EQUATION OF MOTION FOR THE HEISENBERG SPIN CHAIN

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A description in terms of one real variable $q(x, t)$ is proposed for the equation of motion of the classical Heisenberg spin chain with uniaxial anisotropy.

In this letter we consider the classical Heisenberg spin chain in the case of axial symmetry, which in the continuum limit is described by the equation of motion [1,2]

$$\dot{\boldsymbol{S}} = \boldsymbol{S} \times \mathbf{J} \cdot \boldsymbol{S}'' + \boldsymbol{S} \times \boldsymbol{\nabla} F , \tag{1}$$

where

$$\mathbf{J} = \mathrm{diag}\,(1, 1, 1 + c) \tag{2}$$

is the exchange interaction between neighbouring spins. $\boldsymbol{S} = \boldsymbol{S}(x, t)$ is the (normalized) spin density in the continuum limit as a function of the time t and the position x on the chain, and the dot and the prime denote the differentiations with respect to t and x, respectively, e.g. $\dot{\boldsymbol{S}} = \partial \boldsymbol{S}/\partial t$ and $\boldsymbol{S}' = \partial \boldsymbol{S}/\partial x$. $F = F(S^z)$ is an anisotropy field with uniaxial symmetry acting on the spins and arising e.g. from a crystalline field and an external magnetic field in the z-direction, and $\boldsymbol{\nabla} = (0, 0, \partial/\partial S^z)$.

In a two-component ($n = 2$) description with polar angles $\theta(x, t)$, $\phi(x, t)$, i.e.

$$\boldsymbol{S}(x, t) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) , \tag{3}$$

eq. (1) can be rewritten

$$(\partial/\partial t)(\cos\theta) = (\partial/\partial x)(\phi' \sin^2\theta) , \tag{4}$$

$$\dot{\phi} = \theta''/\sin\theta - \phi'^2 \cos\theta + c(\theta'' \sin\theta + \theta'^2 \cos\theta) - \partial F/\partial S^z . \tag{5}$$

Eq. (4) can be formally solved introducing a potential function $q(x, t)$ such that

$$\cos\theta = q'(x, t) . \tag{6}$$

Then

$$\phi' = \dot{q}/(1 - q'^2) , \tag{7}$$

where an integration constant $f(t)$ has been absorbed in q.

Inserting (6) and (7) in (5) we obtain

$$\dot{\phi} = -\frac{q'''}{1 - q'^2} - \frac{(q''^2 + \dot{q}^2)q'}{(1 - q'^2)^2} - cq''' - \frac{\partial}{\partial q'} F(q') . \tag{8}$$

From (7) and (8) it follows that $q(x, t)$ must obey the compatibility condition

$$\frac{\partial}{\partial t}\left(\frac{\dot{q}}{1 - q'^2}\right) = \frac{\partial}{\partial x}\left(\frac{-q'''}{1 - q'^2} - \frac{(q''^2 + \dot{q}^2)q'}{(1 - q'^2)^2} - cq''' - \frac{\partial}{\partial q'} F(q')\right) . \tag{9}$$

The one-component ($n = 1$) description with the variable $q(x, t)$ in eq. (9) is completely equivalent to the $n = 2$ description in eqs. (4) and (5).

In fact, for a given solution $q(x, t)$ of eq. (9), the corresponding $\theta(x, t)$ and $\phi(x, t)$ are given by

$$\theta(x, t) = \arccos q'(x, t) , \tag{10}$$

$$\phi(x, t) = \int_C \{\mathrm{d}l_x\, \phi'(x_1, t_1) + \mathrm{d}l_t\, \dot{\phi}(x_1, t_1)\} + \phi(0, 0) , \tag{11}$$

where (x_1, t_1) denotes a point on an arbitrary curve C in the xt-plane going from (0,0) to (x, t), $(\mathrm{d}l_x, \mathrm{d}l_t)$

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is an infinitesimal two-dimensional vector tangent to C at (x_1, t_1). ϕ' and $\dot{\phi}$ are given by eqs. (7) and (8) and $\phi(0, 0)$ is an arbitrary constant. The result in the right-hand side of (11) is independent of the choice of the curve C, as can be seen using Stokes' theorem and the compatibility relation (9).

On the other hand, from any solution $\theta(x, t)$, $\phi(x, t)$ of eqs. (4), (5), one obtains the following solution of eq. (9):

$$q(x,t) = \int_C \{ \mathrm{d}l_x \cos\theta(x_1, t_1) + \mathrm{d}l_t (\phi' \sin^2\theta)(x_1, t_1)\} + q(0,0) , \tag{12}$$

where $q(0, 0)$ is an arbitrary constant.

In connection with eq. (9) the following remarks can be made:

(i) In the special case

$$c = 0 , \quad F(q') = \tfrac{1}{2} A q'^2 - B q' , \tag{13}$$

it is clear from refs. [3,4] that the fourt-order partial differential equation (9) is completely integrable.

