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BÄCKLUND TRANSFORMATIONS AND THREE-DIMENSIONAL LATTICE EQUATIONS

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A (nonlocal) linear integral equation is studied, which allows for Bäcklund transformations in the measure. The compatibility of three of these transformations leads to an integrable nonlinear three-dimensional lattice equation. In appropriate continuum limits the two-dimensional Toda-lattice equation and the Kadomtsev–Petviashvili equation are derived as special examples.

1. The investigation of nonlinear evolution equations in multi-dimensional space–time has received much interest recently [1–7]. Although a lot of progress has been made in extending the inverse-scattering transform to higher-dimensional equations such as the Kadomtsev–Petviashvili (KP) equation [1], some of the essential features are not completely understood. In order to obtain more insight in the structure of higher-dimensional integrable systems it may be useful to start from higher-dimensional partial difference equations (PDFEs) and their exact linearizations. The reason is that PDFEs with enough free parameters contain the complete information on the system, not only on the lattice, but also in the continuum. In fact, taking appropriate continuum limits one can derive integrable PDEs and their exact linearizations associated with various dispersions.

For a two-dimensional lattice we have shown that the (algebraic) identities for the commutativity of Bäcklund transformations (BTs) can be interpreted as integrable PDFEs [8]. In this letter we show that the same holds true for the three-dimensional case. In order to show this we study a linear integral equation which contains an integration over an arbitrary two-dimensional hypersurface in terms of two complex variables with an arbitrary measure depending on these two complex variables. In this case BTs are generated by nonlocal transformations of the measure, and for each BT one can derive linear relations for the wavefunctions of the problem. The compatibility of three of these relations under three different BTs leads to an algebraic indentity which can be interpreted as a three-dimensional lattice equation. By considering various continuum limits we derive the two-dimensional Toda equation and the KP equation. Finally a reduction to equations of lower dimensionality is discussed.

2. Consider the (linear) integral equation

$$u_k(\alpha) + \mathrm{i}\rho_k \iint_D \mathrm{d}\zeta(l, l')\, \frac{\sigma_{l'} u_l(\alpha)}{k + l'} = \frac{\rho_k}{k + \alpha}\,, \tag{1}$$

in which the wavefunction $u_k(\alpha) \equiv u_{k;n,m,h}(\alpha)$ is defined at the sites (n, m, h) of an infinite three-dimensional lattice, and

$$\rho_k \equiv \rho_{k;n,m,h} = (1-k/p)^n\,(1-k/q)^m(1-k/r)^h, \quad \sigma_{k'} \equiv \sigma_{k';n,m,h} = (1+k'/p)^{-n}\,(1+k'/q)^{-m}\,(1+k'/r)^{-h}\,. \tag{2}$$

In (1) the measure $\mathrm{d}\zeta(l,l')$ is an arbitrary function of two complex variables l and l', and the integration is performed over an arbitrary hypersurface D in the complex hyperplane of l and l'. D and $\mathrm{d}\zeta(l,l')$ are to be chosen such that the solution $u_k(\alpha)$ of (1) for fixed α as a function of k is unique. In terms of the quantity

$$s_{n,m,h} \equiv \mathrm{i} \iint_D \mathrm{d}\zeta(l,l')\,\frac{\sigma_{l';n,m,h}\,u_{l;n,m,h}(\alpha)}{l'+\beta}\,, \tag{3}$$

with the same hypersurface D and measure $\mathrm{d}\zeta(l,l')$ the following PDFE can be derived

$$\begin{aligned}
&[1+(p-\beta)s_{n+1,m+1,h}-(p+\alpha)s_{n,m+1,h}]\,[1+(q-\beta)s_{n,m+1,h+1}-(q+\alpha)s_{n,m,h+1}]\\
&\quad\times[1+(r-\beta)s_{n+1,m,h+1}-(r+\alpha)s_{n+1,m,h}]\\
&\quad=[1+(p-\beta)s_{n+1,m,h+1}-(p+\alpha)s_{n,m,h+1}]\,[1+(q-\beta)s_{n+1,m+1,h}-(q+\alpha)s_{n+1,m,h}]\\
&\quad\times[1+(r-\beta)s_{n,m+1,h+1}-(r+\alpha)s_{n,m+1,h}]\,.
\end{aligned} \tag{4}$$

