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LINEAR INTEGRAL EQUATIONS AND MULTICOMPONENT NONLINEAR INTEGRABLE SYSTEMS I

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A systematic method for obtaining multicomponent generalizations of integrable nonlinear partial differential equations (PDE's) is developed. The method starts from a general type of linear integral equations, containing integrations over an arbitrary contour with an arbitrary measure in the complex plane of k (the spectral parameter). Special k-dependent factors in the integrand are shown to induce an extra coupling between solutions of these integral equations with a different source term. In this way the direct linearization is obtained of multicomponent generalizations of various nonlinear PDE's, such as the nonlinear Schrödinger equation, the derivative nonlinear Schrödinger equations, the isotropic Heisenberg spin chain equation, the (complex) sine-Gordon equation, and the massive Thirring model equations. In the present paper (I) we present the general framework to derive finite-matrix PDE's, and we also discuss Bäcklund transformations and multicomponent lattice versions. In a subsequent paper (II) we treat a variety of examples of multicomponent PDE's, and discuss Miura transformations and gauge equivalences.

1. Introduction

The investigation of soliton systems on the basis of linear integral equations has turned out to be advantageous in many respects. The direct linearization (DL) method, introduced by Fokas and Ablowitz[1]) for the Korteweg–de Vries (KdV) equation, makes use of a singular integral equation in spectral space (i.e.

in terms of the complex spectral parameter k) with an integration over an arbitrary contour and measure. The arbitrariness of the contour and measure in the integral equation enables one to capture a large class of solutions of the associated nonlinear partial differential equation (PDE). The solution ϕ_k of the integral equation depends on the coordinates x and t of the soliton system via plane-wave factors $\exp(ikx - i\omega_k t)$, occurring e.g. in the source term of the (inhomogeneous) integral equation. Under general conditions $\phi_k(x, t)$ can be shown to satisfy a set of two linear equations, one independent of the dispersion ω_k (the scattering or spectral problem), and one dependent on a specific choice of ω_k. The parameter k has the role of the spectral parameter in the scattering problem, and its potential ϕ is given by the integral of ϕ_k over the same measure and contour as in the integral equation (t is a dummy parameter in the scattering problem). The associated nonlinear PDE can be found from the compatibility condition in terms of $\phi(x, t)$ for the linear system, and can also be obtained from it by exploiting the integration of ϕ_k to ϕ.

In subsequent work the DL method was extended into several directions[2–6]). In ref. 2 the DL was formulated for other types of integral equations apart from the one associated to the KdV, and a more general structure was added by including a factor k^{-i} (i integer) in the source term. In this way one can define vector-solutions $\boldsymbol{\phi}_k$ of the integral equations, where the components of $\boldsymbol{\phi}_k$ are the solutions $\phi_k^{(i)}$ with i running from $-\infty$ to ∞. The linear equations for $\boldsymbol{\phi}_k(x, t)$ that are derived from the integral equation obtain a recursive structure in i (such a structure is also present in the recent work of ref. 7b, cf. ref. 7a as quoted in ref. 7b). The recursive structure of the linear equations provides, in the case of polynomial dispersion, a scheme to obtain "closed" equations in terms of the $\infty \times \infty$ potential matrices $\boldsymbol{\Phi}$ with as elements $\phi_k^{(i)} k^{-j}$ integrated over the measure and contour. These equations in terms of $\boldsymbol{\Phi}$ reduce immediately to genuine PDE's in terms of the $(0, 0)$ element of $\boldsymbol{\Phi}$, and with somewhat more effort to genuine PDE's in terms of the $(1, 0)$ and the $(1, 1)$ elements of $\boldsymbol{\Phi}$. Furthermore from the recursive structure one derives the Miura transformations (MT's), i.e. equations relating two of these elements of $\boldsymbol{\Phi}$. Equations that have been treated in this way include the nonlinear Schrödinger equation (NLS), the isotropic Heisenberg spin chain equation (IHSC), both with $\omega_k = k^2$, the (complex) sine-Gordon equation (CSG), a case with non-polynomial $\omega_k = k^{-1}$, and (cf. ref. 3) the derivative nonlinear Schrödinger equation (DNLS), with $\omega_k = k^2$, and the equations of the massive Thirring model (MTM), with $\omega_k = k^{-1}$. A new feature introduced in ref. 3 was a factor f_k in front of the integrations in the integral equation, which for the mentioned equations had to be taken as $f_k = k$.

An application of the DL method is concerned with Bäcklund transformations (BT's). As shown in ref. 4, BT's can be obtained by considering singular transformations of the measure – or, equivalently, of the plane-wave factors – in the integral equation. Furthermore, some linear combinations of the solution of an integral equation and its Bäcklund transform have been shown to yield solutions of integrable, so-called modified versions of the associated PDE. In this way, linearizing integral equations have been found for e.g. the modified sine-Gordon equation (MSG) and the anisotropic Heisenberg spin chain equation (AHSC) with uniaxial anisotropy.

Another application of the DL method is the derivation of lattice versions of the associated PDE's, i.e. integrable nonlinear difference–difference equations (PDfE's). In ref. 5 it is argued that these (dd) equations can be regarded as arising from Bianchi identities expressing the commutativity of BT's. Lattice versions that have been treated in this fashion include the ddNLS, the ddIHSC, the ddCSG, and (cf. ref. 6) the ddMTM.

In the present paper we show how the DL can be formulated for *multicomponent* generalizations of the PDE's (and PDfE's) that were treated in refs. 1–6. For this purpose we introduce the following linear integral equation:

$$\boldsymbol{\phi}_k[A, B] + \int_{C^*} \mathrm{d}\lambda^*(l') \int_{C} \mathrm{d}\lambda(l)\, A_{kl'} B^*_{l'l} \frac{\rho_k \rho^*_{l'}}{(k - l')(l' - l)}\, \boldsymbol{\phi}_l[A, B] = \rho_k \boldsymbol{c}_k\,, \tag{1.1}$$

which apart from the factors $A_{kl'}$, $B^*_{l'l}$ is the integral equation of type I as given in ref. 2. Here we shall focus mainly on this integral equation as an example, but other types of integral equations can be treated likewise, see also section 5 and the appendix.

The solutions $\boldsymbol{\phi}_k[A, B]$ of (1.1) are vectors with components $\phi^{(i)}_k[A, B]$, each associated with a component $c^{(i)}_k \equiv k^{-i}$ of the vector $\boldsymbol{c}_k$ in the source term, and A and B denote the functional dependence on the factors $A_{kk'}$ and $B_{kk'}$. (The dependence of functions on the spectral parameter k is indicated as a subscript.) As before[2]) the integrations in (1.1) are performed over a contour C in the complex k-plane and its complex conjugate C^*, with measures $\mathrm{d}\lambda(k)$ and $\mathrm{d}\lambda^*(k')$, respectively[#]. Contour and measure can be chosen arbitrarily apart from the condition that for given contour and measure the solution of the integral equation must be unique.

We write (1.1) as a coupled set of integral equations:

$$\boldsymbol{\phi}_k[A, B] + \int_{C^*} \mathrm{d}\lambda^*(l')\, A_{kl'} \frac{\rho_k}{k - l'}\, \boldsymbol{\psi}^*_{l'}[A, B] = \rho_k \boldsymbol{c}_k\,, \tag{1.2a}$$

[#] The values of the spectral parameter on C^* will be distinguished from those on C by a prime. For a function $f_k \equiv f(k)$ of the spectral parameter, we use the convention that $f^*_{k'} = (f(k'^*))^*$. For the vectors $\boldsymbol{\phi}_k[A, B]$, $\boldsymbol{\psi}_k[A, B]$ we have $(\boldsymbol{\phi}_k[A, B])^* = \boldsymbol{\phi}^*_{k'}[A, B]$, $(\boldsymbol{\psi}_k[A, B])^* = \boldsymbol{\psi}^*_{k'}[A, B]$ with $k' = k^*$.

$$\boldsymbol{\psi}_k[A, B] - \int_{C^*} \mathrm{d}\lambda^*(l')\, B_{kl'} \frac{\rho_k}{k - l'} \boldsymbol{\phi}^*_{l'}[A, B] = \boldsymbol{0} . \tag{1.2b}$$

The vector $\boldsymbol{\psi}_k[A, B]$ is the solution of the integral equation that is obtained from (1.1) by interchanging the roles of $A_{kl'}$ and $B_{kl'}$ and modifying the source term, viz.

$$\boldsymbol{\psi}_k[A, B] + \int_{C^*} \mathrm{d}\lambda^*(l') \int_{C} \mathrm{d}\lambda(l)\, B_{kl'} A^*_{l'l} \frac{\rho_k \overset{*}{\rho}_{l'}}{(k - l')(l' - l)} \boldsymbol{\psi}_l[A, B]$$

$$= \int_{C^*} \mathrm{d}\lambda^*(l')\, B_{kl'} \frac{\rho_k \overset{*}{\rho}_{l'}}{k - l'} \boldsymbol{c}_{l'} \tag{1.3}$$

(as follows immediately from (1.2)). From the solutions of (1.2) one obtains the $\infty \times \infty$ potential matrices

$$\boldsymbol{\Phi}[A, B] = \int_C \mathrm{d}\lambda(k)\, \boldsymbol{\phi}_k[A, B]\boldsymbol{c}_k , \quad \boldsymbol{\Psi}[A, B] = \int_C \mathrm{d}\lambda(k)\, \boldsymbol{\psi}_k[A, B]\boldsymbol{c}_k \tag{1.4}$$

by integration of dyadic expressions over the contour and with the measure used in (1.1). For the investigation of PDE's we choose ρ_k to be of the form

$$\rho_k \sim \exp(\mathrm{i}kx - \mathrm{i}\omega_k t) , \quad \omega_k = \sum_r \lambda_r k^r . \tag{1.5}$$

In the case of PDfE's we specify the dependence of ρ_k on the lattice-sites later on.

In ref. 2 we have investigated (1.1) in the special case that $A_{kl'} = B_{kl'} = 1$, yielding the DL of the NLS, the IHSC, the CSG and related PDE's. The case $A_{kl'} = k$, $B_{kl'} = 1$ was investigated in ref. 3, leading to the DNLS, the MTM and related PDE's. Furthermore, the investigation of modified PDE's by combining Bäcklund transforms in ref. 4, was shown to yield integral equations of the type (1.1) with $A_{kl'} = |p|^2 + kl'$, $B_{kl'} = 1$ (p is the Bäcklund parameter) associated with the AHSC. It is therefore natural to investigate the case where $A_{kk'}$ and $B_{kk'}$ are given by general expressions in terms of k of the form

$$A_{kl'} = \sum_{n,m} a_{n,m} k^n l'^{\,m} , \quad B_{kl'} = \sum_{n,m} b_{n,m} k^n l'^{\,m} , \tag{1.6}$$

with summations over a *finite* number of terms. For $A_{kl'}$, $B_{kl'}$ of this more general form, it is no longer possible to find closed PDE's in terms of a single element of $\boldsymbol{\Phi}$ by means of the DL method. It is always possible, however, to derive finite-matrix generalizations of the PDE's that were found before. The procedure of obtaining these multicomponent PDE's will be explained in the following sections.

There already exists an extensive literature on the subject of multicomponent integrable systems[8–19]). Most treatments start from a specific linear spectral problem, instead of from an integral equation as we do. The DL method, however, is more convenient to investigate the relations between different integrable systems. In particular, it provides a unifying framework to derive MT's, BT's, and lattice versions associated with a given PDE. Recently, investigations have been carried out on some general $N \times N$ matrix spectral problems[20–22]). Also for some matrix spectral problems a more general Riemann–Hilbert transform has been formulated[23–25]). But then there remains the difficult problem to establish specific reductions[26]) that can be imposed simultaneously on the spectral problem and on the integral transform (reductions only concerned with the spectral problem, for multicomponent systems, have been treated in refs. 11–14). We, therefore, prefer here to investigate the less general integral equation (1.1), which is already explicitly in reduced form. Starting in this way, one can then formulate the more general integral transforms yielding a larger class of solutions of integrable multicomponent systems at a later stage. There is another kind of reduction which might be of interest for obtaining multicomponent PDE's, and that is a reduction from $2+1$ dimensions to $1+1$ dimensions (cf. ref. 27, and also refs. 25, 28–30), but that kind of reduction will also not be considered in the present paper.

The outline of the present paper is as follows. In section 2 we give the constitutive relations, including also Bäcklund relations and some symmetry relations. These equations can be derived immediately from the integral equation (1.1) using (1.2)–(1.6), and they have a recursive structure expressed in terms of an index-shifting matrix. In section 3 this recursive structure is applied to derive an iterative algorithm corresponding to the "recursion operator" Ω (see ref. 2), and we show how the formulation in terms of $\infty \times \infty$ matrices implies a formulation in terms of finite matrices. In section 4 we treat the BT's and the Bianchi identities (leading to lattice versions of the multicomponent PDE's). In section 5 we give some concluding remarks, and discuss also the DL for multicomponent PDE's starting from an integral equation generalizing the one of type II in ref. 2, with two integrations over the same contour with the same measure, instead of over C with $d\lambda(k)$ and C^* with $d\lambda^*(k')$. The general case of an integral equation with two arbitrary contours and measures is considered in the appendix, and the reductions to the special cases of this paper are given.

In the investigation of the DL of multicomponent nonlinear integrable systems, presented in this paper, only the application to the multicomponent NLS and its hierarchy will be worked out explicitly. Further investigations of various other multicomponent PDE's, and MT's, will be presented in the following paper II.

2. Constitutive relations

In this section we shall present the basic relations that can be derived immediately from the integral equations (1.1), (1.3) or equivalently from the set of coupled integral equations (1.2), i.e. relations for the derivatives of $\boldsymbol{\phi}_k[A, B]$ and $\boldsymbol{\psi}_k[A, B]$ with respect to x and t (cf. (1.5)), algebraic equations involving multiplication with powers of k, Bäcklund relations, and some symmetry properties. Together with the basic relations for $\boldsymbol{\phi}_k[A, B]$ and $\boldsymbol{\psi}_k[A, B]$, constitutive relations for the potentials $\boldsymbol{\Phi}[A, B]$ and $\boldsymbol{\Psi}[A, B]$ are obtained by simple integrations as in (1.4). Furthermore there are a few symmetry relations for the potentials that follow from "quadratic identities", which are given at the end of the section.

The following notations will be used: we introduce index-raising and -lowering matrices $\mathbf{J}$ and $\mathbf{J}^{\mathrm{T}}$, with elements

$$(\mathbf{J})_{ij} \equiv \delta_{j,i+1} \,, \quad (\mathbf{J}^{\mathrm{T}})_{ij} \equiv \delta_{i,j+1} \,, \tag{2.1}$$

and a projection matrix $\mathbf{O}$, with elements*

$$(\mathbf{O})_{ij} \equiv \delta_{i,0}\delta_{j,0} \,. \tag{2.2}$$

Then we define a set of matrices $\mathbf{Q}_p$ by

$$\mathbf{Q}_p \equiv \begin{cases} \displaystyle\sum_{j=0}^{p-1} \mathbf{J}^j \cdot \mathbf{O} \cdot \mathbf{J}^{\mathrm{T}p-1-j} \,, & p \geq 0 \quad (2.3a) \\ & \qquad (p \text{ integer}) \,, \\ \displaystyle -\sum_{j=0}^{-p-1} \mathbf{J}^{p+j} \cdot \mathbf{O} \cdot \mathbf{J}^{\mathrm{T}-1-j} \,, & p \leq 0 \quad (2.3b) \end{cases}$$

($\mathbf{Q}_0 \equiv \mathbf{0}$), cf. also ref. 2. Associated with the functions $A_{kl'}$ and $B_{kl'}$ given in (1.6), we use the matrices

$$\mathbf{A} \equiv \sum_{n,m} a_{n,m} \mathbf{J}^m \cdot \mathbf{O} \cdot \mathbf{J}^{\mathrm{T}n} \,, \quad \mathbf{B} \equiv \sum_{n,m} b_{n,m} \mathbf{J}^m \cdot \mathbf{O} \cdot \mathbf{J}^{\mathrm{T}n} \tag{2.4}$$

* This matrix $\mathbf{O}$ should not be confused with the null matrix $\mathbf{0}$.

and the sets of matrices

$$\mathbf{A}_p \equiv \sum_{n,m} a_{n,m} \mathbf{J}^m \cdot \mathbf{Q}_p \cdot \mathbf{J}^{\mathrm{T}n} = \begin{cases} \sum_{j=0}^{p-1} \mathbf{J}^j \cdot \mathbf{A} \cdot \mathbf{J}^{\mathrm{T}p-1-j} , & p \geqslant 0 \quad (2.5a) \\ & (p \text{ integer}) \\ -\sum_{j=0}^{-p-1} \mathbf{J}^{p+j} \cdot \mathbf{A} \cdot \mathbf{J}^{\mathrm{T}-1-j} , & p \leqslant 0 \quad (2.5b) \end{cases}$$

and similarly (changing the associated function $A_{kl'}$ into $B_{kl'}$) $\mathbf{B}_p$. Obviously, $\mathbf{A} \equiv \mathbf{A}_1$, $\mathbf{B} \equiv \mathbf{B}_1$.

