=B_e$, using computer assistance, proofs have been given that a solution of eq. (6) corresponding to the generic period-doubling case exists and the spectral properties of the linearization of the renormalization operator at the fixed point g have been established [‡2] [11,13]. These existence proofs essentially proceed in the following way [14]:

(I) Find a low-order polynomial approximation solution $g_{B_e,\mathrm{low}}$ to eq. (6).

(II) Construct a concentration C in function space.

Numerical evidence indicates in the known cases that the approximate solution need only be very crudely known. The proofs however require much more precision and so two more steps are needed:

(III) Iterate C a number of times (taking $g_{B_e,\mathrm{low}}$ as initial choice), producing a high-order polynomial approximation solution $g_{B_e,\mathrm{high}}$.

(IV) Show that the concentration mapping principle can be applied on a suitable ball around $g_{B_e,\mathrm{high}}$.

Here our aim will be to execute step (I) of the above procedure for the more general case $0\leqslant B_e\leqslant 1$. Cf. ref. [13] for a similar approach to the area preserving case (i.e. $B_e=1$). We look for a solution g_{B_e} to the universal period-doubling eq. (6) that can be expanded in a Taylor series. Since we are interested in the case $\beta_{B_e}\gg 1$, $\alpha^2_{B_e}\gg 1$, we will expand eq. (6) in powers of $1/\alpha_{B_e}$, taking β_{B_e} of the order of $\alpha^2_{B_e}$. We therefore expand g_{B_e} as follows:

$$g_{B_e}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sum_{0\leqslant j+2k\leqslant p+1} a_{1\,jk,B_e}x^jy^k \\ \sum_{0\leqslant j+2k\leqslant p+2} a_{2\,jk,B_e}x^jy^k \end{pmatrix}, \tag{9}$$

and in order to find a low-order polynomial approximate solution we retain the powers $\alpha^2_{B_e}$ through $\alpha^{-1}_{B_e}$ in eq. (6), i.e. take $p=1$ in eq. (9). We write the function g in a new notation

$$g_{B_e}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1+\lambda_1x+\lambda_3y+ux^2 \\ k+\lambda_2y+\lambda_4x+u'x^2+w'xy+q'x^3 \end{pmatrix}, \tag{10}$$

where the dependence of the coefficients k, λ_1, λ_2, λ_3, λ_u, u, u', w' and g' on B_e has been suppressed. In eq. (10) the coefficient $a_{1.0.0}$ has been normalized to 1. We still have one freedom left in the normalization of g_{B_e}, due to the invariance properties (7). We will come back to this point later. Inserting (10) in (6b) and (6a) respectively, and equating like powers of x and y we obtain

$$\lambda_1\lambda_2-\lambda_3\lambda_4=B_e\ ,$$

constant term in $\det Dg_{B_e}$, (11a)

$$\lambda_1w'+2\lambda_2u-2\lambda_3u'=0\ ,$$

x-term in $\det Dg_{B_e}$, (11b)

$$\lambda_3w'=0\ ,$$

y-term in $\det Dg_{B_e}$, (11c)

[‡2] For $B_e=0$ there are also proofs that do not rely on computer assistance [10,12,15,17]. Also monotonic solutions have been found [31].

$$1=\alpha(1+\lambda_1+\lambda_3k+u)\,, \tag{11d}$$

constant term in 1st component ,

$$\tilde{\lambda}_1=\lambda_3\lambda_4+\lambda_1^2+2\lambda_1u\,, \tag{11e}$$

x-term in 1st component ,

$$\tilde{\lambda}_3=(\alpha/\beta)(\lambda_3\lambda_1+\lambda_3\lambda_2+2\lambda_3u)\,, \tag{11f}$$