Eq. (4) is an integrable (three-dimensional) lattice equation in the sense that solutions can be inferred from the linear integral equation (1). Furthermore it will be shown below that in a proper continuum limit the KP equation and its direct linearization (cf. refs. [2,7]) can be derived from (4) and (1).

3. In order to prove (4) it is convenient to introduce the integral equations

$$\boldsymbol{u}_k + \mathrm{i}\rho_k \iint_D \mathrm{d}\zeta(l,l')\,\frac{\sigma_{l'}\boldsymbol{u}_l}{k+l'} = \rho_k\boldsymbol{c}_k, \qquad \boldsymbol{v}_{k'} + \mathrm{i}\sigma_{k'} \iint_D \mathrm{d}\zeta(l,l')\,\frac{\rho_l\boldsymbol{v}_{l'}}{k'+l} = \sigma_{k'}\boldsymbol{c}_{k'}\,, \tag{5a, b}$$

for a wavevector $\boldsymbol{u}_k$ and an associated wavevector $\boldsymbol{v}_{k'}$, respectively. The notation is the same as in (1), but here $\boldsymbol{u}_k$ and $\boldsymbol{v}_{k'}$ are vectors with components $u_k^{(i)}$ and $v_k^{(i)}$ respectively corresponding to a factor $c_k^{(i)} \equiv k^{-i}$ on the right-hand side. Furthermore at this stage we do not specify any dependence of $\rho_k, \sigma_{k'}, \boldsymbol{u}_k$ and $\boldsymbol{v}_{k'}$ on additional variables.

Let us apply a simultaneous transformation of the factors ρ_k and $\sigma_{k'}$, i.e. $\rho_k \to \tilde{\rho}_k = (1-k/p)\,\rho_k$, $\sigma_{k'} \to \tilde{\sigma}_{k'} = (1+k'/p)^{-1}\,\sigma_{k'}$. This is equivalent to a (Bäcklund) transformation of the measure

$$\mathrm{d}\zeta(l,l') \to \mathrm{d}\tilde{\zeta}(l,l') = [(p-l)/(p+l')]\,\mathrm{d}\zeta(l,l')\,,$$

which is nonlocal as it depends essentially on the two variables l and l'. From (5) and the uniqueness of its solution one can derive

$$\boldsymbol{u}_k = (1-p^{-1}\mathbf{J}^\mathrm{T})\cdot\tilde{\boldsymbol{u}}_k + p^{-1}\,\tilde{\mathbf{U}}\cdot\mathbf{O}\cdot\boldsymbol{u}_k, \quad \boldsymbol{v}_{k'} = (1+p^{-1}\,\mathbf{J}^\mathrm{T})\cdot\tilde{\boldsymbol{v}}_{k'} - p^{-1}\,\mathbf{V}\cdot\mathbf{O}\cdot\tilde{\boldsymbol{v}}_{k'}\,, \tag{6}$$

where it is understood that $\tilde{\boldsymbol{u}}_k, \tilde{\boldsymbol{v}}_{k'}$ is the solution of (5a, b) with factors $\tilde{\rho}_k, \tilde{\sigma}_{k'}$ instead of $\rho_k, \sigma_{k'}$. In (6) we have introduced matrices $\mathbf{J}^\mathrm{T}$, $\mathbf{O}$ with elements $J^\mathrm{T}_{ij} = \delta_{i,j+1}$, $\mathrm{O}_{ij} = \delta_{i,0}\delta_{j,0}$ and the dyadics

$$\mathbf{U} \equiv \mathrm{i} \iint_D \mathrm{d}\zeta(l,l')\,\boldsymbol{u}_l\boldsymbol{c}_{l'}\sigma_{l'}, \quad \mathbf{V} \equiv \mathrm{i} \iint_D \mathrm{d}\zeta(l,l')\boldsymbol{v}_{l'}\boldsymbol{c}_l\rho_l = \mathbf{U}^\mathrm{T}\,, \tag{7}$$