2.1. *Algebraic and differential relations*

Multiplying eq. (1.2a) by k^p, we have (p integer)

$$\begin{aligned} k^p \boldsymbol{\phi}_k[A, B] + \int_{C^*} \mathrm{d}\lambda^*(l')\, A_{kl'} \frac{\rho_k}{k - l'}\, l'^{\,p} \boldsymbol{\psi}^*_{l'}[A, B] \\ = k^p \rho_k \boldsymbol{c}_k - \rho_k \int_{C^*} \mathrm{d}\lambda^*(l')\, A_{kl'} \frac{k^p - l'^{\,p}}{k - l'} \boldsymbol{\psi}^*_{l'}[A, B] \\ = \mathbf{J}^{\mathrm{T}p} \cdot \rho_k \boldsymbol{c}_k - \boldsymbol{\Psi}^*[A, B] \cdot \mathbf{A}_p \cdot \rho_k \boldsymbol{c}_k . \end{aligned} \tag{2.6}$$

In the last step of (2.6), use has been made of the fact that

$$A_{kl'} \frac{k^p - l'^{\,p}}{k - l'} = A_{kl'} \boldsymbol{c}_{l'} \cdot \mathbf{Q}_p \cdot \boldsymbol{c}_k = \boldsymbol{c}_{l'} \cdot \mathbf{A}_p \cdot \boldsymbol{c}_k , \tag{2.7}$$

and an integration (1.4) has been applied. Similarly we have from (1.2b)

$$k^p \boldsymbol{\psi}_k[A, B] - \int_{C^*} \mathrm{d}\lambda^*(l')\, B_{kl'} \frac{\rho_k}{k - l'}\, l'^{\,p} \boldsymbol{\phi}^*_{l'}[A, B] = \boldsymbol{\Phi}^*[A, B] \cdot \mathbf{B}_p \cdot \rho_k \boldsymbol{c}_k . \tag{2.8}$$

Inserting $l'^{\,p} \boldsymbol{\psi}^*_{l'}[A, B]$ as determined by (2.8) into the integrand in the left-hand side of (2.6), we obtain an integral equation for $k^p \boldsymbol{\phi}_k[A, B]$, similar to (1.1) but with a different source term, namely

$$\begin{aligned} k^p \boldsymbol{\phi}_k[A, B] + \int_{C^*} \mathrm{d}\lambda^*(l') \int_{C} \mathrm{d}\lambda(l)\, A_{kl'} B^*_{l'l} \frac{\rho_k \rho^*_{l'}}{k - l'}\, l^p \boldsymbol{\phi}_l[A, B] \\ = (\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}^*[A, B] \cdot \mathbf{A}_p) \cdot \rho_k \boldsymbol{c}_k \\ - \boldsymbol{\Phi}[A, B] \cdot \mathbf{B}^*_p \cdot \int_{C^*} \mathrm{d}\lambda^*(l')\, A_{kl'} \frac{\rho_k \rho^*_{l'}}{k - l'} \boldsymbol{c}_{l'} . \end{aligned} \tag{2.9}$$

The first part of the source term in (2.9) is the r.h.s. of (1.1) multiplied by a matrix independent of k, the second part is the r.h.s. of (1.3) with the functions $A_{kl'}$ and $B_{kl'}$ interchanged, likewise multiplied by a matrix independent of k. Hence, taking into account that the homogeneous integral equation has only the zero solution, we obtain the following algebraic relation:

$$k^p \boldsymbol{\phi}_k[A, B] = (\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}^*[A, B] \cdot \mathbf{A}_p) \cdot \boldsymbol{\phi}_k[A, B] - \boldsymbol{\Phi}[A, B] \cdot \mathbf{B}_p^* \cdot \boldsymbol{\psi}_k[B, A] . \tag{2.10}$$

A similar argument applies when we insert $l'^{\,p} \boldsymbol{\phi}_{l'}^*[A, B]$ from (2.6) into (2.8). We then obtain an integral equation for $k^p \boldsymbol{\psi}_k[A, B]$, also with both types of contributions to the source term. Again using (1.1) and (1.3), we obtain the algebraic relation

$$k^p \boldsymbol{\psi}_k[A, B] = (\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}[A, B] \cdot \mathbf{A}_p^*) \cdot \boldsymbol{\psi}_k[A, B] + \boldsymbol{\Phi}^*[A, B] \cdot \mathbf{B}_p \cdot \boldsymbol{\phi}_k[B, A] . \tag{2.11}$$

The derivation of differential relations for $\boldsymbol{\phi}_k[A, B]$ and $\boldsymbol{\psi}_k[A, B]$ proceeds in a similar way. Let us introduce differential operators ∂_p (p integer) which act on ρ_k as follows:

$$\mathrm{i}\partial_p \rho_k = k^p \rho_k , \quad -\mathrm{i}\partial_p \rho_{l'}^* = l'^{\,p} \rho_{l'}^* . \tag{2.12}$$

Letting $\mathrm{i}\partial_p$ act on the integral equation (1.1), and using again the uniqueness of the solution, we obtain in an analogous way as in ref. 2

$$\mathrm{i}\partial_p \boldsymbol{\phi}_k[A, B] = (\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}^*[A, B] \cdot \mathbf{A}_p) \cdot \boldsymbol{\phi}_k[A, B] . \tag{2.13}$$

Similarly letting $\mathrm{i}\partial_p$ act on eq. (1.3), we find

$$\mathrm{i}\partial_p \boldsymbol{\psi}_k[A, B] = \boldsymbol{\Phi}^*[A, B] \cdot \mathbf{B}_p \cdot \boldsymbol{\phi}_k[B, A] . \tag{2.14}$$

The algebraic relations (2.10), (2.11) and the differential relations (2.13), (2.14) can immediately be integrated according to (1.4) to give matrix equations in terms of $\boldsymbol{\Phi}[A, B]$ and $\boldsymbol{\Psi}[A, B]$, namely

$$\boldsymbol{\Phi}[A, B] \cdot \mathbf{J}^p = (\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}^*[A, B] \cdot \mathbf{A}_p) \cdot \boldsymbol{\Phi}[A, B] - \boldsymbol{\Phi}[A, B] \cdot \mathbf{B}_p^* \cdot \boldsymbol{\Psi}[B, A] , \tag{2.15}$$

$$\boldsymbol{\Psi}[A, B] \cdot \mathbf{J}^p = (\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}[A, B] \cdot \mathbf{A}_p^*) \cdot \boldsymbol{\Psi}[A, B] + \boldsymbol{\Phi}^*[A, B] \cdot \mathbf{B}_p \cdot \boldsymbol{\Phi}[B, A] , \tag{2.16}$$

$$\mathrm{i}\partial_p \mathbf{\Phi}[A, B] = (\mathbf{J}^{\mathrm{T}p} - \mathbf{\Psi}^*[A, B] \cdot \mathbf{A}_p) \cdot \mathbf{\Phi}[A, B] , \tag{2.17}$$

$$\mathrm{i}\partial_p \mathbf{\Psi}[A, B] = \mathbf{\Phi}^*[A, B] \cdot \mathbf{B}_p \cdot \mathbf{\Phi}[B, A] . \tag{2.18}$$

The constitutive relations can now be easily written down, identifying ∂_p with the differentiations with respect to x and t, according to (1.5) and (2.12). These relations are a generalization of the ones presented in section 2 of ref. 2. Note that this generalization not only introduces matrices $\mathbf{A}_p, \mathbf{B}_p$ or their complex conjugates instead of $\mathbf{Q}_p$ ($\mathbf{A}, \mathbf{B}$ instead of $\mathbf{O}$ for $p = 1$), but also a coupling between quantities depending functionally on A, B and on B, A.

2.2. *Bäcklund relations*

Bäcklund transformations arise from singular transformations of the measure $\mathrm{d}\lambda(k)$ in the integral equation (1.1)[4]), or, equivalently, from transformations of the plane-wave factor ρ_k. Replacing ρ_k by

$$\tilde{\rho}_k = \theta \frac{p - k}{p^* - k} \rho_k , \quad |\theta| = 1 , \tag{2.19}$$

where p is a free parameter, the solutions $\boldsymbol{\phi}_k[A, B]$, $\boldsymbol{\psi}_k[A, B]$ of the integral equations (1.1), (1.3) become $\tilde{\boldsymbol{\phi}}_k[A, B]$, $\tilde{\boldsymbol{\psi}}_k[A, B]$, and the transformed potentials are defined by (cf. (1.4))

$$\tilde{\mathbf{\Phi}}[A, B] = \int_C \mathrm{d}\lambda(k)\, \tilde{\boldsymbol{\phi}}_k[A, B] \boldsymbol{c}_k , \quad \tilde{\mathbf{\Psi}}[A, B] = \int_C \mathrm{d}\lambda(k)\, \tilde{\boldsymbol{\psi}}_k[A, B] \boldsymbol{c}_k . \tag{2.20}$$

The coupled integral equations for $\tilde{\boldsymbol{\phi}}_k[A, B]$, $\tilde{\boldsymbol{\psi}}_k[A, B]$ (cf. (1.2)), multiplied by $p^* - k$, yield, using (2.19),

$$(p^* - k)\tilde{\boldsymbol{\phi}}_k[A, B] + \int_{C^*} \mathrm{d}\lambda^*(l')\, A_{kl'} \frac{\rho_k}{k - l'} \{\theta(p - l')\tilde{\boldsymbol{\psi}}^*_{l'}[A, B]\}$$

$$= \theta(p\mathbf{1} - \mathbf{J}^{\mathrm{T}}) \cdot \rho_k \boldsymbol{c}_k + \theta \tilde{\mathbf{\Psi}}^*[A, B] \cdot \mathbf{A} \cdot \rho_k \boldsymbol{c}_k , \tag{2.21a}$$

$$\theta^*(p^* - k)\tilde{\boldsymbol{\psi}}_k[A, B] - \int_{C^*} \mathrm{d}\lambda^*(l')\, B_{kl'} \frac{\rho_k}{k - l'} \{(p - l')\tilde{\boldsymbol{\phi}}^*_{l'}[A, B]\}$$

$$= -\tilde{\mathbf{\Phi}}^*[A, B] \cdot \mathbf{B} \cdot \rho_k \boldsymbol{c}_k . \tag{2.21b}$$

Inserting $\theta^*(p^* - k)\tilde{\boldsymbol{\psi}}_k[A, B]$ from (2.21b) into (2.21a), we obtain an integral equation for $(p^* - k)\boldsymbol{\phi}_k[A, B]$, which yields, using again the uniqueness of the solution of (1.1), (1.3),

$$(p^* - k)\tilde{\boldsymbol{\phi}}_k[A, B] = \theta(p\mathbf{1} - \mathbf{J}^{\mathrm{T}} + \tilde{\boldsymbol{\Psi}}^*[A, B] \cdot \mathbf{A}) \cdot \boldsymbol{\phi}_k[A, B] + \tilde{\boldsymbol{\Phi}}[A, B] \cdot \mathbf{B}^* \cdot \boldsymbol{\psi}_k[B, A] . \tag{2.22}$$

Similarly we get from (2.21)

$$(p^* - k)\tilde{\boldsymbol{\psi}}_k[A, B] = (p^*\mathbf{1} - \mathbf{J}^{\mathrm{T}} + \tilde{\boldsymbol{\Psi}}[A, B] \cdot \mathbf{A}^*) \cdot \boldsymbol{\psi}_k[A, B] - \theta\tilde{\boldsymbol{\Phi}}^*[A, B] \cdot \mathbf{B} \cdot \boldsymbol{\phi}_k[B, A] . \tag{2.23}$$

Eqs. (2.22) and (2.23) are the basic Bäcklund relations. Since, according to (2.19),

$$\rho_k = \theta^* \frac{p^* - k}{p - k} \tilde{\rho}_k , \tag{2.24}$$

we may also interchange the quantities with tilde and without tilde, at the same time interchanging $p \leftrightarrow p^*$, $\theta \leftrightarrow \theta^*$, in (2.22) and (2.23), giving

$$(p - k)\boldsymbol{\phi}_k[A, B] = \theta^*(p^*\mathbf{1} - \mathbf{J}^{\mathrm{T}} + \boldsymbol{\Psi}^*[A, B] \cdot \mathbf{A}) \cdot \tilde{\boldsymbol{\phi}}_k[A, B] + \boldsymbol{\Phi}[A, B] \cdot \mathbf{B}^* \cdot \tilde{\boldsymbol{\psi}}_k[B, A] , \tag{2.25}$$

$$(p - k)\boldsymbol{\psi}_k[A, B] = (p\mathbf{1} - \mathbf{J}^{\mathrm{T}} + \boldsymbol{\Psi}[A, B] \cdot \mathbf{A}^*) \cdot \tilde{\boldsymbol{\psi}}_k[A, B] - \theta^*\boldsymbol{\Phi}^*[A, B] \cdot \mathbf{B} \cdot \tilde{\boldsymbol{\phi}}_k[B, A] . \tag{2.26}$$

From (2.22), (2.23), (2.25) and (2.26) the equations for the potential matrices can be obtained by integration according to (1.4) and (2.20), and further elaborations on BT's will be presented in section 4.

2.3. *Symmetry relations*

There are a few relations for the dependence of the solutions $\boldsymbol{\phi}_k[A, B]$, $\boldsymbol{\psi}_k[A, B]$ of the integral equations (1.1), (1.3) on the functions $A_{kl'}$, $B_{kl'}$. Let us denote

$$(F \cdot A \cdot G)_{kl'} \equiv F_k A_{kl'} G_{l'} , \tag{2.27}$$

in which $F_k = \Sigma_n f_n k^n$ and $G_k = \Sigma_n g_n k^n$ (n integer) are arbitrary finite expansions in k. Since we have (cf. (1.2))

$$F_k^{-1}\boldsymbol{\phi}_k[F\cdot A\cdot G, B] + \int_{C^*} \mathrm{d}\lambda^*(l')\, A_{kl'}\, \frac{\rho_k}{k-l'}$$

$$\times \{G_{l'}\boldsymbol{\psi}^*_{l'}[F\cdot A\cdot G, B]\} = F_k^{-1}\rho_k \boldsymbol{c}_k\,, \tag{2.28a}$$

$$G^*_k\boldsymbol{\psi}_k[F\cdot A\cdot G, B] - \int_{C^*} \mathrm{d}\lambda^*(l')\, G^*_k B_{kl'} F^*_{l'}\, \frac{\rho_k}{k-l'}$$

$$\times \{F^{*\,-1}_{l'}\boldsymbol{\phi}^*_{l'}[F\cdot A\cdot G, B]\} = \boldsymbol{0}\,, \tag{2.28b}$$

we can conclude at once that

$$F(\mathbf{J}^{\mathrm{T}})\cdot F_k^{-1}\boldsymbol{\phi}_k[F\cdot A\cdot G, B] = \boldsymbol{\phi}_k[A, G^*\cdot B\cdot F^*]\,, \tag{2.29a}$$

$$F(\mathbf{J}^{\mathrm{T}})\cdot G^*_k\boldsymbol{\psi}_k[F\cdot A\cdot G, B] = \boldsymbol{\psi}_k[A, G^*\cdot B\cdot F^*]\,, \tag{2.29b}$$

in which $F(\mathbf{J}^{\mathrm{T}}) = \Sigma_n f_n \mathbf{J}^{\mathrm{T}^n}$. Moreover we can use the relation

$$\{(F\cdot A\cdot G)_{kl'} - (G\cdot A\cdot F)_{kl'}\}\,\frac{1}{k-l'} \equiv A_{kl'}\,\frac{F_k G_{l'} - G_k F_{l'}}{k-l'}$$

$$= \boldsymbol{c}_{l'}\cdot \mathbf{A}[F, G]\cdot \boldsymbol{c}_k\,, \tag{2.30}$$

with (cf. (2.7))

$$\mathbf{A}[F, G] = \sum_{n,m} f_n g_m \mathbf{J}^m\cdot \mathbf{A}_{n-m}\cdot \mathbf{J}^{\mathrm{T}^m} = -\sum_{n,m} f_n g_m \mathbf{J}^n\cdot \mathbf{A}_{m-n}\cdot \mathbf{J}^{\mathrm{T}^n}\,, \tag{2.31}$$

to derive relations between solutions $\boldsymbol{\phi}_k[F\cdot A\cdot G, B]$ and $\boldsymbol{\phi}_k[G\cdot A\cdot F, B]$, etc. From (1.2a) we get in this way

$$\boldsymbol{\phi}_k[F\cdot A\cdot G, B] + \int_{C^*} \mathrm{d}\lambda^*(l')\, G_k A_{kl'} F_{l'}\, \frac{\rho_k}{k-l'}\, \boldsymbol{\psi}^*_{l'}[F\cdot A\cdot G, B]$$

$$= (\mathbf{1} - \boldsymbol{\Psi}^*[F\cdot A\cdot G, B]\cdot \mathbf{A}[F, G])\cdot \rho_k \boldsymbol{c}_k\,, \tag{2.32}$$

which equation, together with (1.2b) where we substitute $F\cdot A\cdot G$ for A, implies that

$$\boldsymbol{\phi}_k[F\cdot A\cdot G, B] = (\mathbf{1} - \boldsymbol{\Psi}^*[F\cdot A\cdot G, B]\cdot \mathbf{A}[F, G])\cdot \boldsymbol{\phi}_k[G\cdot A\cdot F, B]\,, \tag{2.33a}$$

$$\boldsymbol{\psi}_k[F\cdot A\cdot G, B] = (\mathbf{1} - \boldsymbol{\Psi}[F\cdot A\cdot G, B]\cdot \mathbf{A}^*[F, G])\cdot \boldsymbol{\psi}_k[G\cdot A\cdot F, B]\,. \tag{2.33b}$$

With $\mathbf{B}[F, G]$ defined by the analogue of (2.31), we get from (1.2b)

$$\boldsymbol{\psi}_k[A, F \cdot B \cdot G] - \int\limits_{C^*} \mathrm{d}\lambda^*(l')\, G_k B_{kl'} F_{l'} \frac{\rho_k}{k - l'} \boldsymbol{\phi}^*_{l'}[A, F \cdot B \cdot G]$$

$$= \boldsymbol{\Phi}^*[A, F \cdot B \cdot G] \cdot \mathbf{B}[F, G] \cdot \rho_k \boldsymbol{c}_k\,, \tag{2.34}$$

or, using (1.2a) again,

$$\boldsymbol{\psi}_k[A, F \cdot B \cdot G] - \boldsymbol{\Phi}^*[A, F \cdot B \cdot G] \cdot \mathbf{B}[F, G] \cdot \boldsymbol{\phi}_k[G \cdot B \cdot F, A]$$

$$- \int\limits_{C^*} \mathrm{d}\lambda^*(l')\, G_k B_{kl'} F_{l'} \frac{\rho_k}{k - l'} (\boldsymbol{\phi}^*_{l'}[A, F \cdot B \cdot G]$$

$$+ \boldsymbol{\Phi}^*[A, F \cdot B \cdot G] \cdot \mathbf{B}[F, G] \cdot \boldsymbol{\psi}^*_{l'}[G \cdot B \cdot F, A]) = \boldsymbol{0}\,. \tag{2.35}$$

From this equation and the corresponding coupled equation, it follows that

$$\boldsymbol{\phi}_k[A, F \cdot B \cdot G] + \boldsymbol{\Phi}[A, F \cdot B \cdot G] \cdot \mathbf{B}^*[F, G] \cdot \boldsymbol{\psi}_k[G \cdot B \cdot F, A]$$

$$= \boldsymbol{\phi}_k[A, G \cdot B \cdot F]\,, \tag{2.36a}$$

$$\boldsymbol{\psi}_k[A, F \cdot B \cdot G] - \boldsymbol{\Phi}^*[A, F \cdot B \cdot G] \cdot \mathbf{B}[F, G] \cdot \boldsymbol{\phi}_k[G \cdot B \cdot F, A]$$

$$= \boldsymbol{\psi}_k[A, G \cdot B \cdot F]\,. \tag{2.36b}$$

Eqs. (2.33) and (2.36) can also be integrated by means of (1.4).

