QUISPEL

SCALING OF THE "SUPERSTABLE" FRACTION OF THE 2D PERIOD-DOUBLING INTERVAL

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The scaling properties of a "superstable" parameter interval, $\mathcal{C}$, where the eigenvalues about a period-2^n orbit are complex, are derived for 2D period-doubling maps. The ratio of $\mathcal{C}$ to the whole parameter interval, between the nth and the $(n+1)$st bifurcation, is shown to be a universal function of the effective jacobian, B_e, only ($B_e \equiv B^{2^n}$, B is the jacobian of the map). Unlike the whole period-2^n interval, $\mathcal{C}$ has a convergence rate that behaves as $4.6692016 \times B_e^{-1/4}$ as $B_e \downarrow 0$, while its complement has a convergence rate $8.7210972/4$ as $B_e \uparrow 1$.

1. Introduction. The paradigm of 2D maps exhibiting infinite series of period-doubling bifurcations is the Hénon map [1]:

$$y_{t+1} = 2Cy_t - By_{t-1} + 2y_t^2 . \tag{1}$$

As we change C, keeping B fixed ($|B| < 1$), the period-1 solution (for example) bifurcates into a period-2 solution which remains stable in an interval $\mathcal{I}_1(B)$ and then bifurcates in its turn into a period-4 solution which remains stable in an interval $\mathcal{I}_2(B)$, and so on. In this letter I study the "superstable fraction":

$$F_n(B) \equiv |\mathcal{C}_{n-1}(B)| / |\mathcal{I}_{n-1}(B)| , \tag{2}$$

where $\mathcal{C}_n(B) \subset \mathcal{I}_n(B)$ is the interval where the two eigenvalues about the period-2^n orbit are complex. I find that

$$F_n(B) = F(B_e) , \tag{3}$$

where F is a universal function, of the "effective" jacobian

$$B_e \equiv B^{2^n} , \tag{4}$$

i.e. the jacobian of the 2^nth iteration of the map. I calculate this crossover scaling function F both numerically and using renormalization theory (see fig. 1). As corollaries I find

$$\frac{|\mathcal{C}_{n-1}(B)|}{|\mathcal{C}_n(B)|} = \frac{\delta_{diss}}{B_e^{1/4}} + O(1) , \quad \text{as } B_e \downarrow 0 , \tag{5}$$

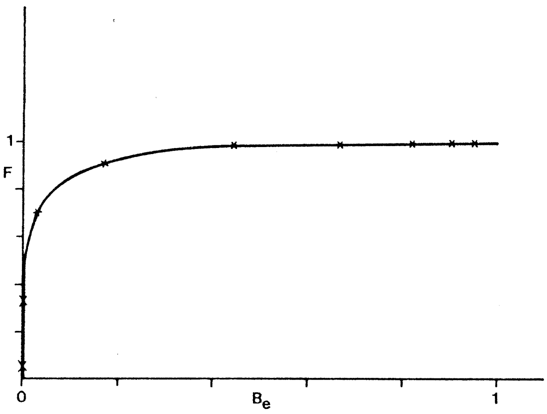

Fig. 1. The superstable fraction, F, as a function of the effective jacobian B_e. The drawn line represents the theoretical prediction, eq. (21); the crosses indicate some numerical data.

$$\frac{|\mathcal{I}_{n-1}(B)| - |\mathcal{C}_{n-1}(B)|}{|\mathcal{I}_n(B)| - |\mathcal{C}_n(B)|} = \tfrac{1}{4}\delta_{cons} + O(1 - B_e) ,$$
$$\text{as } B_e \uparrow 1 . \tag{6}$$

Here δ_{diss} and δ_{cons} are the universal dissipative and conservative Feigenbaum convergence rates respectively [1]:

$$\delta_{diss} = 4.6692016... , \tag{7}$$

$$\delta_{cons} = 8.7210972... . \tag{8}$$

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A similar result for the orbit-scaling factor α is also derived.