(the suffix T denotes matrix transposition). Dividing the first relation in (6) by $(\mathbf{J}^\mathrm{T}+\alpha)\,(l'+\beta)$ and integrating over D including a factor $\sigma_{l'}$ we can derive

$$1 + (p - \beta)\tilde{s} - (p + \alpha)s = [1 - \tilde{u}(\alpha)]\,[1 - v(\beta)]\,, \tag{8}$$

in which the s, as in (3), $u(\alpha)$ and $v(\beta)$ are obtained as the (0,0) elements of the matrices $(\mathbf{J}^{\mathrm{T}} + \alpha)^{-1}\cdot\mathbf{U}\cdot(\mathbf{J} + \beta)^{-1}$, $(\mathbf{J}^{\mathrm{T}} + \alpha)^{-1}\cdot\mathbf{U}$ and $\mathbf{U}\cdot(\mathbf{J} + \beta)^{-1}$ respectively (The matrix $\mathbf{J}$ has elements $J_{ij} = \delta_{j,\,i+1}$). A relation similar to (8) was also valid in the two-dimensional case [8], but in contrast to that case we do not have a relation for the inverse BT due to the nonlocal structure of the BT. Therefore, we need three instead of two separate BTs in order to get a closed equation in terms of one variable. For that purpose we introduce two more transformations: $\rho_k \to \hat{\rho}_k = (1 - k/q)\rho_k$, $\sigma_{k'} \to \hat{\sigma}_{k'} = (1 + k'/q)^{-1}\sigma_{k'}$, and $\rho_k \to \rho_k \to \rho_k' = (1 - k/r)\rho_k$, $\sigma_{k'} \to \sigma_{k'}' = (1 + k'/r)^{-1}\sigma_{k'}$. From (6) and two analogous relations for the two other transformations it is straightforward to derive an algebraic identity in terms of $\tilde{s}, \hat{s}, s', \hat{\tilde{s}}, \tilde{s}'$ and $\hat{s}'$, which is, in fact, a superposition formula for the three BTs, cf. also ref. [9]. Considering the factors ρ_k and $\sigma_{k'}$ as in (2) it is clear that the primitive translations in the n, m and h directions correspond to the three BTs given above. Taking $s \equiv s_{n,m,h}$ and identifying $\tilde{s}, \hat{s}, s'$ with $s_{n+1,m,h}$, $s_{n,m+1,h}$, $s_{n,m,h+1}$ respectively eq. (4) is immediately obtained from the algebraic identity.

4. We now work out three successive continuum limits in order to derive counterparts of (4) with 1, 2 and 3 continuous variables. At each stage the linearizing integral equation for the counterpart of (4) is directly obtained from (1) by inserting the proper factors ρ_k and $\sigma_{l'}$ associated with the continuum limit under consideration.

(i) *First continuum limit*. Consider the case that $m \to \infty$, $q - p \to 0$, $n \to -\infty$, such that $(q - p)m = \tau$ fixed and $n + m = n'$ fixed, implying that

$$\rho_{k;n,m,h} \to \rho_{k;n',h}(\tau) = (1 - k/p)^{n'}\,(1 - k/r)^h \exp[k\tau/p(p - k)]\,,$$

$$\sigma_{k';n,m,h} \to \sigma_{k';n',h}(\tau) = (1 + k'/p)^{-n'}\,(1 + k'/r)^{-h} \exp[k'\tau/p(p + k')]\,, \tag{9}$$