y-term in 1st component,

$$\tilde{u}=(1/\alpha)(\lambda_1u+\lambda_3u'+\lambda_1^2u+2u^2)\,, \tag{11g}$$

x^2-term in 1st component ,

$$\tilde{k}=\beta(k+\lambda_2k+\lambda_4+u'+kw'+k^3q')\,, \tag{11h}$$

constant term in 2nd component ,

$$\begin{aligned}\tilde{\lambda}_2=\lambda_2^2+\lambda_2w'+\lambda_3\lambda_4+2\lambda_3u'\\ +\lambda_3kw'+3\lambda_2k^2q'\,,\end{aligned} \tag{11i}$$

y-term in 2nd component ,

$$\begin{aligned}\tilde{\lambda}_4=(\beta/\alpha)(\lambda_2\lambda_4+\lambda_1\lambda_4+\lambda_4w'\\ +2\lambda_1u'+\lambda_1kw')\,,\end{aligned} \tag{11j}$$

x-term in 2nd component ,

$$\begin{aligned}\tilde{u}'=(\beta/\alpha^2)\lambda_2u'+\lambda_4u+\lambda_1^2u'+2uu'+\lambda_1\lambda_4w'\\ +w'u+w'u'+3k^2u'q'+3k\lambda_4^2q')\,,\end{aligned} \tag{11k}$$

x^2-term in 2nd component ,

$$\begin{aligned}\tilde{w}'=(1/\alpha)(\lambda_2w'+2\lambda_1\lambda_3u'\\ +\lambda_1\lambda_2\mathrm{w}'+\lambda_3\lambda_4\mathrm{w}'+w'^2+3k^2w'q')\,,\end{aligned} \tag{11ℓ}$$

xy-term in 2nd component ,

$$\begin{aligned}\tilde{q}'=(\beta/\alpha^3)(\lambda_2q'+w'q'+3q'k^2+q'\lambda_4^3+2\lambda_1u^2\\ +\lambda_1u'w'+\lambda_2uw')\,,\end{aligned} \tag{11m}$$

x^3-term in 2nd component .

Here $\tilde{\lambda}_i:=\lambda_{i,B_e^2}$, $i=1,2,3,4$; $\tilde{u}:=u_{B_e^2}$;$\tilde{u}':=u'_{B_e^2}$; $\tilde{k}=k_{B_e^2}$, etc . Eqs. (11) can be solved consistently to the order we are working here $(1/\alpha)$. From the fact that $\tilde{\lambda}_1\tilde{\lambda}_2-\tilde{\lambda}_3\tilde{\lambda}_4=B_e^2$ we find, using (11b) and (11c) that $k^2q'(2u+\lambda_1)=0$ and so, for the solution we are interested in,

$$q'=0\,. \tag{12}$$

From (11e), (11a) and (11i) we obtain two similar equations for λ_1 and λ_2

$$\tilde{\lambda}_1/\lambda_1=\lambda_1+\lambda_2+2u-B_e/\lambda_1\,, \tag{13}$$

$$\tilde{\lambda}_2/\lambda_2=\lambda_1+\lambda_2+2u-B_e/\lambda_2\,. \tag{14}$$

Expanding λ_1 and λ_2 as Taylor series in B_e about $B_e=0$ (assuming these series exist and converge for all $B_e\in[0,1]$, cf. eqs. (18) and (19)), i.e. $\lambda_1=\sum_{i=1}^{\infty}a_iB_e^i$, $\lambda_2=\sum_{i=1}^{\infty}A_iB_e^i$ we can prove by induction that the coefficients in these two Taylor series must be equal, $a_i=A_i$, and hence

$$\lambda_1=\lambda_2\,, \tag{15}$$

and hence we can express u in terms of λ_1

$$u=\tilde{\lambda}_1/2\lambda_1-\lambda_1+B_e/2\lambda_1\,. \tag{16}$$

From (11g), using (16), (11b) and (11c), we get the following equation expressing α in terms of λ_1:

$$\alpha=\frac{\tilde{\lambda}_1/\lambda_1-2\lambda_1+B_e/\lambda_1}{\tilde{\tilde{\lambda}}_1/\tilde{\lambda}_1-2\tilde{\lambda}_1+B_e^2/\tilde{\lambda}_1}\left(\lambda_1^2+\frac{B_e}{\lambda_1}+\frac{\tilde{\lambda}_1}{\lambda_1}\right)\,. \tag{17}$$