(ii) The equation of motion (9) can also be derived from the lagrangian density

$$\mathcal{L}(\dot{q}, q', q'') = \frac{1}{2} \frac{\dot{q}^2 - q''^2}{1 - q'^2} - \tfrac{1}{2} c q''^2 + F(q') , \tag{14}$$

using the Lagrange equations

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{q}} + \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial q'} - \frac{\partial^2}{\partial x^2} \frac{\partial \mathcal{L}}{\partial q''} = 0 . \tag{15}$$

(iii) The lagrangian (14) is a convenient starting point for obtaining similarity solutions. In the isotropic case with $c = F = 0$, we have the similarity solutions

$$q(x,t) = px + \tilde{q}(x - vt) , \tag{16a}$$

where p is a constant, and

$$q(x,t) = t^{1/2} \hat{q}(\eta) , \quad \eta = x t^{-1/2} . \tag{16b}$$

Eq. (16a) leads to one-soliton and spin-wave solutions [5,1]. The general one-soliton solution for the isotropic Heisenberg spin chain [5] is equivalent to the following solution of eq. (9) for $c = F = 0$:

$$q(x,t) = \cos\psi \times \{x - 4a(a^2 + v^2)^{-1} \tanh[\tfrac{1}{2} a(x - vt) + \gamma]\} + \sin\psi \frac{4a}{a^2 + v^2} \frac{\cos[\frac{1}{2} vx + \frac{1}{4}(a^2 - v^2)t + d]}{\cosh[\frac{1}{2} a(x - vt) + \gamma]} , \tag{17}$$

where ψ, a, v, γ and d are constants.

Inserting eq. (16b) in (9) we find

$$\left(\frac{\mathrm{d}^2\hat{q}}{\mathrm{d}\eta^2}\right)^2 + \frac{1}{4}\left(\hat{q} - \eta \frac{\mathrm{d}\hat{q}}{\mathrm{d}\eta}\right)^2 = \left(\delta + \epsilon \frac{\mathrm{d}\hat{q}}{\mathrm{d}\eta}\right)\left\{\left(\frac{\mathrm{d}\hat{q}}{\mathrm{d}\eta}\right)^2 - 1\right\} , \tag{18}$$

where δ and ϵ are integration constants. Eq. (18) can be related to the fourth Painlevé transcendent, see ref. [6]. Details will be published in the future [7].

(iv) Recently it has been shown that there is a relation between the isotropic Heisenberg spin chain and the nonlinear Schrödinger equation [8,9]. In connection with this it is of interest to note that, for every solution $q(x, t)$ of eq. (9) with $c = F = 0$, the function

$$y(x,t) = \int_C \{\mathrm{d}l_x\, y'(x_1, t_1) + \mathrm{d}l_t\, \dot{y}(x_1, t_1)\} , \tag{19}$$

cf. (11) for the notation, where

$$y' = \frac{1}{8\alpha}\left(\frac{q''^2 + \dot{q}^2}{1 - q'^2}\right) , \tag{20}$$

$$\dot{y} = \frac{1}{4\alpha}\left\{\frac{q''\dot{q}' - q'''\dot{q}}{1 - q'^2} - \frac{\dot{q}q'(q''^2 + \dot{q}^2)}{(1 - q'^2)^2}\right\} , \tag{21}$$

is a solution of the equations

$$y'^2\ddot{y} + y''\dot{y}^2 - 2y'\dot{y}\dot{y}' + y'^2 y'''' - 2y'y''y''' + y''^3 + 8\alpha y'^3 y'' = 0 . \tag{22}$$

Eq. (22) is the equivalent real form of the nonlinear Schrödinger equation which has been derived (independently) in ref. [10]. Note that the right-hand side of (19) is independent of the choice of the curve C, in view of the compatibility relation $\partial y'/\partial t = \partial \dot{y}/\partial x$ which is identically satisfied if $q(x, t)$ obeys eq. (9) with $c = F = 0$. The proof that $y(x, t)$ is actually a solution of (22) is a matter of straightforward but rather tedious algebra, and will be given in a more extended publication [7].

(v) The considerations in this letter have been restricted to the equation of motion in the continuum limit. The equation of motion for the discrete classical Heisenberg chain with uniaxial anisotropy can also be reformulated in an $n = 1$ description with one real variable $q_m(t)$ assosicated with every site m of the chain [7].

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References

[1] M. Lakshmanan, Th.W. Ruijgrok and C.J. Thompson, Physica 84A (1976) 577.
[2] J. Tjon and J. Wright, Phys. Rev. B15 (1977) 3470.
[3] A.E. Borovik, JETP Lett. 28 (1978) 581.
[4] E.K. Sklyanin, LOMI preprint E-3-1979.
[5] L.A. Takhtadzhyan, Phys. Lett. 64A (1977) 235.
[6] H.T. Davis, Introduction to nonlinear differential and integral equations (Dover Publ., New York, 1962).
[7] G.R.W. Quispel and H.W. Capel, to be published.
[8] M. Lakshmanan, Phys. Lett. 61A (1977) 53.
[9] V.E. Zakharov and L.A. Takhtadzhyan, Theor. Math. Phys. 38 (1979) 17.
[10] M. Boiti, C. Laddomada and F. Pempinelli, Nuovo Cimento 62B (1981) 315.