2.4. *Quadratic identities and resulting symmetry relations*

We conclude this section by deriving symmetry relations for the potential matrices $\boldsymbol{\Phi}[A, B]$ and $\boldsymbol{\Psi}[A, B]$ under transposition and hermitian conjugation, respectively. This is done via so-called quadratic identities, i.e. expressions for the potential matrices in terms of integrations over quadratic forms in the solutions of the integral equations (1.1) and (1.3). Here, associated to the factors $A_{kl'}$ and $B_{kl'}$, we use also $A^\dagger_{kl'}$ and $B^\dagger_{kl'}$, where the operation $A \to A^\dagger$ is defined by

$$A^\dagger_{kl'} \equiv A^*_{l'k}\,, \tag{2.37}$$

and $B \to B^\dagger$ in the same way. From (1.4) and (1.2) we then have

$$\begin{aligned}\boldsymbol{\Phi}[A,B] &= \int_C \mathrm{d}\lambda(k)\,\boldsymbol{\phi}_k[A,B]\Big(\frac{1}{\rho_k}\,\boldsymbol{\phi}_k[B^\dagger,A^\dagger] \\ &\quad + \int_{C^*} \mathrm{d}\lambda^*(l')\,B^\dagger_{kl'}\,\frac{1}{k-l'}\,\boldsymbol{\psi}^*_{l'}[B^\dagger,A^\dagger]\Big) \\ &= \int_C \mathrm{d}\lambda(k)\,\frac{1}{\rho_k}\,\boldsymbol{\phi}_k[A,B]\boldsymbol{\phi}_k[B^\dagger,A^\dagger] \\ &\quad - \int_{C^*} \mathrm{d}\lambda^*(l')\Big(\int_C \mathrm{d}\lambda(k)\,B^*_{l'k}\,\frac{1}{l'-k}\,\boldsymbol{\phi}_k[A,B]\Big)\boldsymbol{\psi}^*_{l'}[B^\dagger,A^\dagger] \\ &= \int_C \mathrm{d}\lambda(k)\,\frac{1}{\rho_k}\,\boldsymbol{\phi}_k[A,B]\boldsymbol{\phi}_k[B^\dagger,A^\dagger] \\ &\quad - \int_{C^*} \mathrm{d}\lambda^*(l')\,\frac{1}{\rho^*_{l'}}\,\boldsymbol{\psi}^*_{l'}[A,B]\boldsymbol{\psi}^*_{l'}[B^\dagger,A^\dagger]\,. \end{aligned} \tag{2.38}$$

Eq. (2.38) is the quadratic identity for $\boldsymbol{\Phi}[A,B]$, and it shows that

$$\boldsymbol{\Phi}[A,B] = (\boldsymbol{\Phi}[B^\dagger,A^\dagger])^{\mathrm{T}} \equiv \boldsymbol{\Phi}^{\mathrm{T}}[B^\dagger,A^\dagger]\,, \tag{2.39}$$

where the superscript T denotes matrix transposition. The expression for $\boldsymbol{\Psi}[A,B]$ in (1.4) can be rewritten as

$$\begin{aligned}\boldsymbol{\Psi}[A,B] &= \int_C \mathrm{d}\lambda(k)\,\boldsymbol{\psi}_k[A,B]\Big(\frac{1}{\rho_k}\,\boldsymbol{\phi}_k[A^\dagger,B^\dagger] \\ &\quad + \int_{C^*} \mathrm{d}\lambda^*(l')\,A^\dagger_{kl'}\,\frac{1}{k-l'}\,\boldsymbol{\psi}^*_{l'}[A^\dagger,B^\dagger]\Big) \\ &= \int_C \mathrm{d}\lambda(k)\int_{C^*} \mathrm{d}\lambda^*(l')\,\frac{1}{k-l'}\,(B_{kl'}\boldsymbol{\phi}^*_{l'}[A,B]\boldsymbol{\phi}_k[A^\dagger,B^\dagger] \\ &\quad + A^*_{l'k}\boldsymbol{\psi}_k[A,B]\boldsymbol{\psi}^*_{l'}[A^\dagger,B^\dagger])\,, \end{aligned} \tag{2.40}$$

where again use has been made of (1.2). From the quadratic identity (2.40) for $\boldsymbol{\Psi}[A,B]$ it follows that

$$\boldsymbol{\Psi}^*[A,B] = -\boldsymbol{\Psi}^{\mathrm{T}}[A^\dagger,B^\dagger] \quad \text{or} \quad \boldsymbol{\Psi}[A,B] = -\boldsymbol{\Psi}^\dagger[A^\dagger,B^\dagger]\,. \tag{2.41}$$

Eqs. (2.39) and (2.41) are generalizations of the symmetry relations resulting from quadratic identities in ref. 2.

3. The multicomponent NLS and its hierarchy

Having obtained the complete set of basic differential and algebraic equations, as well as symmetry relations, we can now derive matrix PDE's. According to (1.5) and (2.12), the differential relations (2.13), (2.14), (2.17) and (2.18) yield

$$-\mathrm{i}\partial_x \boldsymbol{\phi}_k[A, B] = \boldsymbol{F}_k^{(1)}[A, B]\,, \qquad -\mathrm{i}\partial_x \boldsymbol{\psi}_k[A, B] = \boldsymbol{G}_k^{(1)}[A, B]\,, \tag{3.1}$$

$$\mathrm{i}\partial_t \boldsymbol{\phi}_k[A, B] = \sum_r \lambda_r \boldsymbol{F}_k^{(r)}[A, B]\,, \quad \mathrm{i}\partial_t \boldsymbol{\psi}_k[A, B] = \sum_r \lambda_r \boldsymbol{G}_k^{(r)}[A, B]\,, \tag{3.2}$$

where

$$\boldsymbol{F}_k^{(p)}[A, B] \equiv (\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}^*[A, B]\cdot \mathbf{A}_p)\cdot \boldsymbol{\phi}_k[A, B]\,, \tag{3.3}$$

$$\boldsymbol{G}_k^{(p)}[A, B] \equiv \boldsymbol{\Phi}^*[A, B]\cdot \mathbf{B}_p \cdot \boldsymbol{\phi}_k[B, A]\,, \tag{3.4}$$

and the integrated versions of (3.1) and (3.2) expressed in terms of (cf. (1.4))

$$\begin{aligned} \mathbf{F}^{(p)}[A, B] &= \int_C \mathrm{d}\lambda(k)\, \boldsymbol{F}_k^{(p)}[A, B]\boldsymbol{c}_k\,, \\ \mathbf{G}^{(p)}[A, B] &= \int_C \mathrm{d}\lambda(k)\, \boldsymbol{G}_k^{(p)}[A, B]\boldsymbol{c}_k\,. \end{aligned} \tag{3.5}$$

In this section we shall show that there exists an iterative scheme to obtain $\boldsymbol{F}_k^{(p)}[A, B]$ and $\boldsymbol{G}_k^{(p)}[A, B]$ for $p = 2, 3, \ldots$, completely analogous, but with the matrices $\mathbf{A}$, $\mathbf{B}$ instead of $\mathbf{O}$, to the scheme presented in ref. 2. After that we shall show that the results for $\infty \times \infty$ matrices reduce to equations for finite matrices.

3.1. *Iterative scheme*

First we mention a few properties of the quantities defined in (3.3) and (3.4). It is obvious from (3.4) that

$$\mathbf{G}^{(q)*}[A, B]\cdot \mathbf{A}_p \cdot \boldsymbol{\phi}_k[A, B] = \boldsymbol{\Phi}[A, B]\cdot \mathbf{B}_q^* \cdot \boldsymbol{G}_k^{(p)}[B, A]\,. \tag{3.6}$$

Two other properties follow with the help of a recursion relation for the matrices $\mathbf{A}_p$ (or $\mathbf{B}_p$) as defined in (2.5),

$$\mathbf{A}_p \cdot \mathbf{J}^{\mathrm{T}q} + \mathbf{J}^p \cdot \mathbf{A}_q = \mathbf{A}_{p+q}\,. \tag{3.7}$$

Using (3.3), (2.16), (3.7) and (3.4), we find that

$$(\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}^*[A, B]\cdot \mathbf{A}_p)\cdot \boldsymbol{F}_k^{(q)}[A, B] = \boldsymbol{F}_k^{(p+q)}[A, B] + \mathbf{G}^{(p)*}[A, B]\cdot \mathbf{A}_q \cdot \boldsymbol{\phi}_k[A, B]\,, \tag{3.8}$$

and, using (3.3), (2.15), (3.7) and (3.4), that

$$\boldsymbol{\Phi}^*[A, B]\cdot \mathbf{B}_p \cdot \boldsymbol{F}_k^{(q)}[B, A] = \boldsymbol{G}_k^{(p+q)}[A, B] - \mathbf{F}^{(p)*}[A, B]\cdot \mathbf{B}_q \cdot \boldsymbol{\phi}_k[B, A]\,. \tag{3.9}$$

For a differential operator ∂_q satisfying (2.12), we have, according to (2.17), (2.18), (3.3) and (3.4),

$$\mathrm{i}\partial_q \boldsymbol{\phi}_k[A, B] = \boldsymbol{F}_k^{(q)}[A, B]\,, \quad \mathrm{i}\partial_q \boldsymbol{\psi}_k[A, B] = \boldsymbol{G}_k^{(q)}[A, B]\,, \tag{3.10}$$

for instance, when $q = 1$ we have $\partial_q = -\partial_x$ and (3.10) becomes (3.1). Now it follows from (3.3) with (3.10), (3.8) and (3.6) that

$$\boldsymbol{F}_k^{(p+q)}[A, B] = \mathrm{i}\partial_q \boldsymbol{F}_k^{(p)}[A, B] - \mathbf{G}^{(p)*}[A, B]\cdot \mathbf{A}_q \cdot \boldsymbol{\phi}_k[A, B] - \boldsymbol{\Phi}[A, B]\cdot \mathbf{B}_q^* \cdot \boldsymbol{G}_k^{(p)}[B, A]\,. \tag{3.11}$$

Similarly it follows from (3.4) with (3.10), and using the relation which is obtained by subtracting (3.9) from its counterpart with p and q interchanged, that

$$\mathrm{i}\partial_q \boldsymbol{G}_k^{(p)}[A, B] = -\mathbf{F}^{(p)*}[A, B]\cdot \mathbf{B}_q \cdot \boldsymbol{\phi}_k[B, A] + \boldsymbol{\Phi}^*[A, B]\cdot \mathbf{B}_q \cdot \boldsymbol{F}_k^{(p)}[B, A]\,. \tag{3.12}$$

Solving $\boldsymbol{G}_k^{(p)}[A, B]$ formally in terms of $\boldsymbol{F}_k^{(p)}[A, B]$ from (3.12) and inserting the result in (3.11), one can derive an equation which determines $\boldsymbol{F}_k^{(p+q)}[A, B]$ in terms of $\boldsymbol{F}_k^{(p)}[A, B]$.

Restricting ourselves here to the case $q = 1$, $\partial_q = -\partial_x$, we introduce a formal operator ∂_x^{-1} which denotes an indefinite integration over x (i.e. containing an arbitrary integration constant). Then (3.11), (3.12) yield

$$\boldsymbol{F}_k^{(p+1)}[A, B] = \Omega \boldsymbol{F}_k^{(p)}[A, B]\,, \tag{3.13a}$$

$$\begin{aligned}\Omega \boldsymbol{F}_k^{(p)}[A, B] \equiv {} & -\mathrm{i}\partial_x \boldsymbol{F}_k^{(p)}[A, B] \\ & - \mathrm{i}\boldsymbol{\Phi}[A, B]\cdot \mathbf{B}^* \cdot \partial_x^{-1}(\boldsymbol{\Phi}^*[B, A]\cdot \mathbf{A}\cdot \boldsymbol{F}_k^{(p)}[A, B] \\ & - \mathbf{F}^{(p)*}[B, A]\cdot \mathbf{A}\cdot \boldsymbol{\phi}_k[A, B])\end{aligned}$$

$$+\,\mathrm{i}\{\partial_x^{-1}(\mathbf{\Phi}[A,B]\cdot\mathbf{B}^*\cdot\mathbf{F}^{(p)*}[B,A] - \mathbf{F}^{(p)}[A,B]\cdot\mathbf{B}^*\cdot\mathbf{\Phi}^*[B,A])\}\cdot\mathbf{A}\cdot\boldsymbol{\phi}_k[A,B]\,, \tag{3.13b}$$

$$\boldsymbol{G}_k^{(p+1)}[A,B] = \mathrm{i}\partial_x^{-1}(\mathbf{\Phi}^*[A,B]\cdot\mathbf{B}\cdot\boldsymbol{F}_k^{(p+1)}[B,A] - \mathbf{F}^{(p+1)*}[B,A]\cdot\mathbf{B}\cdot\boldsymbol{\phi}_k[B,A])\,. \tag{3.13c}$$

Eqs. (3.13) form the iterative scheme which enables us to evaluate $\boldsymbol{F}_k^{(r)}[A,B]$ and $\boldsymbol{G}_k^{(r)}[A,B]$ for $r\geq 2$ starting from $\boldsymbol{F}_k^{(1)}[A,B]$ as given in (3.1). It can be shown that the undetermined integration constants involved in the definition of ∂_x^{-1} drop out of the result. The proof of this is completely the same as the one given in appendix A of ref. 2, and will not be repeated here. Starting the iterative scheme (3.13a) from $\boldsymbol{F}_k^{(0)}[A,B]\equiv\boldsymbol{\phi}_k[A,B]$, we write (3.2) as

$$\mathrm{i}\partial_t\boldsymbol{\phi}_k[A,B] = \sum_{r\geq 2}\lambda_r\Omega^r\boldsymbol{\phi}_k[A,B] \quad \text{when} \quad \omega_k = \sum_{r\geq 2}\lambda_r k^r\,. \tag{3.14}$$

The operator Ω in (3.13b) is generally referred to as the recursion operator, or (hierarchy-) generating operator. For multicomponent systems such an operator has been treated before in ref. 21.

