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LINEARIZING INTEGRAL TRANSFORM AND PARTIAL DIFFERENCE EQUATIONS

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A linearizing integral transform is proposed which relates solutions of a spectral problem associated with a class of integrable partial difference equations to any given solution of the spectral problem. Examples of this class are lattice versions of the isotropic Heisenberg spin chain, the nonlinear Schrödinger equation and the (complex) sine-Gordon equation.

1. Introduction. Recently a direct linearization method which is based on the use of linear integral equations with singular kernels and arbitrary measures and contours has been developed to obtain solutions of integrable nonlinear systems [1–3]. The integral equations which connect nontrivial solutions of a spectral problem associated with an integrable nonlinear system to so-called free-reference solutions (i.e. solutions of the spectral problem associated with a trivial potential), can be related to the well-known Riemann–Hilbert transformations in special cases [4]. Very recently a generalization of this approach was formulated on the basis of linearizing integral transforms which transform any given solution of the spectral problem into another solution of the same spectral problem [5–7]. This generalization may be regarded as being reminiscent of earlier work relating different solutions of the spectral problem via a Gel'fand–Levitan equation [8–10].

In this note we extend this approach to study the solutions of partial difference equations, instead of the partial differential equations studied in refs. [5–7]. A direct relation between Bianchi identities for the commutativity of Bäcklund transformations (BTs) on the one hand, and integrable partial difference equations on the other hand [3] makes it possible to derive various new results and to interpret these results in terms of BTs for the linearizing integral transform. Furthermore we derive a superposition principle for the linearizing integral transform. The difference–difference equations obtained in this way are of a rather general $N \times N$ matrix structure. Therefore it is of interest to investigate under which constraints on the contours and measures a reduction can be obtained such that solutions of a well-defined subclass of matrix equations are invariant under the linearizing integral transform. In this letter we treat an example of such a reduction for $N = 2$ leading e.g. to lattice versions of the isotropic Heisenberg spin chain, the nonlinear Schrödinger equation and the (complex) sine-Gordon equation. A more exhaustive study of these (reduction) problems may be carried out with the help of the theory of the reduction group of ref. [11].

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2. Linearizing integral transform. Consider the linearizing integral transform

$$\boldsymbol{\Phi}_k - \int_{\mathbf{C}} \boldsymbol{\Phi}_l \cdot \mathrm{d}\boldsymbol{\lambda}(l) \cdot \frac{(\boldsymbol{\Phi}_l^0)^{-1} \cdot \boldsymbol{\Phi}_k^0}{k-l} = \boldsymbol{\Phi}_k^0 , \qquad (1)$$

in which $\boldsymbol{\Phi}_k$ and $\boldsymbol{\Phi}_k^0$ are $N \times N$ matrices depending on a complex variable k and on other, for the moment, unspecified variables. The measure $\mathrm{d}\boldsymbol{\lambda}(l)$ with the associated contour $\mathbf{C}$ is a formal $N \times N$ matrix, meaning that to every element i, j there corresponds an integration over a contour C_{ij} with the measure $\mathrm{d}\lambda_{ij}(l)$. Contours and measures in (1) are chosen in such a way that the solution $\boldsymbol{\Phi}_k$ for a given reference wave function $\boldsymbol{\Phi}_k^0$ is unique. Finally $\boldsymbol{\Phi}_k^0$ is supposed to satisfy a certain linear spectral problem.

Let us impose e.g. the following transformation property on the reference wave function, i.e. $\hat{\boldsymbol{\Phi}}_k^0 = (k\mathbf{J} + \mathbf{Q}^0) \cdot \boldsymbol{\Phi}_k^0$, in which $\mathbf{J}$ and $\mathbf{Q}^0$ are $N \times N$ matrices not depending on k, and in which $\mathbf{J}$, in contrast to $\mathbf{Q}^0$, is a constant matrix not depending on the additional unspecified variables. It is understood that $\hat{\boldsymbol{\Phi}}_k$ is the solution of (1) with the same $\mathrm{d}\boldsymbol{\lambda}(l)$ and $\mathbf{C}$, but with the reference wave function $\hat{\boldsymbol{\Phi}}_k^0$ instead of $\boldsymbol{\Phi}_k^0$.