2. The superstable fraction F. The stability of a period-2^n orbit, $\hat{y}_t$, of the Hénon map [eq. (1)] is determined by the linearization about that orbit:

$$\begin{pmatrix}\Delta y_{t+1}\\ \Delta y_t\end{pmatrix} = \mathbf{M}_t \begin{pmatrix}\Delta y_t\\ \Delta y_{t-1}\end{pmatrix} \equiv \begin{pmatrix}2C+4\hat{y}_t & -B\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Delta y_t\\ \Delta y_{t-1}\end{pmatrix}. \quad (9)$$

The corresponding eigenvalue equation is:

$$\lambda^2 - \lambda \operatorname{Tr}\mathbf{M} + \det \mathbf{M} = 0\,, \quad (10)$$

where

$$\mathbf{M} \equiv \prod_{t=1}^{2^n} \mathbf{M}_t\,, \quad \det \mathbf{M} = B_e\,. \quad (11)$$

The period-2^n solution is born (through bifurcation from a period-2^{n-1} solution) with one of its eigenvalues equal to 1. Hence when

$$\operatorname{Tr}\mathbf{M} = 1 + B_e\,. \quad (12)$$

The period-2^n solution bifurcates into a period-2^{n+1} solution when one of its eigenvalues goes through -1. Hence when

$$\operatorname{Tr}\mathbf{M} = -1 - B_e\,. \quad (13)$$

The interval, $\mathcal{I}_n$, between these 2 bifurcations can be divided into 3 subintervals, $\mathcal{R}_n$, $\mathcal{C}_n$ and $\mathcal{R}'_n$. In $\mathcal{R}_n$ the two eigenvalues are real, in $\mathcal{C}_n$ they are complex, and in $\mathcal{R}'_n$ they are once again real. In $\mathcal{C}_n$ both eigenvalues have absolute value $\sqrt{B_e}$ and this is the interval where convergence to the periodic orbit is fastest, hence I call it the superstable interval, by analogy with the superstable point in 1-dimensional maps [1]. The subintervals are thus defined by respectively:

$$\mathcal{R}_n:\quad 1 + B_e > \operatorname{Tr}\mathbf{M} > 2\sqrt{B_e}\,,$$

$$\mathcal{C}_n:\quad 2\sqrt{B_e} > \operatorname{Tr}\mathbf{M} > 2\sqrt{B_e}\,,$$

$$\mathcal{R}'_n:\quad -2\sqrt{B_e} > \operatorname{Tr}\mathbf{M} > -1 - B_e\,. \quad (14)$$

Note that $F_n(B) \equiv |\mathcal{C}_{n-1}(B)|/|\mathcal{I}_{n-1}(B)| \uparrow 1$ as $B_e \uparrow 1$ (the conservative limit), and $F_n(B) \downarrow 0$ as $B_e \downarrow 0$ (the limit of strong dissipation).

I call the value of C at which the period-2^n solution is born $c_n(B)$; the value of C at which $\operatorname{Tr}\mathbf{M} = 2\sqrt{B_e}$ will be called $a_{n+1}(B)$, and the value of C at which $\operatorname{Tr}\mathbf{M} = -2\sqrt{B_e}$ will be called $b_{n+1}(B)$. The superstable fraction (cf. eq. (2)) is then given by:

$$F_n(B) = \frac{a_n(B) - b_n(B)}{c_{n-1}(B) - c_n(B)}\,. \quad (15)$$

This function can be renormalized using Helleman's formula [2–4], cf. also ref. [5]:

$$c_{n-1}(B^2) = -2c_n^2(B) + 2(1+B)c_n(B) + 2B^2 + 3B + 2\,, \quad (16)$$

(which also holds replacing all c's by a's or all c's by b's). Using (16) we then have for the superstable fraction:

$$\frac{F_n(B)}{F_{n-1}(B^2)} = \frac{c_{n-1}(B) + c_n(B) - 1 - B}{b_n(B) + a_n(B) - 1 - B} \xrightarrow[n\to\infty]{} 1\,. \quad (17)$$

So, for large n the superstable fraction $F_n(B)$ depends only on the effective jacobian $B_e = B^{2^n}$ (for the Hénon map, this holds already for small n):

$$F_n(B) \equiv F(B_e) = F_2(B^{2^{n-2}}) = \frac{a_2(B^{2^{n-2}}) - b_2(B^{2^{n-2}})}{c_1(B^{2^{n-2}}) - c_2(B^{2^{n-2}})}\,. \quad (18)$$

For the Hénon map, eq. (1), the trace of the period-2^0 solution is

$$T_{(0)} = 2C\,, \quad (19)$$

from (14) we therefore can find $c_0(B)$, $a_1(B)$ and $b_1(B)$ and inverting the renormalization formula (16) we find:

$$c_1(B) = -\tfrac{1}{2} - \tfrac{1}{2}B\,,$$

$$a_2(B) = \tfrac{1}{2} + \tfrac{1}{2}B - \tfrac{1}{2}(5 + 6B + 5B^2)^{1/2}\,,$$

$$b_2(B) = \tfrac{1}{2} + \tfrac{1}{2}B - \tfrac{1}{2}\sqrt{5}(1 + B)\,,$$

$$c_2(B) = \tfrac{1}{2} + \tfrac{1}{2}B - \tfrac{1}{2}(6 + 8B + 6B^2)^{1/2}\,. \quad (20)$$