Up to the third order the iterative scheme (3.13) with (3.1) gives

$$\boldsymbol{F}_k^{(2)}[A,B] = -\partial_x^2\boldsymbol{\phi}_k[A,B] - 2\mathbf{\Phi}[A,B]\cdot\mathbf{B}^*\cdot\mathbf{\Phi}^*[B,A]\cdot\mathbf{A}\cdot\boldsymbol{\phi}_k[A,B]\,, \tag{3.15a}$$

$$\boldsymbol{G}_k^{(2)}[A,B] = \mathrm{i}\{(\partial_x\mathbf{\Phi}^*[A,B])\cdot\mathbf{B}\cdot\boldsymbol{\phi}_k[B,A] - \mathbf{\Phi}^*[A,B]\cdot\mathbf{B}\cdot\partial_x\boldsymbol{\phi}_k[B,A]\}\,. \tag{3.15b}$$

$$\boldsymbol{F}_k^{(3)}[A,B] = \mathrm{i}\partial_x^3\boldsymbol{\phi}_k[A,B] + 3\mathrm{i}\{(\partial_x\mathbf{\Phi}[A,B])\cdot\mathbf{B}^*\cdot\mathbf{\Phi}^*[B,A]\cdot\mathbf{A}\cdot\boldsymbol{\phi}_k[A,B] + \mathbf{\Phi}[A,B]\cdot\mathbf{B}^*\cdot\mathbf{\Phi}^*[B,A]\cdot\mathbf{A}\cdot\partial_x\boldsymbol{\phi}_k[A,B]\}\,, \tag{3.16a}$$

$$\begin{aligned} G_k^{(3)}[A,B] = &-(\partial_x^2\mathbf{\Phi}^*[A,B])\cdot\mathbf{B}\cdot\boldsymbol{\phi}_k[B,A] - \mathbf{\Phi}^*[A,B]\cdot\mathbf{B}\cdot\partial_x^2\boldsymbol{\phi}_k[B,A] \\ &+ (\partial_x\mathbf{\Phi}^*[A,B])\cdot\mathbf{B}\cdot\partial_x\boldsymbol{\phi}_k[B,A] \\ &- 3\mathbf{\Phi}^*[A,B]\cdot\mathbf{B}\cdot\mathbf{\Phi}[B,A]\cdot\mathbf{A}^*\cdot\mathbf{\Phi}^*[A,B]\cdot\mathbf{B}\cdot\boldsymbol{\phi}_k[B,A]\,. \end{aligned} \tag{3.16b}$$

Using the results (3.15a) and (3.16a) in (3.14), we have established the DL with $\omega_k = \lambda_2 k^2 + \lambda_3 k^3$ for the equation

$$
\begin{aligned}
\mathrm{i}\partial_t \boldsymbol{\Phi}[A, B] = {} & (-\lambda_2\partial_x^2 + \mathrm{i}\lambda_3\partial_x^3)\boldsymbol{\Phi}[A, B] \\
& - 2\lambda_2 \boldsymbol{\Phi}[A, B]\cdot \mathbf{B}^* \cdot \boldsymbol{\Phi}^*[B, A]\cdot \mathbf{A}\cdot \boldsymbol{\Phi}[A, B] \\
& + 3\mathrm{i}\lambda_3\{(\partial_x \boldsymbol{\Phi}[A, B])\cdot \mathbf{B}^*\cdot \boldsymbol{\Phi}^*[B, A]\cdot \mathbf{A}\cdot \boldsymbol{\Phi}[A, B] \\
& + \boldsymbol{\Phi}[A, B]\cdot \mathbf{B}^*\cdot \boldsymbol{\Phi}^*[B, A]\cdot \mathbf{A}\cdot \partial_x \boldsymbol{\Phi}[A, B]\} . \qquad (3.17)
\end{aligned}
$$

The scattering problem associated to (3.17) is therefore given by (3.1) with (3.3) and (3.4),

$$-\mathrm{i}\partial_x \boldsymbol{\phi}_k[A, B] = k\boldsymbol{\phi}_k[A, B] + \boldsymbol{\Phi}[A, B]\cdot \mathbf{B}^* \cdot \boldsymbol{\psi}_k[B, A] , \qquad (3.18a)$$

$$-\mathrm{i}\partial_x \boldsymbol{\psi}_k[B, A] = \boldsymbol{\Phi}^*[B, A]\cdot \mathbf{A}\cdot \boldsymbol{\phi}_k[A, B] , \qquad (3.18b)$$

where in (3.18a) also (2.10) has been used. Note that (3.17) does not contain the index-shifting matrix $\mathbf{J}$. This feature will enable us to derive immediately a finite-matrix PDE, as will be shown in the second part of this section.

The iterative scheme (3.13) works only for polynomial dispersion. As for other types of dispersion, we consider here only the special case $\omega_k = k^{-1}$. For $p = -q = -1$, (3.7) gives $\mathbf{A}_{-1} = -\mathbf{J}^{\mathrm{T}}\cdot \mathbf{A}\cdot \mathbf{J}$ (and a similar relation for $\mathbf{B}_{-1}$), and (3.11) with (3.4) gives

$$
\begin{aligned}
-\mathrm{i}\partial_x \boldsymbol{F}_k^{(-1)}[A, B] = {} & \boldsymbol{\phi}_k[A, B] + \boldsymbol{\Phi}[A, B]\cdot \mathbf{B}_{-1}^*\cdot \boldsymbol{\Phi}^*[B, A]\cdot \mathbf{A}\cdot \boldsymbol{\phi}_k[A, B] \\
& + \boldsymbol{\Phi}[A, B]\cdot \mathbf{B}^*\cdot \boldsymbol{\Phi}^*[B, A]\cdot \mathbf{A}_{-1}\cdot \boldsymbol{\phi}_k[A, B] . \qquad (3.19)
\end{aligned}
$$

Hence we obtain the equation

$$
\begin{aligned}
\partial_x\partial_t \boldsymbol{\Phi}[A, B] = {} & \boldsymbol{\Phi}[A, B] - \boldsymbol{\Phi}[A, B]\cdot \mathbf{J}^{\mathrm{T}}\cdot \mathbf{B}^*\cdot \mathbf{J}\cdot \boldsymbol{\Phi}^*[B, A]\cdot \mathbf{A}\cdot \boldsymbol{\Phi}[A, B] \\
& - \boldsymbol{\Phi}[A, B]\cdot \mathbf{B}^*\cdot \boldsymbol{\Phi}^*[B, A]\cdot \mathbf{J}^{\mathrm{T}}\cdot \mathbf{A}\cdot \mathbf{J}\cdot \boldsymbol{\Phi}[A, B] , \qquad (3.20)
\end{aligned}
$$

which, in contrast to (3.17), contains the matrices $\mathbf{J}$ and $\mathbf{J}^{\mathrm{T}}$ explicitly. This feature requires a separate treatment which will be given in the following paper. In the remaining part of this section we shall investigate further the case of polynomial dispersion, exemplified by eq. (3.17).

3.2. *Finite matrices*

Let the functions $A_{kl'}$ and $B_{kl'}$, occurring in the integral equation and given in (1.6), be finite sums of the form

$$A_{kl'} = \sum_{\alpha=1}^{N_\alpha} \sum_{\alpha'=1}^{N'_{\alpha'}} a_{-r'_{\alpha'}, -r_\alpha} k^{-r'_{\alpha'}} l'^{\,-r_\alpha} ,$$

$$B_{kl'} = \sum_{\beta=1}^{N_\beta} \sum_{\beta'=1}^{N'_{\beta'}} b_{-s'_{\beta'}, -s_\beta} k^{-s'_{\beta'}} l'^{\,-s_\beta} . \tag{3.21}$$

Then from (2.4) with (2.1)–(2.3) we have

$$\begin{aligned} &(\mathbf{A})_{r_\alpha r'_{\alpha'}} = a_{-r'_{\alpha'}, -r_\alpha} \quad \text{and} \quad (\mathbf{A})_{ij} = 0 \quad \text{for } i \neq r_\alpha,\ j \neq r'_{\alpha'} \\ &\qquad (\alpha = 1, \ldots, N_\alpha;\ \alpha' = 1, \ldots, N'_{\alpha'}) , \\ &(\mathbf{B})_{s_\beta s'_{\beta'}} = b_{-s'_{\beta'}, -s_\beta} \quad \text{and} \quad (\mathbf{B})_{ij} = 0 \quad \text{for } i \neq s_\beta,\ j \neq s'_{\beta'} \\ &\qquad (\beta = 1, \ldots, N_\beta;\ \beta' = 1, \ldots, N'_{\beta'}) . \end{aligned} \tag{3.22}$$

In other words, the $\infty \times \infty$ matrices $\mathbf{A}$ and $\mathbf{B}$ are vanishing except for finite submatrices $\mathbf{a}$ and $\mathbf{b}$, respectively, where the $N_\alpha \times N'_{\alpha'}$ matrix $\mathbf{a}$ and the $N_\beta \times N'_{\beta'}$ matrix $\mathbf{b}$ are defined by

$$(\mathbf{a})_{\alpha\alpha'} \equiv a_{-r'_{\alpha'}, -r_\alpha} , \quad (\mathbf{b})_{\beta\beta'} \equiv b_{-s'_{\beta'}, -s_\beta} . \tag{3.23}$$

Also corresponding to (3.21), we introduce finite submatrices of the $\infty \times \infty$ potential matrices $\Phi[A, B]$ and $\Phi[B, A]$: the $N'_{\alpha'} \times N_\beta$ matrix $\Phi(\mathbf{a}, \mathbf{b})$ and the $N'_{\beta'} \times N_\alpha$ matrix $\Phi(\mathbf{b}, \mathbf{a})$, respectively, with elements

$$(\Phi(\mathbf{a}, \mathbf{b}))_{\alpha'\beta} \equiv (\Phi[A, B])_{r'_{\alpha'} s_\beta} , \quad (\Phi(\mathbf{b}, \mathbf{a}))_{\beta'\alpha} \equiv (\Phi[B, A])_{s'_{\beta'} r_\alpha} . \tag{3.24}$$

Now, restricting the equation (3.17) for $\infty \times \infty$ matrices to the $(r'_{\alpha'}, s_\beta)$ elements, we get, with $\alpha' = 1, \ldots, N'_{\alpha'}$ and $\beta = 1, \ldots, N_\beta$, the equation for finite matrices:

$$\begin{aligned} \mathrm{i}\partial_t \Phi(\mathbf{a}, \mathbf{b}) = {} & (-\lambda_2 \partial_x^2 + \mathrm{i}\lambda_3 \partial_x^3) \Phi(\mathbf{a}, \mathbf{b}) \\ & - 2\lambda_2 \Phi(\mathbf{a}, \mathbf{b}) \cdot \mathbf{b}^* \cdot \Phi^*(\mathbf{b}, \mathbf{a}) \cdot \mathbf{a} \cdot \Phi(\mathbf{a}, \mathbf{b}) \\ & + 3\mathrm{i}\lambda_3 \{ (\partial_x \Phi(\mathbf{a}, \mathbf{b})) \cdot \mathbf{b}^* \cdot \Phi^*(\mathbf{b}, \mathbf{a}) \cdot \mathbf{a} \cdot \Phi(\mathbf{a}, \mathbf{b}) \\ & + \Phi(\mathbf{a}, \mathbf{b}) \cdot \mathbf{b}^* \cdot \Phi^*(\mathbf{b}, \mathbf{a}) \cdot \partial_x \Phi(\mathbf{a}, \mathbf{b}) \} . \end{aligned} \tag{3.25}$$

This result is the generalization, in the case of arbitrary $A_{kl'}$ and $B_{kl'}$, of the equation that is obtained by taking the $(0, 0)$ element in ref. 2 (where $N_\alpha = N'_{\alpha'} = N_\beta = N'_{\beta'} = 1$). One can also consider other reductions, generalizing the equations obtained by taking of $(0, 1)$ and $(1, 1)$ elements in ref. 2, but this will be treated in the following paper.

The scattering problem associated to (3.25) can be formulated in terms of an $(N'_{\alpha'} + N'_{\beta'})$-dimensional vector $\boldsymbol{\chi}_k(\mathbf{a}, \mathbf{b})$, defined by

$$\boldsymbol{\chi}_k(\mathbf{a}, \mathbf{b}) = \begin{bmatrix} (\boldsymbol{\phi}_k[A, B])_{r'_{\alpha'}} \\ (\boldsymbol{\psi}_k[B, A])_{s'_{\beta'}} \end{bmatrix} \mathrm{e}^{-\mathrm{i}(kx - \omega_k t)/2} \tag{3.26}$$

From (3.18) we have

$$-\mathrm{i}\partial_x \boldsymbol{\chi}_k(\mathbf{a}, \mathbf{b}) = \begin{bmatrix} \frac{1}{2}\, k\, \mathbf{1}^{(N'_{\alpha'})} & \boldsymbol{\Phi}(\mathbf{a}, \mathbf{b}) \cdot \mathbf{b}^* \\ \boldsymbol{\Phi}^*(\mathbf{b}, \mathbf{a}) \cdot \mathbf{a} & -\frac{1}{2}\, k\, \mathbf{1}^{(N'_{\beta'})} \end{bmatrix} \cdot \boldsymbol{\chi}_k(\mathbf{a}, \mathbf{b})\,, \tag{3.27}$$

in which $\mathbf{1}^{(N)}$ is the $N \times N$ unit matrix. It is possible to obtain the complete Lax representation of (3.25) by computing also $\mathrm{i}\partial_t \boldsymbol{\chi}_k(a, b)$ from (3.26) with (3.15) and (3.16).

Due to the occurrence of $\boldsymbol{\Phi}(\mathbf{b}, \mathbf{a})$ as well as $\boldsymbol{\Phi}(\mathbf{a}, \mathbf{b})$, eq. (3.25) is in general not a closed matrix PDE, but it forms with its counterpart interchanging $\mathbf{a}$ and $\mathbf{b}$ a set of two coupled matrix PDE's. There are two cases where (3.25) is a closed matrix PDE. The first case occurs when $\mathbf{a} = \mathbf{b}$, i.e. in (3.21) $N_\alpha = N_\beta$, $N'_{\alpha'} = N'_{\beta'}$ and $a_{n,m} = b_{n,m}$. We then obtain a (closed) matrix PDE for $\boldsymbol{\Phi}(\mathbf{a}, \mathbf{a})$, where $\mathbf{a}$ can be a general $N_\alpha \times N'_{\alpha'}$ matrix. This matrix PDE is a matrix generalization of Hirota's equation[31]) which is the combination of NLS ($\lambda_3 = 0$) and the complex modified Korteweg–de Vries equation ($\lambda_2 = 0$). The second case is that $\mathbf{a}$ and $\mathbf{b}$ are (square and) hermitian matrices, i.e. in (3.21) $N_\alpha = N'_{\alpha'}$, $N_\beta = N'_{\beta'}$, $a_{n,m} = a^*_{m,n}$ and $b_{n,m} = b^*_{m,n}$. In the latter case we apply the symmetry relation (2.39) (cf. also (2.37)), and (3.25) becomes

$$\begin{aligned} \mathrm{i}\partial_t \boldsymbol{\Phi}(\mathbf{a}, \mathbf{b}) = {} & (-\lambda_2 \partial_x^2 + \mathrm{i}\lambda_3 \partial_x^3)\boldsymbol{\Phi}(\mathbf{a}, \mathbf{b}) - 2\lambda_2 \boldsymbol{\Phi}(\mathbf{a}, \mathbf{b}) \cdot \mathbf{b}^* \cdot \boldsymbol{\Phi}^\dagger(\mathbf{a}, \mathbf{b}) \cdot \mathbf{a} \cdot \boldsymbol{\Phi}(\mathbf{a}, \mathbf{b}) \\ & - 3\mathrm{i}\lambda_3 \{(\partial_x \boldsymbol{\Phi}(\mathbf{a}, \mathbf{b})) \cdot \mathbf{b}^* \cdot \boldsymbol{\Phi}^\dagger(\mathbf{a}, \mathbf{b}) \cdot \mathbf{a} \cdot \boldsymbol{\Phi}(\mathbf{a}, \mathbf{b}) \\ & + \boldsymbol{\Phi}(\mathbf{a}, \mathbf{b}) \cdot \mathbf{b}^* \cdot \boldsymbol{\Phi}^\dagger(\mathbf{a}, \mathbf{b}) \cdot \mathbf{a} \cdot \partial_x \boldsymbol{\Phi}(\mathbf{a}, \mathbf{b})\}\,, \end{aligned} \tag{3.28}$$

which is another finite-matrix generalization of Hirota's equation. In the case that the matrices $\mathbf{a}$ and $\mathbf{b}$ are positive definite, so that their square roots exists, the matrix $\mathbf{a}^{1/2} \cdot \boldsymbol{\Phi}(\mathbf{a}, \mathbf{b}) \cdot \mathbf{b}^{*\,1/2}$ can be shown to satisfy a simplified version of (3.28) where $\mathbf{a}$ and $\mathbf{b}$ are replaced by unit matrices. This version can also be found in ref. 16. Furthermore, in the special case that $N_\beta = 1$ in (3.28), $\boldsymbol{\Phi}(\mathbf{a}, \mathbf{b})$ becomes an N_α-dimensional vector $\boldsymbol{\phi}(\mathbf{a})$ for which

$$\begin{aligned} \mathrm{i}\partial_t \boldsymbol{\phi}(\mathbf{a}) = {} & (-\lambda_2 \partial_x^2 + \mathrm{i}\lambda_3 \partial_x^3)\boldsymbol{\phi}(\mathbf{a}) - 2\lambda_2 (\boldsymbol{\phi}^*(\mathbf{a}) \cdot \mathbf{a} \cdot \boldsymbol{\phi}(\mathbf{a}))\boldsymbol{\phi}(\mathbf{a}) \\ & - 3\mathrm{i}\lambda_3 \{(\boldsymbol{\phi}^*(\mathbf{a}) \cdot \mathbf{a} \cdot \boldsymbol{\phi}(\mathbf{a}))\partial_x \boldsymbol{\phi}(\mathbf{a}) + (\boldsymbol{\phi}^*(\mathbf{a}) \cdot \mathbf{a} \cdot \partial_x \boldsymbol{\phi}(\mathbf{a}))\boldsymbol{\phi}(\mathbf{a})\}\,. \end{aligned} \tag{3.29}$$

Of course, taking $N_\alpha = 1$ instead, gives essentially the same vector generalization of Hirota's equation (for the vector-NLS, see refs. 10 and 12). The scattering problems for these special cases of (3.25) follow from (3.27).