Using the relation

$$(\hat{\boldsymbol{\Phi}}_l^0)^{-1} \cdot \hat{\boldsymbol{\Phi}}_k^0 = (\boldsymbol{\Phi}_l^0)^{-1} \cdot \boldsymbol{\Phi}_k^0 + (k-l)(\hat{\boldsymbol{\Phi}}_l^0)^{-1} \cdot \mathbf{J} \cdot \boldsymbol{\Phi}_k^0 \qquad (2)$$

and the uniqueness of the solution of (1), it can be shown that

$$\hat{\boldsymbol{\Phi}}_k = (k\mathbf{J} + \mathbf{Q}) \cdot \boldsymbol{\Phi}_k , \qquad (3)$$

with

$$\mathbf{Q} - \mathbf{Q}^0 = \int_{\mathbf{C}} \hat{\boldsymbol{\Phi}}_l \cdot \mathrm{d}\boldsymbol{\lambda}(l) \cdot (\hat{\boldsymbol{\Phi}}_l^0)^{-1} \cdot \mathbf{J} - \mathbf{J} \cdot \int_{\mathbf{C}} \boldsymbol{\Phi}_l \cdot \mathrm{d}\boldsymbol{\lambda}(l) \cdot (\boldsymbol{\Phi}_l^0)^{-1} , \qquad (4)$$

or equivalently

$$\mathbf{Q} - \mathbf{Q}^0 = \mathbf{Q} \cdot \int_{\mathbf{C}} l^{-1} \boldsymbol{\Phi}_l \cdot \mathrm{d}\boldsymbol{\lambda}(l) \cdot (\boldsymbol{\Phi}_l^0)^{-1} - \int_{\mathbf{C}} l^{-1} \hat{\boldsymbol{\Phi}}_l \cdot \mathrm{d}\boldsymbol{\lambda}(l) \cdot (\hat{\boldsymbol{\Phi}}_l^0)^{-1} \cdot \mathbf{Q}^0 . \qquad (5)$$

The transformation $\boldsymbol{\Phi}_k \to \hat{\boldsymbol{\Phi}}_k$, cf. (3), may be regarded as a Bäcklund transformation (BT) associated with the integral equation (1), cf. ref. [3].

Let us consider a second BT of the type $\boldsymbol{\Phi}_k^0 \to \tilde{\boldsymbol{\Phi}}_k^0 = (k\mathbf{J}' + \mathbf{Q}'^0) \cdot \boldsymbol{\Phi}_k^0$, then the solution of (1) with $\tilde{\boldsymbol{\Phi}}_k^0$ instead of $\boldsymbol{\Phi}_k^0$ satisfies

$$\tilde{\boldsymbol{\Phi}}_k = (k\mathbf{J}' + \mathbf{Q}') \cdot \boldsymbol{\Phi}_k , \qquad (6)$$

in which $\mathbf{Q}' - \mathbf{Q}'^0$ satisfies two relations which can be found from (4) and (5) replacing $\hat{\boldsymbol{\Phi}}_l$ and $\hat{\boldsymbol{\Phi}}_l^0$ by $\tilde{\boldsymbol{\Phi}}_l$ and $\tilde{\boldsymbol{\Phi}}_l^0$ and $\mathbf{J}$ by $\mathbf{J}'$.