and $s_{n,m,h} \to s_{n',h}(\tau)$. Then $s_{n',h}(\tau)$ satisfies the differential–partial difference equation

$$\frac{(p - \beta)\partial_\tau s_{n'+1,h+1} + (s_{n'+1,h+1} - s_{n',h+1})}{1 + (p - \beta)s_{n'+1,h+1} - (p + \alpha)s_{n',h+1}} - \frac{(p + \alpha)\partial_\tau s_{n'+1,h} + (s_{n'+2,h} - s_{n'+1,h})}{1 + (p - \beta)s_{n'+2,h} - (p + \alpha)s_{n'+1,h}}$$
$$= \frac{(r - \beta)\partial_\tau s_{n'+1,h+1} - (r + \alpha)\partial_\tau s_{n'+1,h}}{1 + (r - \beta)s_{n'+1,h+1} - (r + \alpha)s_{n'+1,h}}\,. \tag{10}$$

(ii) *Second continuum limit*. Consider the case that $r \to \infty$, $h \to \infty$ such that $r^{-1}h = \eta$ fixed, implying that

$$\rho_{k;n',h}(\tau) \to \rho_{k;n'}(\eta, \tau) = (1 - k/p)^{n'} \exp[-k\eta + k\tau/p(p - k)]\,,$$

$$\sigma_{k';n',h}(\tau) \to \sigma_{k';n'}(\eta, \tau) = (1 + k'/p)^{-n'} \exp[-k'\eta + k'\tau/p(p + k')]\,, \tag{11}$$

and $s_{n',h}(\tau) \to s_{n'}(\eta, \tau)$. Then $s_{n'}(\eta, \tau)$ satisfies the partial differential–difference equation

$$\frac{(p - \beta)\,\partial_\tau s_{n'} - (s_{n'-1} - s_{n'})}{1 + (p - \beta)s_{n'} - (p + \alpha)s_{n'-1}} - \frac{(p + \alpha)\,\partial_\tau s_{n'} + (s_{n'+1} - s_{n'})}{1 + (p - \beta)s_{n'+1} - (p + \alpha)s_{n'}} = \frac{\partial_\eta\partial_\tau s_{n'} - (\alpha + \beta)\,\partial_\tau s_{n'}}{1 + \partial_\eta s_{n'} - (\alpha + \beta)s_{n'}}\,. \tag{12}$$

(iii) *Third continuum limit*. Consider finally the case that $n' \to \infty$, $p \to \infty$ such that $p^{-1}n' \to \xi$, with $-\eta - \xi + p^{-2}\tau = x$, $\frac{1}{2}p^{-1}\,\xi + p^{-3}\tau = y$, $-\frac{1}{3}p^{-2}\,\xi + p^{-4}\,\tau = t$ fixed, such that

$$\rho_{k;n'}(\eta, \tau) \to \rho_k(x, y, t) = \exp(kx - k^2y + k^3t)\,,$$

$$\sigma_{k',n'}(\eta, \tau) \to \sigma_{k'}(x, y, t) = \exp(k'x + k'^2y + k'^3t)\,, \tag{13}$$

and $s_{n'}(\eta, \tau) \to s(x, y, t)$. Then s satisfies

$$\tfrac{1}{3}\,\partial_x\,\partial_t\,s-\tfrac{1}{4}\,\partial_y^2\,s-\tfrac{1}{12}\partial_x^4\,s=-\,[1-(\alpha+\beta)\,s-\partial_x s]^{-1}\,(\beta\partial_x s+\tfrac{1}{2}\,\partial_y s+\tfrac{1}{2}\,\partial_x^2 s)\,[-\tfrac{1}{2}\,\alpha(\partial_y s+\partial_x^2 s)+\tfrac{1}{3}\,(\partial_t s-\partial_x^3 s)]$$
$$-\,[1-(\alpha+\beta)s-\partial_x s]^{-1}\,(\alpha\partial_x s-\tfrac{1}{2}\,\partial_y s+\tfrac{1}{2}\,\partial_x^2 s)\,[\tfrac{1}{2}\,\beta(\partial_y s-\partial_x^2 s)+\tfrac{1}{3}(\partial_t s-\partial_x^3 s)]$$
$$+\,[1-(\alpha+\beta)s-\partial_x s]^{-2}\,[(\alpha+\beta)\,\partial_x s+\partial_x^2 s]\,(\alpha\partial_x s-\tfrac{1}{2}\,\partial_y s+\tfrac{1}{2}\,\partial_x^2 s)\,(\beta\partial_x s+\tfrac{1}{2}\,\partial_y s+\tfrac{1}{2}\,\partial_x^2 s)\,. \quad (14)$$