In this section we have seen that the assumption of hermitian A and B (cf. (2.37)) is important in order to derive closed matrix PDE's (the multicomponent NLS and its hierarchy). In the following paper we will show that this goal can also be reached in cases of 'minimal departure from hermiticity', such as e.g. $A_{kl'} = k\bar{A}_{kl'}$, $B_{kl'} = \bar{B}_{kl'}$ with $\bar{A}$ and $\bar{B}$ hermitian (leading to the multicomponent DNLS and its hierarchy).

4. The Bäcklund transformations and the associated multicomponent PDfE

In this section we shall derive the Bianchi identity expressing the commutativity of two BT's, and investigate its interpretation as a lattice version of the matrix PDE (3.25). After that we shall elaborate the BT for (3.25) and also a modified vector PDE corresponding to (3.29).

4.1. *The Bianchi identity*

Consider two BT's, one induced by $\rho_k \to \tilde{\rho}_k$ given in (2.19), and the other by $\rho_k \to \hat{\rho}_k$ with

$$\hat{\rho}_k = \theta' \frac{q-k}{q^*-k}\, \rho_k\,, \quad |\theta'| = 1\,. \tag{4.1}$$

As a matter of fact, we then have the basic Bäcklund relations (2.22), (2.23), (2.25) and (2.26) also in a version with p, θ and the tildes replaced by q, θ' and hats, respectively. Proceeding in this way, we consider also the BT's induced by $\tilde{\rho}_k \to \hat{\tilde{\rho}}_k$ and by $\hat{\rho}_k \to \tilde{\hat{\rho}}_k$, and note that $\hat{\tilde{\rho}}_k = \tilde{\hat{\rho}}_k$. Hence we find that $\hat{\tilde{\boldsymbol{\phi}}}_k[A, B] = \tilde{\hat{\boldsymbol{\phi}}}_k[A, B]$, meaning that the two BT's commute. The Bianchi identity, which is based on this commutativity, is

$$\begin{aligned} &-\mathbf{J}^{\mathrm{T}} \cdot (\hat{\tilde{\boldsymbol{\phi}}}_k[A,B] - \theta' \tilde{\boldsymbol{\phi}}_k[A,B]) - \theta' \mathbf{J}^{\mathrm{T}} \cdot (\tilde{\boldsymbol{\phi}}_k[A,B] - \theta \boldsymbol{\phi}_k[A,B]) \\ &\quad + \mathbf{J}^{\mathrm{T}} \cdot (\tilde{\hat{\boldsymbol{\phi}}}_k[A,B] - \theta \hat{\boldsymbol{\phi}}_k[A,B]) + \theta \mathbf{J}^{\mathrm{T}} \cdot (\hat{\boldsymbol{\phi}}_k[A,B] - \theta' \boldsymbol{\phi}_k[A,B]) = \boldsymbol{0}\,. \end{aligned} \tag{4.2}$$

Using the algebraic relation (2.10) for $k\tilde{\boldsymbol{\phi}}_k[A, B]$ in (2.22), one obtains

$$\begin{aligned} -\mathbf{J}^{\mathrm{T}} \cdot (\tilde{\boldsymbol{\phi}}_k[A,B] - \theta \boldsymbol{\phi}_k[A,B]) &= -(p^* \tilde{\boldsymbol{\phi}}_k[A,B] - \theta p \boldsymbol{\phi}_k[A,B]) \\ &\quad - \tilde{\boldsymbol{\Psi}}^*[A,B] \cdot \mathbf{A} \cdot (\tilde{\boldsymbol{\phi}}_k[A,B] - \theta \boldsymbol{\phi}_k[A,B]) \\ &\quad - \tilde{\boldsymbol{\Phi}}[A,B] \cdot \mathbf{B}^* \cdot (\tilde{\boldsymbol{\psi}}_k[B,A] - \boldsymbol{\psi}_k[B,A])\,, \end{aligned} \tag{4.3}$$

and, using again (2.10) for $k\boldsymbol{\phi}_k[A, B]$, in (2.25),

$$-\mathbf{J}^{\mathrm{T}}\cdot(\tilde{\boldsymbol{\phi}}_k[A, B] - \theta\boldsymbol{\phi}_k[A, B]) = -(p^*\tilde{\boldsymbol{\phi}}_k[A, B] - \theta p\boldsymbol{\phi}_k[A, B])$$
$$- \boldsymbol{\Psi}^*[A, B]\cdot\mathbf{A}\cdot(\tilde{\boldsymbol{\phi}}_k[A, B] - \theta\boldsymbol{\phi}_k[A, B])$$
$$- \theta\boldsymbol{\Phi}[A, B]\cdot\mathbf{B}^*\cdot(\tilde{\boldsymbol{\psi}}_k[B, A] - \boldsymbol{\psi}_k[B, A])\,. \tag{4.4}$$

Related to the BT (4.1), there are also relations similar to (4.3) and (4.4) where p, θ and the tildes are replaced by q, θ' and hats, respectively. Inserting (4.3), (4.4) and the two other relations in (4.2), we get the Bianchi identity in the form

$$(p^* - q^*)\hat{\tilde{\boldsymbol{\phi}}}_k[A, B] + \theta'(q - p^*)\tilde{\boldsymbol{\phi}}_k[A, B]$$
$$+ \theta(q^* - p)\hat{\boldsymbol{\phi}}_k[A, B] + \theta\theta'(p - q)\boldsymbol{\phi}_k[A, B]$$
$$- (\tilde{\boldsymbol{\Psi}}^*[A, B] - \hat{\boldsymbol{\Psi}}^*[A, B])\cdot\mathbf{A}\cdot(\hat{\tilde{\boldsymbol{\phi}}}_k[A, B] - \theta\theta'\boldsymbol{\phi}_k[A, B])$$
$$- (\theta'\tilde{\boldsymbol{\Phi}}[A, B] - \theta\hat{\boldsymbol{\Phi}}[A, B])\cdot\mathbf{B}^*\cdot(\hat{\tilde{\boldsymbol{\psi}}}_k[B, A] - \boldsymbol{\psi}_k[B, A]) = \boldsymbol{0}\,. \tag{4.5}$$

Subtracting (4.3) and (4.4) it furthermore follows that

$$(\tilde{\boldsymbol{\Psi}}^*[A, B] - \boldsymbol{\Psi}^*[A, B])\cdot\mathbf{A}\cdot(\tilde{\boldsymbol{\phi}}_k[A, B] - \theta\boldsymbol{\phi}_k[A, B])$$
$$+ (\tilde{\boldsymbol{\Phi}}[A, B] - \theta\boldsymbol{\Phi}[A, B])\cdot\mathbf{B}^*\cdot(\tilde{\boldsymbol{\psi}}_k[B, A] - \boldsymbol{\psi}_k[B, A]) = \boldsymbol{0}\,. \tag{4.6}$$

Using (4.6) (once with the tildes, and once with hats instead), the Bianchi identity (4.5) can be written in still another form, which in the integrated version reads

$$\{p^*\,\mathbf{1} - (\tilde{\boldsymbol{\Psi}}^*[A, B] - \boldsymbol{\Psi}^*[A, B])\cdot\mathbf{A}\}\cdot(\hat{\tilde{\boldsymbol{\Phi}}}[A, B] - \theta'\tilde{\boldsymbol{\Phi}}[A, B])$$
$$- \{q^*\,\mathbf{1} - (\hat{\boldsymbol{\Psi}}^*[A, B] - \boldsymbol{\Psi}^*[A, B])\cdot\mathbf{A}\}\cdot(\hat{\tilde{\boldsymbol{\Phi}}}[A, B] - \theta\hat{\boldsymbol{\Phi}}[A, B])$$
$$+ \theta'(\tilde{\boldsymbol{\Phi}}[A, B] - \theta\boldsymbol{\Phi}[A, B])\cdot\{q\,\mathbf{1} - \mathbf{B}^*\cdot(\hat{\tilde{\boldsymbol{\Psi}}}[B, A] - \tilde{\boldsymbol{\Psi}}[B, A])\}$$
$$- \theta(\hat{\boldsymbol{\Phi}}[A, B] - \theta'\boldsymbol{\Phi}[A, B])\cdot\{p\,\mathbf{1} - \mathbf{B}^*\cdot(\hat{\tilde{\boldsymbol{\Psi}}}[B, A] - \hat{\boldsymbol{\Psi}}[B, A])\} = \mathbf{0}\,. \tag{4.7}$$

From the Bäcklund relations (2.23) and (2.26) we obtain with the algebraic relation (2.11)

$$-\mathbf{J}^{\mathrm{T}}\cdot(\tilde{\boldsymbol{\psi}}_k[A, B] - \boldsymbol{\psi}_k[A, B]) = -p^*(\tilde{\boldsymbol{\psi}}_k[A, B] - \boldsymbol{\psi}_k[A, B])$$
$$- \tilde{\boldsymbol{\Psi}}[A, B]\cdot\mathbf{A}^*\cdot(\tilde{\boldsymbol{\psi}}_k[A, B] - \boldsymbol{\psi}_k[A, B])$$
$$+ \tilde{\boldsymbol{\Phi}}^*[A, B]\cdot\mathbf{B}\cdot(\tilde{\boldsymbol{\phi}}_k[B, A] - \theta\boldsymbol{\phi}_k[B, A])\,, \tag{4.8}$$

and

$$-\mathbf{J}^{\mathrm{T}}\cdot(\tilde{\psi}_k[A,B]-\psi_k[A,B])=-p(\tilde{\psi}_k[A,B]-\psi_k[A,B])$$
$$-\Psi[A,B]\cdot\mathbf{A}^*\cdot(\tilde{\psi}_k[A,B]-\psi_k[A,B])$$
$$+\theta^*\Phi^*[A,B]\cdot\mathbf{B}\cdot(\tilde{\phi}_k[B,A]-\theta\phi_k[B,A])\,, \tag{4.9}$$

respectively. It is straightforward to write down a Bianchi identity based on $\hat{\tilde{\psi}}_k[A,B]=\tilde{\hat{\psi}}_k[A,B]$, analogous to (4.5), with the use of (4.8) and (4.9), but we shall not give this identity here. Subtracting (4.8) and (4.9) we find also that

$$(p^*-p)(\tilde{\psi}_k[A,B]-\psi_k[A,B])$$
$$+(\tilde{\Psi}[A,B]-\Psi[A,B])\cdot\mathbf{A}^*\cdot(\tilde{\psi}_k[A,B]-\psi_k[A,B])$$
$$-(\tilde{\Phi}^*[A,B]-\theta^*\Phi^*[A,B])\cdot\mathbf{B}\cdot(\tilde{\phi}_k[B,A]-\theta\phi_k[B,A])=\mathbf{0}\,. \tag{4.10}$$

Consider now the reduction from $\infty\times\infty$ matrices to finite matrices as described in (3.21)–(3.24). We also define the $N'_{\alpha'}\times N_\alpha$ matrix $\Psi(\mathbf{a},\mathbf{b})$ and the $N'_{\beta'}\times N_\beta$ matrix $\Psi(\mathbf{b},\mathbf{a})$ by

$$(\Psi(\mathbf{a},\mathbf{b}))_{\alpha'\alpha}\equiv(\Psi[A,B])_{r'_{\alpha'}r_\alpha}\,,\quad(\Psi(\mathbf{b},\mathbf{a}))_{\beta'\beta}\equiv(\Psi[B,A])_{s'_{\beta'}s_\beta}\,. \tag{4.11}$$

Then the integrated version of (4.10) yields

$$(p^*-p)\{\tilde{\Psi}^*(\mathbf{a},\mathbf{b})-\Psi^*(\mathbf{a},\mathbf{b})\}$$
$$-\{\tilde{\Psi}^*(\mathbf{a},\mathbf{b})-\Psi^*(\mathbf{a},\mathbf{b})\}\cdot\mathbf{a}\cdot\{\tilde{\Psi}^*(\mathbf{a},\mathbf{b})-\Psi^*(\mathbf{a},\mathbf{b})\}$$
$$+\{\tilde{\Phi}(\mathbf{a},\mathbf{b})-\theta\Phi(\mathbf{a},\mathbf{b})\}\cdot\mathbf{b}^*\cdot\{\tilde{\Phi}^*(\mathbf{b},\mathbf{a})-\theta^*\Phi^*(\mathbf{b},\mathbf{a})\}=\mathbf{0}\,. \tag{4.12}$$

From (4.11) one can solve (assuming $p\neq p^*$)

$$\{\tilde{\Psi}^*(\mathbf{a},\mathbf{b})-\Psi^*(\mathbf{a},\mathbf{b})\}\cdot\mathbf{a}=\tfrac{1}{2}(p^*-p)[\mathbf{1}^{(N'_{\alpha'})}-\{\mathbf{1}^{(N'_{\alpha'})}-\Gamma(\mathbf{a},\mathbf{b})\cdot\mathbf{a}\}^{1/2}]\,, \tag{4.13}$$

with

$$\Gamma(\mathbf{a},\mathbf{b})=\frac{4}{|p-p^*|^2}\{\tilde{\Phi}(\mathbf{a},\mathbf{b})-\theta\Phi(\mathbf{a},\mathbf{b})\}\cdot\mathbf{b}^*\cdot\{\tilde{\Phi}^*(\mathbf{b},\mathbf{a})-\theta^*\Phi^*(\mathbf{b},\mathbf{a})\}\,. \tag{4.14}$$

The square root in (4.13) has to be understood in the sense of the expansion $(1-x)^{1/2}=1-\frac{1}{2}x+\ldots$, and the sign before the square root has been determined by observing that the left-hand side of (4.13) tends to zero for $\mathbf{a}\to\mathbf{0}$. Similarly, from (4.10) with A and B interchanged, one has

$$\mathbf{b}^* \cdot \{\tilde{\boldsymbol{\Psi}}(\mathbf{b},\mathbf{a}) - \boldsymbol{\Psi}(\mathbf{b},\mathbf{a})\} = -\tfrac{1}{2}(p^* - p)[\mathbf{1}^{(N_\beta)} - \{\mathbf{1}^{(N_\beta)} - \mathbf{b}^* \cdot \boldsymbol{\Gamma}^*(\mathbf{b}\cdot\mathbf{a})\}^{1/2}] . \tag{4.15}$$

Note that the integrated and reduced version of (4.6) is identically satisfied by (4.13) and (4.15), since it follows from (4.14) that

$$\begin{aligned} &\{\boldsymbol{\Gamma}(\mathbf{a},\mathbf{b})\cdot\mathbf{a}\}^n \cdot \{\tilde{\boldsymbol{\Phi}}(\mathbf{a},\mathbf{b}) - \theta\boldsymbol{\Phi}(\mathbf{a},\mathbf{b})\} \\ &\quad = \{\tilde{\boldsymbol{\Phi}}(\mathbf{a},\mathbf{b}) - \theta\boldsymbol{\Phi}(\mathbf{a},\mathbf{b})\} \cdot \{\mathbf{b}^* \cdot \boldsymbol{\Gamma}^*(\mathbf{b},\mathbf{a})\}^n , \quad n = 0, 1, 2, \ldots . \end{aligned} \tag{4.16}$$