Eliminating λ_3, β and k from (11d), (11f) and (11h) and using (16) for $\tilde{u}$ and u, and (17) for α and $\tilde{\alpha}$, we obtain the following equation for λ_1:

$$\begin{aligned}&\frac{\tilde{\tilde{\tilde{\lambda}}}_1/\tilde{\tilde{\lambda}}_1-2\tilde{\tilde{\lambda}}_1+B_e^4/\tilde{\tilde{\lambda}}_1}{(\tilde{\tilde{\lambda}}_1/\tilde{\lambda}_1-2\tilde{\lambda}_1+B_e^2/\tilde{\lambda}_1)(\tilde{\lambda}_1^2+B_e^2/\tilde{\lambda}_1+\tilde{\tilde{\lambda}}_1/\tilde{\lambda}_1)}\\ &\quad-1-\frac{\tilde{\tilde{\lambda}}_1}{2\tilde{\lambda}_1}-\frac{B_e^2}{2\tilde{\lambda}_1}\\ &=\frac{\tilde{\lambda}_1+B_e}{\lambda_1}\left[\lambda_1+1-\left(B_e+1+\frac{\tilde{\lambda}_1}{2\lambda_1}+\lambda_1+\frac{B_e}{2\lambda_1}\right)\right.\\ &\quad\left.\times\frac{\tilde{\lambda}_1/\lambda_1-2\lambda_1+B_e/\lambda_1}{\tilde{\tilde{\lambda}}_1/\tilde{\lambda}_1-2\tilde{\lambda}_1+B_e^2/\tilde{\lambda}_1}\left(\lambda_1^2+\frac{B_e}{\lambda_1}+\frac{\tilde{\lambda}_1}{\lambda_1}\right)\right]\,.\end{aligned} \tag{18}$$

(In eqs. (17) and (18) $\tilde{\tilde{\lambda}}_1:=\lambda_{1,B_e^4}$,$\tilde{\tilde{\tilde{\lambda}}}_1:=\lambda_{1,B_e^8}$.) Eq.

(18) can be solved numerically as follows: We expand eq. (18) in a Taylor series about $B_e=0$. From this we obtain

$$\lambda_1=\lambda_{1,B_e}=(\tfrac{1}{2}-\tfrac{1}{2}\sqrt{3})B_e+O(B_e^2)\ . \tag{19}$$

For small enough values of B_e we can use this linear approximation for λ_{1,B_e^2}, λ_{1,B_e^4} and λ_{1,B_e^8} and obtain λ_{1,B_e} from (18) using Newton's method. Repeating this procedure we find $\lambda_{1,B_e^{1/2}}$, $\lambda_{1,B_e^{1/4}}$ etc., until we reach $B_e^{2^{-n}}\approx 1$. (Unfortunately the procedure starting from $B_e=1$ and iterating towards $B_e=0$, for which we would *not* need to use Newton's method, is unstable.) A plot of λ_1 as a function of B_e is given in fig. 2B. Now that λ_1 is known as a function of B_e, the orbit-scaling factor α is given by eq. (17) (see fig. 1A), u by eq. (16) (see fig. 2D), and λ_2 by eq. (15) (see fig. 2B). To determine the other coefficients we choose the normalization