Eqs. (10), (12) and (14) are integrable in the sense that solutions can be obtained from the linear integral equation (1) with (3), by inserting the proper factors ρ_k, $\sigma_{k'}$, i.e. (9), (11) and (13) respectively. From (6) it is also possible to derive BTs for eqs. (10), (12) and (14) in terms of s and $\widetilde{s}$, where $\widetilde{s}$ is considered to be the Bäcklund transformed of s.

For different values of α and β various equations can be found. We mention in particular two examples related to the function u, which is the limit of $\alpha\beta s$ as $\alpha, \beta \to \infty$ (or equivalently the (0,0) element of the matrix **U**). In that limit one can derive from (12) for the function $v_n \equiv \ln(1+\partial_\tau u_n)$,

$$\partial_\eta\,\partial_\tau v_n+\exp(v_{n+1})+\exp(v_{n-1})-2\,\exp(v_n)=0\,, \quad (15)$$

which is related to the two-dimensional Toda equation, i.e.

$$\partial_\eta\,\partial_\tau \phi_n+\exp(\phi_{n+1}-\phi_n)-\exp(\phi_n-\phi_{n-1})=0\,, \quad (16)$$

(cf. ref. [10], and also refs. [11–14]) by the substitution $v_n=\phi_n-\phi_{n-1}$. It is easily noted that eq. (16) can be obtained directly from (12) by taking $\beta=p$ and the limit $(p+\alpha)s_n\to 1-\exp(\phi_n)$ as $\alpha\to\infty$. Furthermore from (14) we have

$$\partial_x\,[\partial_t u-\tfrac{1}{4}\,\partial_x^3 u-\tfrac{3}{2}\,(\partial_x u)^2]=\tfrac{3}{4}\,\partial_y^2 u\,, \quad (17)$$

which is the (potential) KP equation [1]. The BTs for the KP and Toda equation can be derived from the explicit expressions for the transformation $s\to\widetilde{s}$, cf. refs. [9,13,15].

5. So far we have considered in (1) a reference-free integral equation in which the free-reference solutions $\rho_k c_k$ and $\sigma_{k'} c_{k'}$ of the linear relations (6) are simple functions associated with a trivial potential. In order to obtain a more general type of integral equation, which relates arbitrary given solutions $\boldsymbol{u}_k^0, \boldsymbol{v}_{k'}^0$ of (6) with potentials $\mathbf{U}^0$, $\mathbf{V}^0$ to new solutions $\boldsymbol{u}_k, \boldsymbol{v}_{k'}$ of (6), we introduce the function

$$G^0_{kl'}(r)=\sum_{\Gamma} p_\nu^{-1}\,v^0_{l'}(r'+e_\nu)\,u^0_k(r'),\quad (\nu=n,m,h)\,, \quad (18)$$