Inserting (4.13) and (4.15), with the necessary alterations for the various BT's, in the left-hand side of the reduced version of (4.7), we obtain the relation

$$\begin{aligned} &[(p^* + p)\mathbf{1}^{(N'_{\alpha'})} + (p^* - p)\{\mathbf{1}^{(N'_{\alpha'})} - \boldsymbol{\Gamma}(\mathbf{a},\mathbf{b})\cdot\mathbf{a}\}^{1/2}] \cdot \{\hat{\tilde{\boldsymbol{\Phi}}}(\mathbf{a},\mathbf{b}) - \theta'\tilde{\boldsymbol{\Phi}}(\mathbf{a},\mathbf{b})\} \\ &\quad - [(q^* + q)\mathbf{1}^{(N'_{\alpha'})} \\ &\quad + (q^* - q)\{\mathbf{1}^{(N'_{\alpha'})} - \boldsymbol{\Gamma}'(\mathbf{a},\mathbf{b})\cdot\mathbf{a}\}^{1/2}] \cdot \{\hat{\tilde{\boldsymbol{\Phi}}}(\mathbf{a},\mathbf{b}) - \theta\hat{\boldsymbol{\Phi}}(\mathbf{a},\mathbf{b})\} \\ &\quad + \theta'\{\tilde{\boldsymbol{\Phi}}(\mathbf{a},\mathbf{b}) - \theta\boldsymbol{\Phi}(\mathbf{a},\mathbf{b})\} \cdot [(q^* + q)\mathbf{1}^{(N_\beta)} \\ &\quad - (q^* - q)\{\mathbf{1}^{(N_\beta)} - \mathbf{b}^* \cdot \tilde{\boldsymbol{\Gamma}}'^*(\mathbf{b},\mathbf{a})\}^{1/2}] \\ &\quad - \theta\{\hat{\boldsymbol{\Phi}}(\mathbf{a},\mathbf{b}) - \theta'\boldsymbol{\Phi}(\mathbf{a},\mathbf{b})\} \cdot [(p^* + p)\mathbf{1}^{(N_\beta)} \\ &\quad - (p^* - p)\{\mathbf{1}^{(N_\beta)} - \mathbf{b}^* \cdot \hat{\boldsymbol{\Gamma}}^*(\mathbf{b},\mathbf{a})\}^{1/2}] = \mathbf{0} \end{aligned} \tag{4.17}$$

in which $\boldsymbol{\Gamma}'(\mathbf{a},\mathbf{b})$ can be inferred from (4.14) replacing p, θ and the tildes by q, θ' and hats, respectively, $\hat{\boldsymbol{\Gamma}}(\mathbf{b},\mathbf{a})$ is the expression for $\boldsymbol{\Gamma}(\mathbf{b},\mathbf{a})$ with an additional hat on each occurring $\boldsymbol{\Phi}$, and $\tilde{\boldsymbol{\Gamma}}'(\mathbf{b},\mathbf{a})$ is the expression for $\boldsymbol{\Gamma}'(\mathbf{b},\mathbf{a})$ with an additional tilde on each occurring $\boldsymbol{\Phi}$. Eq. (4.17) and the relation that can be obtained from it by interchanging A and B are Bianchi identities expressing the commutativity of the two BT's generated by (2.19) and (4.1)*). Moreover, using the identifications

$$\begin{aligned} &\boldsymbol{\Phi}(\mathbf{a},\mathbf{b}) \equiv \boldsymbol{\Phi}_{n,m}(\mathbf{a},\mathbf{b}) , \quad &&\tilde{\boldsymbol{\Phi}}(\mathbf{a},\mathbf{b}) \equiv \boldsymbol{\Phi}_{n+1,m}(\mathbf{a},\mathbf{b}) , \\ &\hat{\boldsymbol{\Phi}}(\mathbf{a},\mathbf{b}) \equiv \boldsymbol{\Phi}_{n,m+1}(\mathbf{a},\mathbf{b}) , \quad &&\hat{\tilde{\boldsymbol{\Phi}}}(\mathbf{a},\mathbf{b}) \equiv \boldsymbol{\Phi}_{n+1,m+1}(\mathbf{a},\mathbf{b}) , \end{aligned} \tag{4.18}$$

and similar ones for $\boldsymbol{\Phi}(\mathbf{b},\mathbf{a})$ etc., eq. (4.17) and its analogue with A and B interchanged are two coupled partial difference equations for the $N'_\alpha \times N_\beta$ matrices $\boldsymbol{\Phi}_{n,m}(\mathbf{a},\mathbf{b})$ and the $N'_{\beta'} \times N_\alpha$ matrices $\boldsymbol{\Phi}_{n,m}(\mathbf{b},\mathbf{a})$, defined at the sites (n, m) of a two-dimensional lattice. The set of PDfE's is integrable, since solutions of it can be found from the linear integral equation (1.1) with ρ_k given by

*) In the special case that $A_{kl'} = B_{kl'} = 1$, eq. (4.17) has been given in eq. (7.18) of ref. 5, apart from some misprints.

$$\rho_k \equiv \rho_k(n, m) = \left(\theta \frac{p-k}{p^*-k}\right)^n \left(\theta' \frac{q-k}{q^*-k}\right)^m \rho_k(0,0) \tag{4.19}$$

instead of by (1.5). According to (4.19), the primitive translations in the n- and m-directions can be interpreted as BT's in the way of (4.18). In the case that A and B are hermitian we have $\boldsymbol{\Phi}^*_{n,m}(\mathbf{b}, \mathbf{a}) = \boldsymbol{\Phi}^\dagger_{n,m}(\mathbf{a}, \mathbf{b})$, and eq. (4.17) with (4.18) becomes a PDfE in terms of the $N'_{\alpha'} \times N_\beta$ matrices $\boldsymbol{\Phi}_{n,m}(\mathbf{a}, \mathbf{b})$, which is a lattice version of the multicomponent PDE (3.28). As shown in ref. 5, the integrable PDE's can be obtained from their associated lattice version by applying two appropriate continuum limits, and, applying only one of these limits, one obtains integrable nonlinear differential–difference equations. One may expect this to remain true in the multicomponent case. Multicomponent differential–difference equations have been studied in ref. 19.

4.2. *The Bäcklund transform*

Let us investigate a further consequence of (4.3): using the constitutive relation (2.13) with $p = 1$ ($\partial_1 = -\partial_x$) for $\mathbf{J}^{\mathrm{T}} \cdot \tilde{\boldsymbol{\phi}}_k[A, B]$ and for $\mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{\phi}_k[A, B]$, one obtains

$$\begin{aligned} -\mathrm{i}\partial_x(\tilde{\boldsymbol{\phi}}_k[A, B] - \theta\boldsymbol{\phi}_k[A, B]) = &-(p^*\tilde{\boldsymbol{\phi}}_k[A, B] - \theta p\boldsymbol{\phi}_k[A, B]) \\ &+ \theta(\tilde{\boldsymbol{\Psi}}^*[A, B] - \boldsymbol{\Psi}^*[A, B]) \cdot \mathbf{A} \cdot \boldsymbol{\phi}_k[A, B] \\ &- \tilde{\boldsymbol{\Phi}}[A, B] \cdot \mathbf{B}^* \cdot (\tilde{\boldsymbol{\psi}}_k[B, A] - \boldsymbol{\psi}_k[B, A]) \,. \end{aligned} \tag{4.20}$$

Reducing the integrated version of (4.20) to finite matrices as defined by (3.21)–(3.24) and (4.11), we can use (4.13) and (4.15) with the result that

$$\begin{aligned} &-\mathrm{i}\partial_x\{\tilde{\boldsymbol{\Phi}}(\mathbf{a}, \mathbf{b}) - \theta\boldsymbol{\Phi}(\mathbf{a}, \mathbf{b})\} \\ &= \tfrac{1}{2}\theta[(p^*+p)\mathbf{1}^{(N'_{\alpha'})} - (p^*-p)\{\mathbf{1}^{(N'_{\alpha'})} - \boldsymbol{\Gamma}(\mathbf{a}, \mathbf{b}) \cdot \mathbf{a}\}^{1/2}] \cdot \boldsymbol{\Phi}(\mathbf{a}, \mathbf{b}) \\ &\quad - \tfrac{1}{2}\tilde{\boldsymbol{\Phi}}(\mathbf{a}, \mathbf{b}) \cdot [(p^*+p)\mathbf{1}^{(N_\beta)} + (p^*-p)\{\mathbf{1}^{(N_\beta)} - \mathbf{b}^* \cdot \boldsymbol{\Gamma}^*(\mathbf{b}, \mathbf{a})\}^{1/2}] \,. \end{aligned} \tag{4.21}$$

Eq. (4.21) and its counterpart with A and B interchanged are the spatial part of the BT for the coupled PDE's given by (3.25) and its counterpart with A and B interchanged. The time-dependent part of the BT can be inferred from (3.25) for $\boldsymbol{\Phi}(\mathbf{a}, \mathbf{b})$ and for $\tilde{\boldsymbol{\Phi}}(\mathbf{a}, \mathbf{b})$, with (4.21).

Furthermore, eq. (4.21) can be used to derive the modified equation to eq. (3.25), in the sense of ref. 4 and also ref. 32. This modified equation is an equation for

$$\boldsymbol{\Phi}^-(\mathbf{a}, \mathbf{b}) \equiv \tilde{\boldsymbol{\Phi}}(\mathbf{a}, \mathbf{b}) - \theta\boldsymbol{\Phi}(\mathbf{a}, \mathbf{b}) \,. \tag{4.22}$$

Using (4.22), eq. (4.21) can be rewritten as

$$-\mathrm{i}\partial_x \boldsymbol{\Phi}^-(\mathbf{a},\mathbf{b}) + \tfrac{1}{2}\boldsymbol{\Phi}^-(\mathbf{a},\mathbf{b})\cdot[(p^*+p)\mathbf{1}^{(N_\beta)} + (p^*-p)\{\mathbf{1}^{(N_\beta)} - \mathbf{b}^*\cdot\boldsymbol{\Gamma}^*(\mathbf{b},\mathbf{a})\}^{1/2}]$$
$$= -\tfrac{1}{2}\theta(p^*-p)[\{\mathbf{1}^{(N'_{\alpha'})} - \boldsymbol{\Gamma}(\mathbf{a},\mathbf{b})\cdot\mathbf{a}\}^{1/2}\cdot\boldsymbol{\Phi}(\mathbf{a},\mathbf{b}) + \boldsymbol{\Phi}(\mathbf{a},\mathbf{b})\cdot\{\mathbf{1}^{(N_\beta)} - \mathbf{b}^*\cdot\boldsymbol{\Gamma}^*(\mathbf{b},\mathbf{a})\}^{1/2}]\,, \tag{4.23}$$

in which, according to (4.14) and (4.22),

$$\boldsymbol{\Gamma}(\mathbf{a},\mathbf{b}) = \frac{4}{|p-p^*|^2}\,\boldsymbol{\Phi}^-(\mathbf{a},\mathbf{b})\cdot\mathbf{b}^*\cdot\boldsymbol{\Phi}^{-*}(\mathbf{b},\mathbf{a})\,. \tag{4.24}$$

From (4.23) with (4.24), $\boldsymbol{\Phi}(\mathbf{a},\mathbf{b})$ can be solved to be

$$\boldsymbol{\Phi}(\mathbf{a},\mathbf{b}) = \frac{\theta^*}{p^*-p}\int_0^\infty \mathrm{d}\tau\,\exp[-\tau\{\mathbf{1}^{(N'_{\alpha'})} - \boldsymbol{\Gamma}(\mathbf{a},\mathbf{b})\cdot\mathbf{a}\}^{1/2}]$$
$$\cdot[2\mathrm{i}\partial_x\boldsymbol{\Phi}^-(\mathbf{a},\mathbf{b}) - (p^*+p)\boldsymbol{\Phi}^-(\mathbf{a},\mathbf{b}) - (p^*-p)\boldsymbol{\Phi}^-(\mathbf{a},\mathbf{b})\cdot\{\mathbf{1}^{(N_\beta)} - \mathbf{b}^*\cdot\boldsymbol{\Gamma}^*(\mathbf{b},\mathbf{a})\}^{1/2}]$$
$$\cdot\exp[-\tau\{\mathbf{1}^{(N_\beta)} - \mathbf{b}^*\cdot\boldsymbol{\Gamma}^*(\mathbf{b},\mathbf{a})\}^{1/2}]\,, \tag{4.25}$$

which enables one to determine the modified equation to eq. (3.25), but only in a formal way unless the integral in (4.25) can be evaluated.

In the case that $N_\alpha = N'_{\alpha'}$, $N_\beta = N'_{\beta'} = 1$, and $\mathbf{a} = \mathbf{a}^\dagger$, we deal with N_α-dimensional vectors $\boldsymbol{\phi}(\mathbf{a})$, $\tilde{\boldsymbol{\phi}}(\mathbf{a})$ and $\boldsymbol{\phi}^-(\mathbf{a})$, and we find the modified equation to the vector-PDE (3.29) in the following way. Combining (3.29) for $\boldsymbol{\phi}(\mathbf{a})$ and for $\tilde{\boldsymbol{\phi}}(\mathbf{a}) \equiv \boldsymbol{\phi}^-(\mathbf{a}) + \theta\boldsymbol{\phi}(\mathbf{a})$, we get

$$\mathrm{i}\partial_t\boldsymbol{\phi}^-(a) = (-\lambda_2\partial_x^2 + \mathrm{i}\lambda_3\partial_x^3)\boldsymbol{\phi}^-(\mathbf{a}) + 2\lambda_2\theta(\boldsymbol{\phi}^*(\mathbf{a})\cdot\mathbf{a}\cdot\boldsymbol{\phi}(\mathbf{a}))\boldsymbol{\phi}(\mathbf{a})$$
$$-2\lambda_2[\{\boldsymbol{\phi}^{-*}(\mathbf{a}) + \theta^*\boldsymbol{\phi}^*(\mathbf{a})\}\cdot\mathbf{a}\cdot\{\boldsymbol{\phi}^-(\mathbf{a}) + \theta\boldsymbol{\phi}(\mathbf{a})\}]\{\boldsymbol{\phi}^-(\mathbf{a}) + \theta\boldsymbol{\phi}(\mathbf{a})\}$$
$$+3\mathrm{i}\lambda_3\theta\{(\boldsymbol{\phi}^*(\mathbf{a})\cdot\mathbf{a}\cdot\boldsymbol{\phi}(\mathbf{a}))\partial_x\boldsymbol{\phi}(\mathbf{a}) + (\boldsymbol{\phi}^*(\mathbf{a})\cdot\mathbf{a}\cdot\partial_x\boldsymbol{\phi}(\mathbf{a}))\boldsymbol{\phi}(\mathbf{a})\}$$
$$-3\mathrm{i}\lambda_3[\{\boldsymbol{\phi}^{-*}(\mathbf{a}) + \theta^*\boldsymbol{\phi}^*(\mathbf{a})\}\cdot\mathbf{a}\cdot\{\boldsymbol{\phi}^-(\mathbf{a}) + \theta\boldsymbol{\phi}(\mathbf{a})\}]\partial_x\{\boldsymbol{\phi}^-(\mathbf{a}) + \theta\boldsymbol{\phi}(\mathbf{a})\}$$
$$-3\mathrm{i}\lambda_3[\{\boldsymbol{\phi}^{-*}(\mathbf{a}) + \theta^*\boldsymbol{\phi}^*(\mathbf{a})\}\cdot\mathbf{a}\cdot\partial_x\{\boldsymbol{\phi}^-(\mathbf{a}) + \theta\boldsymbol{\phi}(\mathbf{a})\}]\{\boldsymbol{\phi}^-(\mathbf{a}) + \theta\boldsymbol{\phi}(\mathbf{a})\}\,, \tag{4.26}$$

and, in this case, we see that (4.25) with (4.24) reduces to

$$\boldsymbol{\phi}(\mathbf{a}) = \frac{\theta^*}{p^*-p}\left[\left\{\mathbf{1}^{(N_\alpha)} - \frac{4}{|p-p^*|^2}\,\boldsymbol{\phi}^-(\mathbf{a})\boldsymbol{\phi}^{-*}(\mathbf{a})\cdot\mathbf{a}\right\}^{1/2}\right.$$

$$+\left\{1-\frac{4}{|p-p^*|^2}\boldsymbol{\phi}^{-*}(\mathbf{a})\cdot\mathbf{a}\cdot\boldsymbol{\phi}^{-}(\mathbf{a})\right\}^{1/2}\mathbf{1}^{(N_\alpha)}\Bigg]^{-1}\cdot\Bigg[2\mathrm{i}\partial_x-(p^*+p)$$

$$-(p^*-p)\left\{1-\frac{4}{|p-p^*|^2}\boldsymbol{\phi}^{-*}(\mathbf{a})\cdot\mathbf{a}\cdot\boldsymbol{\phi}^{-}(\mathbf{a})\right\}^{1/2}\Bigg]\boldsymbol{\phi}^{-}(\mathbf{a})\,. \tag{4.27}$$

Substituting (4.27) in (4.26), one obtains a PDE in terms of $\boldsymbol{\phi}^{-}(\mathbf{a})$, which is the modified equation associated with the vector-Hirota equation (3.29). Eq. (4.27) can be regarded as a Miura transformation mapping a solution $\boldsymbol{\phi}^{-}(\mathbf{a})$ of the modified equation on a solution $\boldsymbol{\phi}(\mathbf{a})$ of (3.29). The modified equation is a vector-generalization of eq. (4.41) of ref. 4, and this equation, i.e. the scalar version with $N_\alpha = 1$, has been related to the AHSC, cf. refs. 4 and 32.

5. Concluding remarks

In this paper we have treated the direct linearization (DL) of finite-matrix generalizations of the nonlinear Schrödinger equation (NLS) and the complex modified Korteweg–de Vries equation (CMKdV). For that purpose we have introduced the linear integral equation (1.1) with $A_{kl'}$ and $B_{kl'}$ given by (1.6). This linear integral equation is a generalization of the integral equation of type I treated in ref. 2 which is obtained from (1.1) for $A_{kl'} = B_{kl'} = 1$. Starting from (1.1) we have derived in section 2 the constitutive relations including also Bäcklund relations and some symmetry properties. In section 3 we have derived a recursion operator Ω which may be used to generate finite-matrix PDE's for various dispersions ω_k. Choosing $\omega_k = \lambda_2 k^2 + \lambda_3 k^3$, we have treated two finite-matrix generalizations of Hirota's equation in the special cases that $A_{kl'} = B_{kl'}$ and that $A_{kl'} = A^*_{l'k}$, $B_{kl'} = B^*_{l'k}$. These equations include the finite-matrix generalizations of the NLS for $\lambda_3 = 0$ and of the CMKdV for $\lambda_2 = 0$. It is interesting to note that there is an alternative way to obtain the DL of multicomponent nonlinear integrable systems. In fact, one may start from the treatment of ref. 23 for a general $N \times N$ spectral problem, and apply special reductions[26]) leading to the spectral problem of the multicomponent case. This approach, however, leads to a large set of coupled matrix integral equations, whereas in the present treatment we have a single integral equation (1.1) for $\boldsymbol{\phi}_k[A, B]$, with the factors $A_{kl'}$ and $B_{kl'}$ inducing the coupling between the components of the system. The questions connected with the alternative method have not been investigated here.