Inserting (20) into (18) we get the following analytical formula for the superstable fraction F:

$$F(B_e) = \frac{\sqrt{5}(1 + B_e^{1/4}) - (5 + 6B_e^{1/4} + 5B_e^{1/2})^{1/2}}{(6 + 8B_e^{1/4} + 6B_e^{1/2})^{1/2} - 2 - 2B_e^{1/4}}\,. \quad (21)$$

In fig. 1 this first-order renormalization curve is seen to fit the numerical data very well.

Of special interest is the behaviour of $F(B_e)$ near $B_e = 1$ (the conservative limit) and near $B_e = 0$ (the limit of strong dissipation). We therefore make two Taylor expansions about these two limiting points:

$$F(B_e) = 1 - (1/64 + 9/1280\sqrt{5})(1 - B_e)^2 + O(1 - B_e)^3$$
$$\approx 1 - 0.31347(1 - B_e)^2 + O(1 - B_e)^3 ,$$
$$B_e \uparrow 1 , \tag{22}$$

$$F(B_e) = (2/5\sqrt{5} + 1/5\sqrt{30})B_e^{1/4} + O(B_e^{1/2})$$
$$\approx 1.9898723 B_e^{1/4} + O(B_e^{1/2}) , \quad B_e \downarrow 0 . \tag{23}$$

These theoretical expansions for the asymptotic behaviour of the superstable fraction F are compared with numerical data in figs. 2 and 3 respectively.

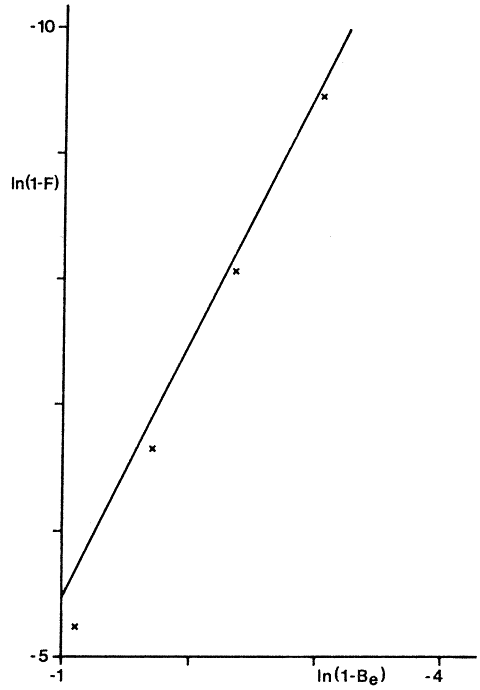

Fig. 2. The superstable fraction, F, in the conservative limit: $B_e \uparrow 1$. The drawn line represents the theoretical prediction, eq. (22); the crosses indicate some numerical data.

3. The period-doubling exponents $\delta_{\mathcal{R}}$, $\delta_{\mathcal{C}}$ *and* $\alpha_{\mathcal{R}}$. Due to the fact that $F(B_e)$ is not a constant, $\mathcal{C}_n(B)$ and its complement, $\mathcal{G}_n(B) - \mathcal{C}_n(B)$, have different

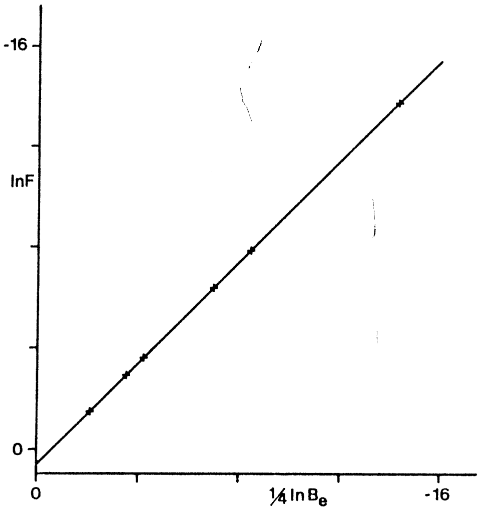

Fig. 3. The superstable fraction, F, in the limit of strong dissipation: $B_e \downarrow 0$. The drawn line represents the theoretical prediction, eq. (23); the crosses indicate some numerical data.