$$\lambda_{3,B_e}=1\ . \tag{20}$$

Note that the choice of normalization has great influence since we do not normalize constants but functions of B_e, in this way getting rid of e.g. factors $\lambda_{3,B_e^2}/\lambda_{3,B_e}$. Using eq. (20), u' is given by eq. (11b) (see fig. 2E), λ_4 by eq. (11a) (see fig. 2C), β by eq. (11f) (see fig. 1B), and k by eq. (11d) (see fig. 2A). Note that at $B_e=0$ we have $\lambda_1=\lambda_2=\lambda_4=k=u'=w'=q'=0$, $\beta=\alpha^2$, in agreement with the exact solution given in ref. [30]. The accuracy of our approximation can be gauged from the exact values at $B_e=0$ and $B_e=1$, indicated in the figures by crosses. It is interesting to note that our approximate solution g_{B_e} can be transformed into the two-dimensional dissipative Hénon map [32,24–26] by a linear coordinate transformation S_{B_e} (cf. eq. (7)). Note however that since this transformation depends explicitly on B_e, the scaling matrix $\mathbf{A}_{B_e}$ is certainly not invariant under this transformation. Finally it should be noted that another numerical solution g_{B_e} to the universal functional equation (6) can be found in an analogous way to the above, starting from

$$\lambda_1=(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3})B_e+O(B_e^2)\ ,\quad B_e\downarrow 0\ , \tag{21}$$

which crosses over to the following function near $B_e=1$:

$$\lambda_1=1-\tfrac{1}{2}(1-B_e)+a(1-B_e)^2+b(1-B_e)^{\ln 6/\ln 2}$$
$$+O((1-B_e)^{\ln 6/\ln 2+1})\ ,\quad B_e\uparrow 1\ ,$$

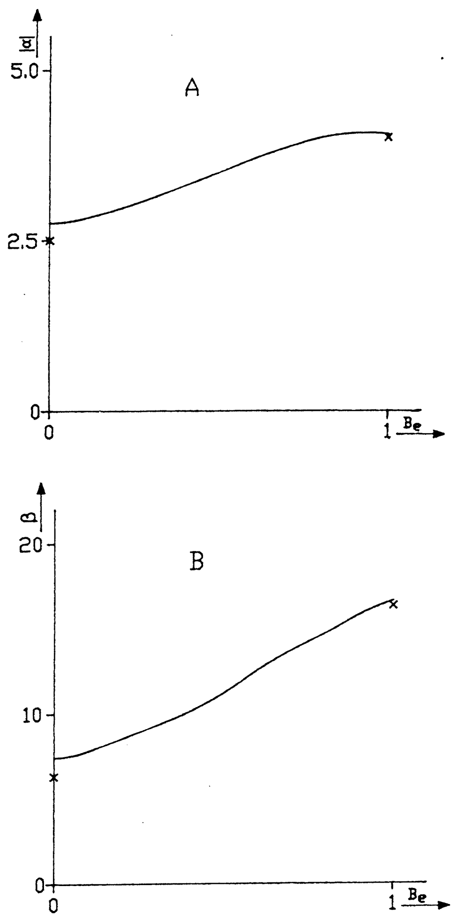

Fig. 1. The universal crossover scaling exponents α_{B_e} and β_{B_e} as functions of the effective jacobian B_e:(A) $|\alpha_{B_e}|$;(B) β_{B_e} (the crosses at $B_e=0$ and $B_e=1$ indicate exact values).

where the coefficients a and b can be determined numerically. This solution for g_{B_e} has a reverse tangential bifurcation at $B_e=1$ with two exact linear solutions of eq. (6), i.e. with

$$g_{B_e}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\sqrt{B_e} & 1\\ 0 & \sqrt{B_e}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}, \tag{22}$$

$$g_{B_e}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\tfrac{1}{2}+\tfrac{1}{2}B_e & 1\\ (\tfrac{1}{2}-\tfrac{1}{2}B_e)^2 & \tfrac{1}{2}+\tfrac{1}{2}B_e\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}. \tag{23}$$