As a straightforward application of the DL of multicomponent systems, based on (1.1), we have derived in section 4 a Bianchi identity (4.17) expressing the commutativity of two Bäcklund transformations (BT's). With the identifications (4.18), this identity can be interpreted as an integrable lattice version of the finite-matrix generalization of the NLS. Furthermore, working out the BT, we have also obtained a so-called modified equation (4.25) with (4.26), which is a vector generalization of a nonlinear partial differential equation (PDE) that has been shown to be equivalent to the equation of the motion for the (classical) Heisenberg chain with uniaxial anisotropy.

Finite-matrix generalizations of PDE's can be derived from other types of integral equations as well. As an example, we present some results for the integral equation

$$\boldsymbol{v}_k[C, D] + \int_C \mathrm{d}\lambda(l) \int_C \mathrm{d}\lambda(l')\, C_{kl'} D_{l'l} \frac{\rho_k \rho_{l'}}{(k+l')(l'+l)}\, \boldsymbol{v}_l[C, D] = \rho_k \boldsymbol{c}_k\,, \tag{5.1}$$

where C, $\mathrm{d}\lambda(k)$, ρ_k and $\boldsymbol{c}_k$ have the same meaning as in section 1, and $C_{kl'}$ and $D_{l'l}$ are again given by finite sums of the form (1.6),

$$C_{kl'} = \sum_{n,m} c_{n,m} k^n l'^{\,m}\,, \quad D_{l'l} = \sum_{n,m} d_{n,m} l'^{\,n} l^m\,. \tag{5.2}$$

The contour C and the measure $\mathrm{d}\lambda(k)$ in the integral equation (5.1) can be chosen arbitrarily, apart from the condition that its solution must be unique. The functional dependence of the solution on $C_{kl'}$ and $D_{l'l}$ is denoted by $\boldsymbol{v}_k[C, D]$*. In the special case that $C_{kl'} = D_{l'l} = 1$, eq. (5.1) is the integral equation of type II treated in ref. 2.

Eq. (5.1) can be expressed as a coupled set of integral equations (cf. section 1) for $\boldsymbol{v}_k[C, D]$ and

$$\boldsymbol{w}_k[C, D] = \int_C \mathrm{d}\lambda(l')\, D_{kl'} \frac{\rho_k}{k+l'}\, \boldsymbol{v}_{l'}[C, D] \tag{5.3}$$

and the potential matrices $\mathbf{V}[C, D]$ and $\mathbf{W}[C, D]$ are obtained by integrating over the same contour with the same measure as in (5.1), (5.3):

$$\mathbf{V}[C, D] = \int_C \mathrm{d}\lambda(k)\, \boldsymbol{v}_k[C, D] \boldsymbol{c}_k\,, \quad \mathbf{W}[C, D] = \int_C \mathrm{d}\lambda(k)\, \boldsymbol{w}_k[C, D] \boldsymbol{c}_k\,. \tag{5.4}$$

Introducing matrices $\mathbf{C}_p$ and $\mathbf{D}_p$ by

$$\mathbf{C}_p = \sum_{n,m} c_{n,m} \mathbf{J}^m \cdot \mathbf{R}_p \cdot \mathbf{J}^{\mathrm{T}n}\,, \quad \mathbf{D}_p = \sum_{n,m} d_{n,m} \mathbf{J}^m \cdot \mathbf{R}_p \cdot \mathbf{J}^{\mathrm{T}n}\,, \tag{5.5}$$

* The C in $\boldsymbol{v}_k[C, D]$ denoting the factor $C_{kl'}$ in (5.1) should not be confused with the contour C of integration.

with

$$\mathbf{R}_p = \begin{cases} \sum_{j=0}^{p-1} \mathbf{J}^j \cdot \mathbf{O} \cdot (-\mathbf{J}^{\mathrm{T}})^{p-1-j}, & p \geq 0, \\ -\sum_{j=0}^{-p-1} \mathbf{J}^{p+j} \cdot \mathbf{O} \cdot (-\mathbf{J}^{\mathrm{T}})^{-1-j}, & p \leq 0 \end{cases} \quad (p \text{ integer}), \tag{5.6}$$

one can derive the algebraic relations

$$\begin{aligned} k^p \boldsymbol{v}_k[C, D] &= \{\mathbf{J}^{\mathrm{T}p} + (-1)^p \mathbf{W}[C, D] \cdot \mathbf{C}_p\} \cdot \boldsymbol{v}_k[C, D] \\ &\quad + \mathbf{V}[C, D] \cdot \mathbf{D}_p \cdot \boldsymbol{w}_k[D, C], \\ k^p \boldsymbol{w}_k[C, D] &= \{(-\mathbf{J}^{\mathrm{T}})^p + \mathbf{W}[C, D] \cdot \mathbf{C}_p\} \cdot \boldsymbol{w}_k[C, D] \\ &\quad - (-1)^p \mathbf{V}[C, D] \cdot \mathbf{D}_p \cdot \boldsymbol{v}_k[D, C], \end{aligned} \tag{5.7}$$

and the differential relations

$$\begin{aligned} \mathrm{i}\partial_p \boldsymbol{v}_k[C, D] &= (\mathbf{J}^{\mathrm{T}p} - \mathbf{W}[C, D] \cdot \mathbf{C}_p) \cdot \boldsymbol{v}_k[C, D], \\ \mathrm{i}\partial_p \boldsymbol{w}_k[C, D] &= \mathbf{V}[C, D] \cdot \mathbf{D}_p \cdot \boldsymbol{v}_k[D, C], \end{aligned} \quad \text{if } p \text{ is odd}, \tag{5.8}$$

in which ∂_p is a differential operator satisfying (2.12). For the matrices $\mathbf{V}[C, D]$ and $\mathbf{W}[C, D]$ one has the symmetry properties

$$\mathbf{V}[C, D] = \mathbf{V}^{\mathrm{T}}[D^{\mathrm{T}}, C^{\mathrm{T}}], \quad \mathbf{W}[C, D] = \mathbf{W}^{\mathrm{T}}[C^{\mathrm{T}}, D^{\mathrm{T}}], \tag{5.9}$$

where $C^{\mathrm{T}}_{kl'} \equiv C_{l'k}$, $D^{\mathrm{T}}_{kl'} \equiv D_{l'k}$. Eqs. (5.7)–(5.9) can be derived directly from (5.1), but follow also as a special case from a more general treatment which is presented in the appendix.

The integral equations in the appendix contain an integration over a contour C with the measure $\mathrm{d}\lambda(k)$ and an integration over a contour C' with the measure $\mathrm{d}\lambda'(k')$, without the requirement that these integrations are related as in (1.1) and (5.1). Starting from these integral equations we derive constitutive relations (including Bäcklund relations and symmetry properties), as well as recursion relations similar to the ones presented in section 3. With these recursion relations one can derive in a systematic way coupled multicomponent PDE's of the AKNS-type. We also show in the appendix that the results obtained by starting from (1.1) or (5.1) are contained as special cases in the treatment of the

AKNS-type, i.e. as reductions of (A.9)–(A.18). The DL based on (5.1) is restricted to PDE's for which the dispersion ω_k is an odd function of k, and therefore the first PDE resulting from the recursive scheme is the one with $\omega_k = k^3$,

$$(\partial_t - \partial_x^3)\mathbf{V}[C, D] + 3(\partial_x \mathbf{V}[C, D]) \cdot \mathbf{D} \cdot \mathbf{V}[D, C] \cdot \mathbf{C} \cdot \mathbf{V}[C, D] + 3\mathbf{V}[C, D] \cdot \mathbf{D} \cdot \mathbf{V}[D, C] \cdot \mathbf{C} \cdot \partial_x \mathbf{V}[C, D] = \mathbf{0} . \quad (5.10)$$

which can be used to deduce finite-matrix generalizations of the Modified Korteweg–de Vries equation (MKdV).

In this paper we have shown that the DL method based on the integral equation (1.1) or (5.1) leads to a systematic treatment of integrable multicomponent versions of the NLS, the CMKdV and the MKdV. Their solutions can be obtained as follows. Having specified the dispersion ω_k and the factors $A_{kl'}$ and $B_{kl'}$ or $C_{kl'}$ and $D_{kl'}$, we solve the integral equation for special choices of the measure $d\lambda(k)$ and the contour C. Then, for the same measure and contour, we calculate the potential matrix defined in (1.4) or (5.4). The solutions of the various components are given by special elements of this potential matrix (cf. (3.24)). A still remaining subject is the derivation of more multicomponent nonlinear systems likewise from (1.1) and (1.4), but considering other specifications of the dispersion ω_k, the factors $A_{kl'}$ and $B_{kl'}$, and the special elements of the potential matrix $\mathbf{\Phi}[A, B]$. In the following paper II we will investigate this further, and show that the DL method based on the integral equation (1.1) also leads to a systematic treatment of integrable multicomponent versions of such equations as the IHSC, the CSG, the Getmanov equation, the DNLS, and the MTM. For the generalizations of the IHSC, the CSG and the Getmanov equation, the factors $A_{kl'}$ and $B_{kl'}$ are of the 'hermitian' type, as for (3.28), but for the generalizations of the DNLS and the MTM we must consider a 'minimal departure from hermiticity'.

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Appendix

In this appendix we show how the results presented in sections 2 and 3 can be generalized in a straightforward way to a more general type of integral equation.

Instead of eq. (1.1) we consider an integral equation with a two-fold integration over contours C' and C, with measures $\mathrm{d}\lambda'(k')$ and $\mathrm{d}\lambda(k)$, respectively, which need not be related to each other, i.e.

$$\boldsymbol{\phi}_k[A, B] + \int_{C'} \mathrm{d}\lambda'(l') \int_{C} \mathrm{d}\lambda(l)\, A_{kl'} B^*_{l'l} \frac{\rho_k \rho'_{l'}}{(k-l')(l'-l)}\, \boldsymbol{\phi}_l[A, B] = \rho_k \boldsymbol{c}_k \,. \tag{A.1}$$

Eq. (A.1) can be rewritten as a coupled system as follows:

$$\begin{aligned} &\boldsymbol{\phi}_k[A, B] + \int_{C'} \mathrm{d}\lambda'(l')\, A_{kl'} \frac{\rho_k}{k-l'}\, \boldsymbol{\psi}'_{l'}[A, B] = \rho_k \boldsymbol{c}_k \,, \\ &\boldsymbol{\psi}'_{k'}[A, B] - \int_{C} \mathrm{d}\lambda(l) B^*_{k'l} \frac{\rho'_{k'}}{k'-l}\, \boldsymbol{\phi}_l[A, B] = \boldsymbol{0} \,. \end{aligned} \tag{A.2}$$

There is also a related integral equation with the roles of $\mathrm{d}\lambda'(k')$ and $\mathrm{d}\lambda(k)$ interchanged, namely

$$\boldsymbol{\phi}'_{k'}[A, B] + \int_{C} \mathrm{d}\lambda(l) \int_{C'} \mathrm{d}\lambda'(l')\, A^*_{k'l} B_{ll'} \frac{\rho'_{k'} \rho_l}{(k'-l)(l-l')}\, \boldsymbol{\phi}'_{l'}[A, B] = \rho'_{k'} \boldsymbol{c}_{k'} \,, \tag{A.3}$$

with the coupled system

$$\begin{aligned} &\boldsymbol{\phi}'_{k'}[A, B] + \int_{C} \mathrm{d}\lambda(l)\, A^*_{k'l} \frac{\rho'_{k'}}{k'-l}\, \boldsymbol{\psi}_l[A, B] = \rho'_{k'} \boldsymbol{c}_{k'} \,, \\ &\boldsymbol{\psi}_k[A, B] - \int_{C'} \mathrm{d}\lambda'(l')\, B_{kl'} \frac{\rho_k}{k-l'}\, \boldsymbol{\phi}'_{l'}[A, B] = \boldsymbol{0} \,. \end{aligned} \tag{A.4}$$

The integral equations for $\boldsymbol{\psi}_k[A, B]$ and $\boldsymbol{\psi}'_{k'}[A, B]$ read

$$\begin{aligned} &\boldsymbol{\psi}_k[A, B] + \int_{C'} \mathrm{d}\lambda'(l') \int_{C} \mathrm{d}\lambda(l)\, B_{kl'} A^*_{l'l} \frac{\rho_k \rho'_{l'}}{(k-l')(l'-l)}\, \boldsymbol{\psi}_l[A, B] \\ &\quad = \int_{C'} \mathrm{d}\lambda'(l')\, B_{kl'} \frac{\rho_k \rho'_{l'}}{k-l'}\, \boldsymbol{c}_{l'} \end{aligned} \tag{A.5}$$

and

$$\boldsymbol{\psi}'_{k'}[A,B] + \int_C \mathrm{d}\lambda(l) \int_{C'} \mathrm{d}\lambda'(l')\, B^*_{k'l} A_{ll'}\, \frac{\rho'_{k'}\rho_l}{(k'-l)(l-l')}\, \boldsymbol{\psi}'_{l'}[A,B]$$

$$= \int_C \mathrm{d}\lambda(l)\, B^*_{k'l}\, \frac{\rho'_{k'}\rho_l}{k'-l}\, \boldsymbol{c}_l\,, \tag{A.6}$$

due to (A.4) and (A.2), respectively.

Following the same line of reasoning as in sections 2 and 3, one can derive the relations involving the solutions of the integral equations (A.1), (A.3), (A.5) and (A.6), and the corresponding $\infty \times \infty$ potential matrices

$$\boldsymbol{\Phi}[A,B] = \int_C \mathrm{d}\lambda(k)\, \boldsymbol{\phi}_k[A,B]\boldsymbol{c}_k\,, \quad \boldsymbol{\Phi}'[A,B] = \int_{C'} \mathrm{d}\lambda'(k')\, \boldsymbol{\phi}'_{k'}[A,B]\boldsymbol{c}_{k'}\,, \tag{A.7}$$

$$\boldsymbol{\Psi}[A,B] = \int_C \mathrm{d}\lambda(k)\, \boldsymbol{\psi}_k[A,B]\boldsymbol{c}_k\,, \quad \boldsymbol{\Psi}'[A,B] = \int_{C'} \mathrm{d}\lambda'(k')\, \boldsymbol{\psi}'_{k'}[A,B]\boldsymbol{c}_{k'}\,. \tag{A.8}$$

We omit a detailed derivation, and give only the results.

A.1. *Algebraic and differential relations*

Defining the matrices $\mathbf{A}_p$ and $\mathbf{B}_p$ according to (1.6) and (2.5), we have the algebraic relations (p integer)

$$\begin{aligned}
k^p\boldsymbol{\phi}_k[A,B] &= (\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}'[A,B]\cdot\mathbf{A}_p)\cdot\boldsymbol{\phi}_k[A,B] \\
&\quad - \boldsymbol{\Phi}[A,B]\cdot\mathbf{B}^*_p\cdot\boldsymbol{\psi}_k[B,A]\,, \\
k'^p\boldsymbol{\phi}'_{k'}[A,B] &= (\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}[A,B]\cdot\mathbf{A}^*_p)\cdot\boldsymbol{\phi}'_{k'}[A,B] \\
&\quad - \boldsymbol{\Phi}'[A,B]\cdot\mathbf{B}_p\cdot\boldsymbol{\psi}'_{k'}[B,A]\,, \\
k^p\boldsymbol{\psi}_k[A,B] &= (\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}[A,B]\cdot\mathbf{A}^*_p)\cdot\boldsymbol{\psi}_k[A,B] \\
&\quad + \boldsymbol{\Phi}'[A,B]\cdot\mathbf{B}_p\cdot\boldsymbol{\phi}_k[B,A]\,, \\
k'^p\boldsymbol{\psi}'_{k'}[A,B] &= (\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}'[A,B]\cdot\mathbf{A}_p)\cdot\boldsymbol{\psi}'_{k'}[A,B] \\
&\quad + \boldsymbol{\Phi}[A,B]\cdot\mathbf{B}^*_p\cdot\boldsymbol{\phi}'_{k'}[B,A]\,.
\end{aligned} \tag{A.9}$$

For a differential operator ∂_p satisfying

$$\mathrm{i}\partial_p\rho_k = k^p\rho_k\,, \quad -\mathrm{i}\partial_p\rho'_{k'} = k'^p\rho'_{k'}\,, \tag{A.10}$$

we have the relations

$$
\begin{aligned}
&\mathrm{i}\partial_p \boldsymbol{\phi}_k[A,B] = (\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}'[A,B]\cdot \mathbf{A}_p)\cdot \boldsymbol{\phi}_k[A,B]\,,\\
&-\mathrm{i}\partial_p \boldsymbol{\phi}'_{k'}[A,B] = (\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}[A,B]\cdot \mathbf{A}^*_p)\cdot \boldsymbol{\phi}'_{k'}[A,B]\,,\\
&\mathrm{i}\partial_p \boldsymbol{\psi}_k[A,B] = \boldsymbol{\Phi}'[A,B]\cdot \mathbf{B}_p\cdot \boldsymbol{\phi}_k[B,A]\,,\\
&-\mathrm{i}\partial_p \boldsymbol{\psi}'_{k'}[A,B] = \boldsymbol{\Phi}[A,B]\cdot \mathbf{B}^*_p\cdot \boldsymbol{\phi}'_{k'}[B,A]\,.
\end{aligned}
\tag{A.11}
$$

Eqs. (A.9) and (A.11) are generalizations of the constitutive relations given in (2.10), (2.11), (2.13) and (2.14). By simple integrations over C or C' one gets the analogues of eqs. (2.15)–(2.18).