rates of convergence. It turns out, numerically, that $|\mathcal{R}_{n-1}(B)|/|\mathcal{R}_n(B)|$ and $|\mathcal{R}'_{n-1}(B)|/|\mathcal{R}'_n(B)|$ are (almost) equal, hence the convergence rate $\delta_{\mathcal{R}}$ of the interval $\mathcal{R}$ is given by

$$\delta_{\mathcal{R},n}(B) = \frac{|\mathcal{R}_{n-1}(B)|}{|\mathcal{R}_n(B)|} = \frac{1 - F_n(B)}{1 - F_{n+1}(B)}\, \delta_{\mathcal{G},n}(B)$$
$$= \frac{1 - F(B_e)}{1 - F(B_e^2)}\, \delta_{\mathcal{G}}(B_e) , \tag{24}$$

where an analytical formula for the convergence rate of the whole interval $\mathcal{G}$, $\delta_{\mathcal{G}}(B_e)$, was derived in ref. [3], cf. also refs. [4,6–8]. Analogously we obtain the rate of convergence $\delta_{\mathcal{C}}$ for the interval $\mathcal{C}$:

$$\delta_{\mathcal{C},n}(B) = \frac{|\mathcal{C}_{n-1}(B)|}{|\mathcal{C}_n(B)|} = \frac{F(B_e)}{F(B_e^2)}\, \delta_{\mathcal{G}}(B_e) . \tag{25}$$

Just as we can assign a convergence rate $\delta_{\mathcal{R}}$ to the interval $\mathcal{R}$ we can also assign an orbit scaling factor $\alpha_{\mathcal{R}}$ to it:

$$\alpha_{\mathcal{R},n}(B) \equiv \Delta y_n(a_n(B))/\Delta y_{n+1}(a_{n+1}(B)) . \tag{26}$$

It turns out numerically that

$$\delta_{\mathcal{R}}(B_e)/\alpha^2_{\mathcal{R}}(B_e) = \delta_{\mathcal{G}}(B_e)/\alpha^2_{\mathcal{G}}(B_e) , \tag{27}$$

hence the orbit scaling factor for the interval $\mathcal{R}$ is given by

$$\alpha_{\mathcal{R}}(B_e) = \alpha_{\mathcal{G}}(B_e)\left(\frac{1-F(B_e)}{1-F(B_e^2)}\right)^{1/2}, \qquad (28)$$

where $\alpha_{\mathcal{G}}(B_e)$ was determined numerically in ref. [6] (strictly speaking the α determined there was defined slightly differently). Analytically $\alpha_{\mathcal{G}}(B_e)$ can be inferred from the formulas given in ref. [3]. Inserting the Taylor expansions (22) and (23) in the equations for $\delta_{\mathcal{R}}$(24), $\alpha_{\mathcal{R}}$ (28) and $\delta_{\mathcal{C}}$(25) we get:

$$\delta_{\mathcal{R}}(B_e) = \tfrac{1}{4}\delta_{\rm cons} + O(B_e - 1), \quad B_e \uparrow 1, \qquad (29)$$

$$\alpha_{\mathcal{R}}(B_e) = \tfrac{1}{2}\alpha_{\rm cons} + O(B_e - 1), \quad B_e \uparrow 1, \qquad (30)$$

$$\delta_{\mathcal{C}}(B_e) = \frac{\delta_{\rm diss}}{B_e^{1/4}} + O(1), \quad B_e \downarrow 0, \qquad (31)$$

where $\delta_{\rm cons}$, $\alpha_{\rm cons}$ and $\delta_{\rm diss}$ are the well-known conservative and dissipative Feigenbaum exponents respectively,

$$\delta_{\rm cons} = 8.7210972..., \quad \alpha_{\rm cons} = -4.0180767...,$$

$$\delta_{\rm diss} = 4.6692016...,$$

The factors $\frac{1}{4}$ respectively $\frac{1}{2}$ in eqs. (29) and (30) are direct consequences of the exponent 2 of the second term in the Taylor expansion (22). The accuracy of eq. (22) may be inferred from fig. 2. Analogously for eq. (31). I have calculated $\alpha_{\mathcal{R}}(B_e)$ also for two other maps, viz. for $y_{t+1} = 2Cy_t - By_{t-1} + y_t^4$ and $y_{t+1} = 2Cy_t - B \tanh y_{t-1} + 2y_t^2$. It turns out that these maps possess the same orbit-scaling function $\alpha_{\mathcal{R}}(B_e)$ as the Hénon map. This indicates the universality of results.

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