defined at the sites $\boldsymbol{r}=n\boldsymbol{e}_n+m\boldsymbol{e}_m+h\boldsymbol{e}_h$ of a 3-dimensional lattice, where the $\boldsymbol{e}_\nu$ are unit vectors in the ν-direction and $p_\nu=p,q,r$ for $\nu=n,m,h$. In (18) the summation is over nearest neighbour steps $\boldsymbol{r}'\to\boldsymbol{r}'+\boldsymbol{e}_\nu$ along a path Γ starting at an arbitrary lattice point P and with a last step for which $\boldsymbol{r}'+\boldsymbol{e}_\nu=\boldsymbol{r}$. The scalars $u_k^0, v_{k'}^0$ are the $i=0$ components of the vectors $\boldsymbol{u}_k^0, \boldsymbol{v}_{k'}^0$ of the given solutions. For given P the function $G_{kl'}$ is independent of the path Γ, as a consequence of the relation

$$p_\nu^{-1}\,[v^0_{l'}(r+e_\nu+e_{\nu'})u^0_k(r+e_{\nu'})-v^0_{l'}(r+e_\nu)u^0_k(r)]$$
$$=p_{\nu'}^{-1}\,[v^0_{l'}(r+e_\nu+e_{\nu'})u^0_k(r+e_\nu)-v^0_{l'}(r+e_{\nu'})u^0_k(r)]\quad(\nu,\nu'=n,m,h)\,, \quad (19)$$

which follows from (6). We can now write down the integral equations

$$\boldsymbol{u}_k=\boldsymbol{u}^0_k+\iint_D d\zeta(l,l')\,\boldsymbol{u}_l\,G^0_{kl'},\quad \boldsymbol{v}_{k'}=\boldsymbol{v}^0_{k'}+\iint_D d\zeta(l,l')\,\boldsymbol{v}_{l'}\,G^0_{lk'}\,, \quad (20)$$

and define new potentials by

$$\mathbf{U}-\mathbf{U}^0=\iint_D \mathrm{d}\zeta(l,l')\,\boldsymbol{u}_l\boldsymbol{v}^0_{l'},\qquad \mathbf{V}-\mathbf{V}^0=\iint_D \mathrm{d}\zeta(l,l')\,\boldsymbol{v}_{l'}\boldsymbol{u}^0_l\,. \tag{21}$$

If $\boldsymbol{u}^0_k, \boldsymbol{v}^0_{k'}$ are arbitrary given solutions of (6) with potentials $\mathbf{U}^0, \mathbf{V}^0$, then the solutions $\boldsymbol{u}_k, \boldsymbol{v}_{k'}$ of (20) are solutions of (6) with potentials $\mathbf{U}, \mathbf{V}$ as defined in (21). The proof is straightforward using the same condition of uniqueness as for the free-reference integral equation (1).

The compatibility of (6) with the analogous relation for $\boldsymbol{u}_k \to \hat{\boldsymbol{u}}_k$ leads to

$$[-\mathbf{J}^{\mathrm{T}}\cdot(\hat{\mathbf{U}}-\tilde{\mathbf{U}})+p\hat{\mathbf{U}}-q\tilde{\mathbf{U}}-(p-q)\hat{\tilde{\mathbf{U}}}+\hat{\tilde{\mathbf{U}}}\cdot\mathbf{O}\cdot(\hat{\mathbf{U}}-\tilde{\mathbf{U}})]\cdot\mathbf{O}\cdot\hat{\boldsymbol{u}}_k=0\,. \tag{22}$$

From (22) and similar expressions involving the transformation $\boldsymbol{u}_k \to \boldsymbol{u}'_k$, we obtain

$$p(\tilde{u}'-\hat{\tilde{u}}+\hat{u}-u')+q(\hat{\tilde{u}}-\hat{u}'+u'-\tilde{u})+r(\hat{u}'-\tilde{u}'+\tilde{u}-\hat{u})+\tilde{u}'(\tilde{u}-u')+\hat{\tilde{u}}(\hat{u}-\tilde{u})+\hat{u}'(u'-\hat{u})=0\,, \tag{23}$$

where $u \equiv U_{0,0}$ is the (0,0) element of the matrix $\mathbf{U}$. Eq. (23) is the lattice KP, cf. also ref. [16], and the integral equation (20) may be used to derive more general solutions for it than can be obtained from (1), cf. also ref. [7] for the continuous case.

