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ANALYTICAL RENORMALIZATION RESULTS FOR THE CROSS-OVER BEHAVIOR OF PERIOD DOUBLING, FROM CONSERVATIVE TO DISSIPATIVE SYSTEMS

G. Reinout W. QUISPEL
Center for Theoretical Physics, Twente University of Technology, 7500 AE Enschede, The Netherlands

Extended abstract

It has been shown that there is a universal scaling function describing the cross-over of the effective Feigenbaum convergence rate δ from its conservative value ($\delta = 8.721097\ldots.$) to its dissipative value ($\delta = 4.669201\ldots.$), as a function of the "effective dissipation". Using renormalization theory I obtain an *explicit analytical* expression for this cross-over function and show that it's not monotonic but has a *minimum*, just before it reaches its asymptotic dissipative value. I also derive an *analytical* expression for the (period-doubling) bifurcation values in a particular map (the Hénon map), at all values of the Jacobian.

1. Introduction

Many systems display infinite series of period-doubling bifurcations [1]. Using renormalization theory I derive explicit expressions for the dependence, of the parameter values at bifurcation and their "effective" rate of convergence δ (i.e. at the nth bifurcation), on B and n, for planar maps of constant Jacobian B.

Zisook has shown that there is a universal scaling function describing the cross-over of the effective rate of convergence δ from its conservative value ($\delta = 8.721097\ldots.$) to its dissipative value ($\delta = 4.669201\ldots.$), as a function of the "effective dissipation", i.e. of $B_e = B^{2^n}$. Using a technique of Ghendrih and renormalization theory, I obtain an analytical expression for this scaling function and show that it's not monotonic but has a minimum at $B_{e,\min} \approx 3 \times 10^{-6}$. The true existence of this minimum has been demonstrated numerically for the Hénon map [3]. Finally, an expression for the bifurcation values of a particular map, the Hénon map, is derived. A more extended version of this work, including results on the orbit-scaling factor α, are presented in ref. 4.

2. The universal rate of convergence δ

The behavior of almost every planar mapping, of constant Jacobian B, can be approximated locally by a quadratic mapping that can be brought into the standard form [5]

$$y_{t+1} + By_{t-1} = 2Cy_t + 2y_t^2, \quad t = 0,1,2,\ldots, \tag{1}$$

the two-dimensional Hénon map. The parameter values C_n at which a period 2^n is born are determined in first-order renormalization theory by [5]

$$\begin{aligned} C_{n-1}(B^2) &= -2C_n^2(B) + 2(1+B)C_n(B) \\ &\quad + 2B^2 + 3B + 2, \quad C_0(B) = \tfrac{1}{2} + \tfrac{1}{2}B. \end{aligned} \tag{2}$$

The effective rate of convergence is defined as

$$\delta_n(B) \equiv \frac{C_{n-1}(B) - C_n(B)}{C_n(B) - C_{n+1}(B)}. \tag{3}$$

Using (2) it can be shown that for large n the effective rate of convergence $\delta_n(B)$ depends only on $B_e = B^{2^n}$ [2] and that this universal function

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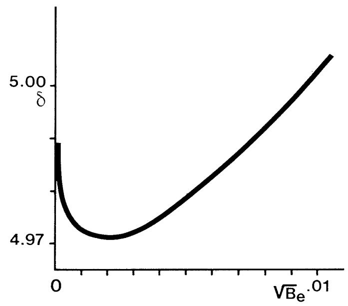

Fig. 1. The minimum in the effective convergence rate for small B_e.

can be approximated by

$$\delta_n(B) = \delta(B_e)$$

$$\approx \left\{\left[6 + 8\sqrt[4]{B_e} + 6\sqrt[2]{B_e}\right]^{1/2} - 2 - 2\sqrt[4]{B_e}\right\}$$

$$\Big/ \left\{\left[4 + 8\sqrt[4]{B_e} + 4\sqrt[2]{B_e} + \left[6 + 8\sqrt[2]{B_e} + 6B_e\right]^{1/2}\right]^{1/2} - \left[6 + 8\sqrt[4]{B_e} + 6\sqrt[2]{B_e}\right]^{1/2}\right\}, \quad (4)$$

which I derive in ref. 4, cf. also ref. 6. The function $\delta(B_e)$ is not monotonic but has a minimum at $B_e \approx 3 \times 10^{-6}$. This can be seen from fig. 1, where $\delta(B_e)$ is plotted at small B_e. This behavior agrees with numerical simulation [3].

3. The bifurcation values $C_n(B)$ for the Hénon map

Defining the distance to the fixed point,

$$D_n(B) \equiv C_n(B) - C_\infty(B), \quad (5)$$

the linearization of eq. (2) about its fixed point is

$$D_n(B) = k(B) D_{n-1}(B^2), \quad (6)$$

where

$$k(B) = (-4C_\infty(B) + 2 + 2B)^{-1}. \quad (7)$$

Iterating eq. (6) we get

$$D_n(B) = k(B)k(B^2) \cdots k(B^{2^{n-2}}) D_1(B^{2^{n-1}}). \quad (8)$$

The product in eq. (8) can be simplified introducing a function $f(B)$ such that

$$k(B) \equiv \frac{f(B)}{f(B^2)}. \quad (9)$$

Eq. (8) then becomes

$$D_n(B) = \frac{f(B)}{f(B^{2^{n-1}})} D_1(B^{2^{n-1}}). \quad (10)$$

Since D_1 can be found (as a Taylor series) using eq. (2), it remains to determine $f(B)$. In order to do this we split off two factors that are non-analytic at $B = 1$ and $B = 0$, respectively. The remaining factor is analytic and is found recursively in a Taylor series using (9), (7) and (2). The $C_n(B)$ are then found for all n and B, using eq. (5), cf. ref. 4.

References

[1] P. Cvitanovic, ed., Universality in Chaos, a reprint selection (Hilger, Bristol, 1982).
[2] A.B. Zisook, Phys. Rev. A24 (1981) 1640.
[3] J.P. van der Weele and H.W. Capel, Institute for Theor. Phys., Univ. of Amsterdam, to be published (1985).
[4] G.R.W. Quispel, Phys. Rev. A31 (1985) 3924.
[5] R.H.G. Helleman, in: Long Time Prediction in Dynamics, W. Horton et al., eds. (Wiley, New York, 1983) p. 95.
[6] P. Ghendrih, "Universality in the Transition to Chaos of Dissipative Systems", CEN-SACLAY preprint (1983).