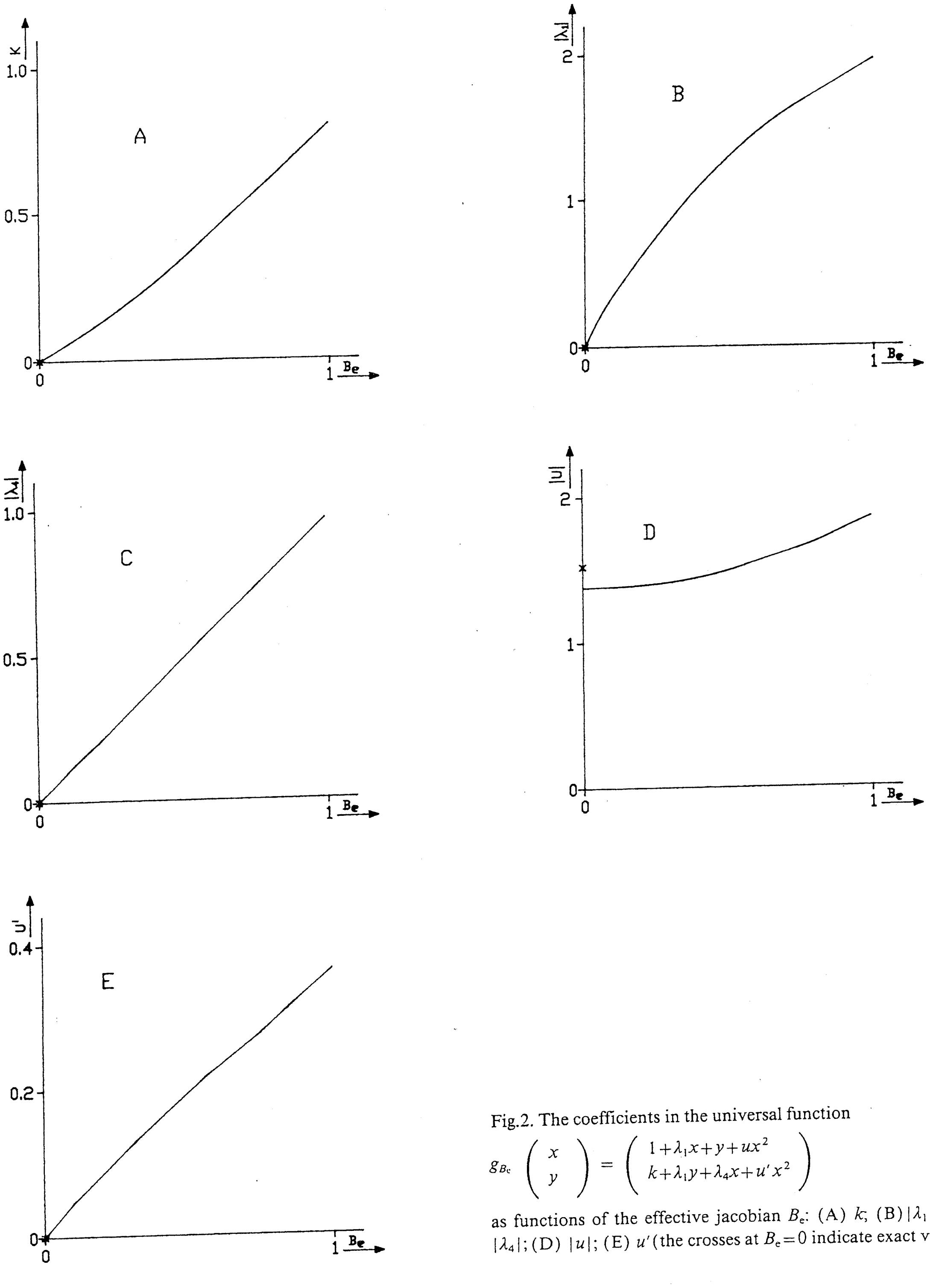

Fig.2. The coefficients in the universal function

$$g_{B_e}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1+\lambda_1 x+y+ux^2 \\ k+\lambda_1 y+\lambda_4 x+u'x^2 \end{pmatrix}$$

as functions of the effective jacobian B_e: (A) k; (B) $|\lambda_1|$; (C) $|\lambda_4|$; (D) $|u|$; (E) u' (the crosses at $B_e=0$ indicate exact values).

It is not quite clear how the latter three solutions for g_{B_c} should be interpreted.

Finally it should be remarked that α and β as defined by eqs. (6) and (8) are not identical to the α and β studied in ref. [27].

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