A.2. *Bäcklund relations*

Definition of the transformation

$$
\rho_k \to \tilde{\rho}_k = \theta\,\frac{p-k}{p'-k}\,\rho_k\,,\quad \rho'_{k'} \to \tilde{\rho}'_{k'} = \theta^{-1}\,\frac{p'-k'}{p-k'}\,\rho'_{k'}
\tag{A.12}
$$

in the integral equations (A.1), (A.3), (A.5) and (A.6), leads to the Bäcklund relations (with two parameters p and p')

$$
\begin{aligned}
(p'-k)\tilde{\boldsymbol{\phi}}_k[A,B] &= \theta(p\mathbf{1} - \mathbf{J}^{\mathrm{T}} + \tilde{\boldsymbol{\Psi}}'[A,B]\cdot \mathbf{A})\cdot \boldsymbol{\phi}_k[A,B]\\
&\quad + \tilde{\boldsymbol{\Phi}}[A,B]\cdot \mathbf{B}^*\cdot \boldsymbol{\psi}_k[B,A]\,,\\
(p-k')\tilde{\boldsymbol{\phi}}'_{k'}[A,B] &= \theta^{-1}(p'\mathbf{1} - \mathbf{J}^{\mathrm{T}} + \tilde{\boldsymbol{\Psi}}[A,B]\cdot \mathbf{A}^*)\cdot \boldsymbol{\phi}'_{k'}[A,B]\\
&\quad + \tilde{\boldsymbol{\Phi}}'[A,B]\cdot \mathbf{B}\cdot \boldsymbol{\psi}'_{k'}[B,A]\,,\\
(p'-k)\tilde{\boldsymbol{\psi}}_k[A,B] &= (p'\mathbf{1} - \mathbf{J}^{\mathrm{T}} + \tilde{\boldsymbol{\Psi}}[A,B]\cdot \mathbf{A}^*)\cdot \boldsymbol{\psi}_k[A,B]\\
&\quad - \theta\tilde{\boldsymbol{\Phi}}'[A,B]\cdot \mathbf{B}\cdot \boldsymbol{\phi}_k[B,A]\,,\\
(p-k')\tilde{\boldsymbol{\psi}}'_{k'}[A,B] &= (p\mathbf{1} - \mathbf{J}^{\mathrm{T}} + \tilde{\boldsymbol{\Psi}}'[A,B]\cdot \mathbf{A})\cdot \boldsymbol{\psi}'_{k'}[A,B]\\
&\quad - \theta^{-1}\tilde{\boldsymbol{\Phi}}[A,B]\cdot \mathbf{B}^*\cdot \boldsymbol{\phi}'_{k'}[B,A]\,.
\end{aligned}
\tag{A.13}
$$

Eqs. (A.13) are a generalization of (2.22) and (2.23). According to (A.12), the inverse Bäcklund relations that generalize (2.25) and (2.26) are obtained from (A.13) by interchanging $p \leftrightarrow p'$, $\theta \leftrightarrow \theta^{-1}$, and the quantities with tildes and without tildes.

A.3. *Symmetry relations arising from quadratic identities*

For the potentials defined by (A.7) and (A.8), one can derive the quadratic identities

$$\begin{aligned}
\boldsymbol{\Phi}[A,B] &= \int_C \mathrm{d}\lambda(k)\,\frac{1}{\rho_k}\,\boldsymbol{\phi}_k[A,B]\boldsymbol{\phi}_k[B^\dagger,A^\dagger] \\
&\quad - \int_{C'} \mathrm{d}\lambda'(k')\,\frac{1}{\rho'_{k'}}\,\boldsymbol{\psi}'_{k'}[A,B]\boldsymbol{\psi}'_{k'}[B^\dagger,A^\dagger]\,, \\
\boldsymbol{\Phi}'[A,B] &= \int_{C'} \mathrm{d}\lambda'(k')\,\frac{1}{\rho'_{k'}}\,\boldsymbol{\phi}'_{k'}[A,B]\boldsymbol{\phi}'_{k'}[B^\dagger,A^\dagger] \\
&\quad - \int_C \mathrm{d}\lambda(k)\,\frac{1}{\rho_k}\,\boldsymbol{\psi}_k[A,B]\boldsymbol{\psi}_k[B^\dagger,A^\dagger]\,, \\
\boldsymbol{\Psi}[A,B] &= \int_C \mathrm{d}\lambda(k)\int_{C'} \mathrm{d}\lambda'(l')\,\frac{1}{k-l'}\,(B_{kl'}\boldsymbol{\phi}'_{l'}[A,B]\boldsymbol{\phi}_k[B^\dagger,A^\dagger] \\
&\quad + A^*_{l'k}\boldsymbol{\psi}_k[A,B]\boldsymbol{\psi}'_{l'}[A^\dagger,B^\dagger])\,, \\
\boldsymbol{\Psi}'[A,B] &= \int_{C'} \mathrm{d}\lambda'(k')\int_C \mathrm{d}\lambda(l)\,\frac{1}{k'-l}\,(B^*_{k'l}\boldsymbol{\phi}_l[A,B]\boldsymbol{\phi}'_{k'}[A^\dagger,B^\dagger] \\
&\quad + A_{lk'}\boldsymbol{\psi}'_{k'}[A,B]\boldsymbol{\psi}_l[A^\dagger,B^\dagger])\,,
\end{aligned} \tag{A.14}$$

which generalize (2.38) and (2.40). From (A.14) we obtain the symmetry relations

$$\boldsymbol{\Phi}[A,B] = \boldsymbol{\Phi}^{\mathrm{T}}[B^\dagger,A^\dagger]\,,\quad \boldsymbol{\Phi}'[A,B] = \boldsymbol{\Phi}'^{\mathrm{T}}[B^\dagger,A^\dagger]\,, \tag{A.15}$$

$$\boldsymbol{\Psi}'[A,B] = -\boldsymbol{\Psi}^{\mathrm{T}}[A^\dagger,B^\dagger]\,. \tag{A.16}$$

Eqs. (A.15) and (A.16) are a generalization of (2.39) and (2.41).

A.4. *Recursion relations*

In analogy with (3.3), (3.4) and (3.5) we define

$$\boldsymbol{F}^{(p)}_k[A,B] \equiv (\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}'[A,B]\cdot\mathbf{A}_p)\cdot\boldsymbol{\phi}_k[A,B]\,,$$

$$\boldsymbol{F}'^{(p)}_{k'}[A,B] \equiv (\mathbf{J}^{\mathrm{T}p} - \boldsymbol{\Psi}[A,B]\cdot\mathbf{A}^*_p)\cdot\boldsymbol{\phi}'_{k'}[A,B]\,,$$

$$
\begin{aligned}
&\boldsymbol{G}_k^{(p)}[A,B] \equiv \boldsymbol{\Phi}'[A,B]\cdot \mathbf{B}_p \cdot \boldsymbol{\phi}_k[B,A]\,,\\
&\boldsymbol{G}_{k'}'^{(p)}[A,B] \equiv \boldsymbol{\Phi}[A,B]\cdot \mathbf{B}_p^* \cdot \boldsymbol{\phi}_{k'}'[B,A]\,,
\end{aligned}
\tag{A.17}
$$

and the corresponding matrices $\mathbf{F}^{(p)}[A,B]$, $\mathbf{F}'^{(p)}[A,B]$, $\mathbf{G}^{(p)}[A,B]$ and $\mathbf{G}'^{(p)}[A,B]$ (cf. (A.7) and (A.8)). Then the recursion relations generalizing (3.11) and (3.12) are

$$
\begin{aligned}
\boldsymbol{F}_k^{(p+q)}[A,B] &= \mathrm{i}\partial_q \boldsymbol{F}_k^{(p)}[A,B] - \mathbf{G}'^{(p)}[A,B]\cdot \mathbf{A}_q \cdot \boldsymbol{\phi}_k[A,B]\\
&\qquad - \boldsymbol{\Phi}[A,B]\cdot \mathbf{B}_q^* \cdot \boldsymbol{G}_k^{(p)}[B,A]\,,\\
\boldsymbol{F}_{k'}'^{(p+q)}[A,B] &= -\mathrm{i}\partial_q \boldsymbol{F}_{k'}'^{(p)}[A,B] - \mathbf{G}^{(p)}[A,B]\cdot \mathbf{A}_q^* \cdot \boldsymbol{\phi}_{k'}'[A,B]\\
&\qquad - \boldsymbol{\Phi}'[A,B]\cdot \mathbf{B}_q \cdot \boldsymbol{G}_{k'}'^{(p)}[B,A]\,,\\
\mathrm{i}\partial_q \boldsymbol{G}_k^{(p)}[A,B] &= -\mathbf{F}'^{(p)}[A,B]\cdot \mathbf{B}_q \cdot \boldsymbol{\phi}_k[B,A] + \boldsymbol{\Phi}'[A,B]\cdot \mathbf{B}_q \cdot \boldsymbol{F}_k^{(p)}[B,A]\,,\\
-\mathrm{i}\partial_q \boldsymbol{G}_{k'}'^{(p)}[A,B] &= -\mathbf{F}^{(p)}[A,B]\cdot \mathbf{B}_q^* \cdot \boldsymbol{\phi}_{k'}'[B,A] + \boldsymbol{\Phi}[A,B]\cdot \mathbf{B}_q^* \cdot \boldsymbol{F}_{k'}'^{(p)}[B,A]\,,
\end{aligned}
\tag{A.18}
$$

in which ∂_q satisfies (A.10).

A.5. *Special cases leading to reductions*

For the choice

$$
C' = C^*\,,\quad \mathrm{d}\lambda'(k') = \mathrm{d}\lambda^*(k')\,,\quad \rho_{k'}' = \rho_{k'}^*\,,
\tag{A.19}
$$

it follows from (A.1) and (A.3) that

$$
\boldsymbol{\phi}_{k'}'[A,B] = \boldsymbol{\phi}_{k'}^*[A,B]\,.
\tag{A.20}
$$

Hence the set of integral equations (A.1) and (A.3) reduces to one equation, eq. (1.1). With (A.20) and

$$
\begin{aligned}
&\boldsymbol{\psi}_{k'}'[A,B] = \boldsymbol{\psi}_{k'}^*[A,B]\,,\quad \boldsymbol{\Phi}'[A,B] = \boldsymbol{\Phi}^*[A,B]\,,\\
&\quad \boldsymbol{\Psi}'[A,B] = \boldsymbol{\Psi}^*[A,B]\,,
\end{aligned}
\tag{A.21}
$$

the results presented in this appendix reduce to the ones of sections 2 and 3 in this special case.

A different reduction is obtained choosing C' and $\mathrm{d}\lambda'(k')$ such that

$$
\int_{C'} \mathrm{d}\lambda'(k')\, f_{k'} = \int_{C} \mathrm{d}\lambda(k')\, f_{-k'}
\tag{A.22}
$$

for an arbitrary function f_k depending on the spectral parameter, together with the requirement that

$$\rho'_{-k'} = -\rho_{k'} \,. \tag{A.23}$$

For the following it is convenient to introduce the $\infty \times \infty$ matrix $\boldsymbol{\varepsilon}$ with elements

$$(\boldsymbol{\varepsilon})_{ij} \equiv (-1)^i \delta_{i,j} \,, \tag{A.24}$$

so that $\boldsymbol{c}_{-k} = \boldsymbol{\varepsilon} \cdot \boldsymbol{c}_k$, $\boldsymbol{\varepsilon}^2 = \mathbf{1}$. For the solutions of the integral equations (A.1) and (A.3) we then have

$$\boldsymbol{\phi}'_{-k'}[A', B'] = -\boldsymbol{\varepsilon} \cdot \boldsymbol{\phi}_{k'}[A, B] \,, \tag{A.25}$$

with

$$A'_{kl'} = A^*_{-k,-l'} \,, \quad B'_{kl'} = B^*_{-k,-l'} \,. \tag{A.26}$$

Therefore, also in this special case, the set of integral equations (A.1) and (A.3) reduces to one equation, which is the generalization of the integral equation of type II as given in ref. 2, with the extra factors

$$A_{k,-l'} = A'^*_{-k,l'} \equiv C_{kl'} \,, \quad B^*_{-l',l} = B'_{l',-l} \equiv D_{l'l} \,. \tag{A.27}$$

In terms of the functions $C_{kl'}$ and $D_{kl'}$ defined in (A.27), we obtain here the integral equation (5.1), with

$$\boldsymbol{\phi}_k[A, B] = -\boldsymbol{\varepsilon} \cdot \boldsymbol{\phi}'_{-k}[A', B'] \equiv \boldsymbol{v}_k[C, D] \,. \tag{A.28}$$

The reduction implies also that

$$\begin{aligned}
\boldsymbol{\psi}'_{-k}[A, B] &= -\boldsymbol{\varepsilon} \cdot \boldsymbol{\psi}_k[A', B'] \equiv \boldsymbol{w}_k[C, D] \,, \\
\boldsymbol{\Phi}[A, B] &= -\boldsymbol{\varepsilon} \cdot \boldsymbol{\Phi}'[A', B'] \cdot \boldsymbol{\varepsilon} \equiv \mathbf{V}[C, D] \,, \\
\boldsymbol{\Psi}'[A, B] &= -\boldsymbol{\varepsilon} \cdot \boldsymbol{\Psi}[A', B'] \cdot \boldsymbol{\varepsilon} \equiv \mathbf{W}[C, D] \cdot \boldsymbol{\varepsilon} \,,
\end{aligned} \tag{A.29}$$

with definitions given in (5.3) and (5.4). From (A.27)–(A.29) it also follows that

$$\begin{aligned}
\boldsymbol{\phi}_k[B', A'] &= -\boldsymbol{\varepsilon} \cdot \boldsymbol{\phi}'_{-k}[B, A] \equiv \boldsymbol{v}_k[D, C] \,, \\
\boldsymbol{\psi}'_{-k}[B', A'] &= -\boldsymbol{\varepsilon} \cdot \boldsymbol{\psi}_k[B, A] \equiv \boldsymbol{w}_k[D, C] \,,
\end{aligned}$$

$$\boldsymbol{\Phi}[B', A'] = -\boldsymbol{\varepsilon} \cdot \boldsymbol{\Phi}'[B, A] \cdot \boldsymbol{\varepsilon} \equiv \mathbf{V}[D, C] , \tag{A.30}$$

$$\boldsymbol{\Psi}'[B', A'] = -\boldsymbol{\varepsilon} \cdot \boldsymbol{\Psi}[B, A] \cdot \boldsymbol{\varepsilon} \equiv \mathbf{W}[D, C] \cdot \boldsymbol{\varepsilon} .$$

For the coefficients in (1.6) and (5.2), we have, according to (A.27),

$$(-1)^m a_{n,m} = (-1)^n a'^{\,*}_{n,m} \equiv c_{n,m} , \quad (-1)^n b^*_{n,m} = (-1)^m b'_{n,m} \equiv d_{n,m} . \tag{A.31}$$

Furthermore we use the properties of $\boldsymbol{\varepsilon}$ in the multiplication with $\mathbf{J}, \mathbf{J}^{\mathrm{T}}$, viz. (A.24) and (2.1),

$$\boldsymbol{\varepsilon} \cdot \mathbf{J} = -\mathbf{J} \cdot \boldsymbol{\varepsilon} , \quad \boldsymbol{\varepsilon} \cdot \mathbf{J}^{\mathrm{T}} = -\mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{\varepsilon} , \tag{A.32}$$

and with $\mathbf{O}$, viz. (A.24) and (2.2),

$$\boldsymbol{\varepsilon} \cdot \mathbf{O} = \mathbf{O} \cdot \boldsymbol{\varepsilon} = \mathbf{O} . \tag{A.33}$$

From (A.32) and (A.33) it follows that

$$\boldsymbol{\varepsilon} \cdot \mathbf{Q}_p = (-1)^{p-1} \mathbf{Q}_p \cdot \boldsymbol{\varepsilon} \equiv (-1)^{p-1} \mathbf{R}_p , \tag{A.34}$$

with $\mathbf{Q}_p$ defined in (2.3) and $\mathbf{R}_p$ defined in (5.6). Now, using (2.5), (A.32), (A.31), (A.34) and (5.5), we find that

$$\boldsymbol{\varepsilon} \cdot \mathbf{A}_p = (-1)^{p-1} \mathbf{A}'^{\,*}_p \cdot \boldsymbol{\varepsilon} \equiv (-1)^{p-1} \mathbf{C}_p , \quad \mathbf{B}^*_p \cdot \boldsymbol{\varepsilon} = (-1)^{p-1} \boldsymbol{\varepsilon} \cdot \mathbf{B}'_p \equiv \mathbf{D}_p , \tag{A.35}$$

and applying (A.28)–(A.30), (A.32), and (A.35), eqs. (5.7)–(5.9) can be derived from (A.9), (A.11), (A.15) and (A.16). Note that in this special case (A.23) implies that (A.10) can only be satisfied for p odd, so that the differential relations (5.8) are restricted to odd p.

Eq. (5.10) can be derived evaluating $\boldsymbol{F}^{(3)}_k[A, B]$ with the recursion relations (A.18) and using the reduction (A.28)–(A.30). Moreover the Bäcklund relations, the quadratic identities and the recursion relations for the integral equation (5.1) (i.e. (A.1), (A.3) in the special case (A.22), (A.23)) can be worked out from (A.13), (A.14) and (A.18), respectively, using the reduction (A.28)–(A.30).

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