6. It is clear that special solutions of the KP (17) also solve equations of lower dimensionality such as the (potential) Korteweg–de Vries (KdV) equation and Boussinesq (BSQ) equation. The integral equation (1) is particularly well suited for obtaining reductions in the dimensionality by implementing special choices for the measure $\mathrm{d}\zeta(l,l')$ and hypersurface D. In this way explicit connections between the KP and various two-dimensional equations can be obtained. In the following we shall give a few examples, the details of which will be worked out in the future [17].

(i) *Reduction to the KdV equation*. Consider a measure

$$\mathrm{d}\zeta(l,l')=\frac{1}{2\pi\mathrm{i}}\,\frac{\mathrm{d}l'}{l'-l}\,\mathrm{d}\lambda(l)\,, \tag{24}$$

and a hypersurface D = C × C′ consisting of a contour C in the complex l-plane and a contour C′ for l' surrounding the poles $l'=l$, $l\in$ C. Then (1) with $\sigma_l u_l(\alpha)=s_l(\alpha)$ reduces to the integral equation for the KdV and its lattice analogues including the Toda equation [8], cf. also refs. [18,19], and (6) reduces to the basic BT relation and its inverse for the two-dimensional case.

(ii) *Reduction to the BSQ equation*. Consider a measure

$$\mathrm{d}\zeta(l,l')=\frac{1}{2\pi\mathrm{i}}\,\frac{\mathrm{d}l'}{l'+\omega l}\,\mathrm{d}\lambda(l),\qquad \omega=\exp(2\pi\mathrm{i}/3)\,, \tag{25}$$

and a hypersurface D = C × C′, in which C′ for l' surrounds the poles $l'=-\omega l$, $l\in$ C. Then (5) with $\sigma_{-\omega l}\,u_l=\boldsymbol{\phi}_l$ reduces to the integral equation for the BSQ [20]. A lattice version for the BSQ may be worked out also with the help of this integral equation.

(iii) *Reduction to the two-dimensional periodic Toda chain*. Consider the measure

$$\mathrm{d}\zeta(l,l')=\frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{N}\frac{\mathrm{d}l'}{l'+\omega^j l}\,\mathrm{d}\lambda_j(l),\qquad \omega=\exp(2\pi\mathrm{i}/N)\,, \tag{26}$$

and a hypersurface D = C × C′, in which C′ for l' surrounds all points $-\omega^j l$, $l\in$ C. If we take $\rho_k=\rho_{k;n}(\eta,\tau)$, $\sigma_{k';n}(\eta,\tau)$ according to (11) and $p\to 0$, the function $\phi_n(\eta,\tau)=\ln(1-u_{0,1})$, in which $u_{0,1}$ is the (0,1)-element of the matrix $\mathbf{U}$, satisfies the equation for the twodimensional *periodic* Toda chain, i.e. (16) with the constraint $\phi_{n+N}=\phi_n$, cf. refs. [11,12]. (From a formal limit $N\to\infty$, $j\to\infty$, such that $\omega^j l\to l'$ $\mathrm{d}\lambda_j(l)\to\mathrm{d}\zeta(l,l')$ one recovers the integral equation for the infinite Toda chain).

Finally different types of reductions may be obtained for a factorizing measure, e.g.

$$d\zeta(l, l') = -i\,(l + l')^{-1}\, d\zeta_1(l)\, d\zeta_2(l')\,,$$

where $D = C_1 \times C_2$ consists of two separate contours C_1 and C_2 in the complex l-plane and complex l'-plane respectively. Eq. (1) then reduces to integral equations of the types discussed in ref. [19] for the special cases that either $d\lambda_2 = d\lambda_1$ or $d\lambda_2 = d\lambda_1^*$. More generally one may establish explicit connections between KP-type of equations considered here and equations belonging to the AKNS system [21].

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