Assuming that both BTs commute, i.e. $\hat{\tilde{\boldsymbol{\Phi}}}_k^0 = \tilde{\hat{\boldsymbol{\Phi}}}_k^0$, or equivalently $\tilde{\mathbf{Q}}^0 \cdot \mathbf{Q}'^0 = \hat{\mathbf{Q}}'^0 \cdot \mathbf{Q}^0$, $\tilde{\mathbf{Q}}^0 \cdot \mathbf{J}' + \mathbf{J} \cdot \mathbf{Q}'^0 = \hat{\mathbf{Q}}'^0 \cdot \mathbf{J} + \mathbf{J}' \cdot \mathbf{Q}^0$ and

$$[\mathbf{J}, \mathbf{J}'] = 0 , \qquad (7)$$

we also have

$$\hat{\tilde{\boldsymbol{\Phi}}}_k = \tilde{\hat{\boldsymbol{\Phi}}}_k \qquad (8)$$

and the compatibility of (3) and (6) gives the equations

$$\tilde{\mathbf{Q}} \cdot \mathbf{Q}' = \hat{\mathbf{Q}}' \cdot \mathbf{Q} , \qquad (9a)$$

$$\tilde{\mathbf{Q}} \cdot \mathbf{J}' + \mathbf{J} \cdot \mathbf{Q}' = \hat{\mathbf{Q}}' \cdot \mathbf{J} + \mathbf{J}' \cdot \mathbf{Q} . \qquad (9b)$$

These two coupled equations can be expressed in terms of one single equation in the following two ways. Firstly one can define a potential matrix $\mathbf{g}$ such that

$$\mathbf{Q} = \hat{\mathbf{g}} \cdot \mathbf{g}^{-1} , \quad \mathbf{Q}' = \tilde{\mathbf{g}} \cdot \mathbf{g}^{-1} , \qquad (10)$$

so that (9a) is identically satisfied and from (9b) we have

$$\mathbf{J} \cdot \tilde{\mathbf{g}} \cdot \mathbf{g}^{-1} + \hat{\tilde{\mathbf{g}}} \cdot \tilde{\mathbf{g}}^{-1} \cdot \mathbf{J}' = \mathbf{J}' \cdot \hat{\mathbf{g}} \cdot \mathbf{g}^{-1} + \hat{\tilde{\mathbf{g}}} \cdot \hat{\mathbf{g}}^{-1} \cdot \mathbf{J} . \qquad (11)$$

We can also define a potential matrix $\mathbf{H}$ by

$$\mathbf{Q} = \mathbf{I} + \hat{\mathbf{H}} \cdot \mathbf{J} - \mathbf{J} \cdot \mathbf{H} , \quad \mathbf{Q}' = \mathbf{I}' + \tilde{\mathbf{H}} \cdot \mathbf{J}' - \mathbf{J}' \cdot \mathbf{H} , \qquad (12)$$

in which $\mathbf{I}$ and $\mathbf{I}'$ are taken to be invariant under the BTs, and commuting with each other and also with $\mathbf{J}$ and $\mathbf{J}'$. Eq. (12) ensures that (9b) is satisfied identically and eq. (9a) yields

$$(\mathbf{I} + \hat{\tilde{\mathbf{H}}} \cdot \mathbf{J} - \mathbf{J} \cdot \tilde{\mathbf{H}}) \cdot (\mathbf{I}' + \tilde{\mathbf{H}} \cdot \mathbf{J}' - \mathbf{J}' \cdot \mathbf{H}) = (\mathbf{I}' + \hat{\tilde{\mathbf{H}}} \cdot \mathbf{J}' - \mathbf{J}' \cdot \hat{\mathbf{H}}) \cdot (\mathbf{I} + \hat{\mathbf{H}} \cdot \mathbf{J} - \mathbf{J} \cdot \mathbf{H}) . \qquad (13)$$

Eqs. (4) and (5) can be expressed as

$$\mathbf{g} = \left(\mathbf{1} - \int_{\mathbf{C}} l^{-1} \boldsymbol{\Phi}_l \cdot \mathrm{d}\boldsymbol{\lambda}(l) \cdot (\boldsymbol{\Phi}_l^0)^{-1} \right) \cdot \mathbf{g}^0 , \qquad (14)$$

in which an arbitrary (invariant) matrix has been included in $\mathbf{g}^0$, and

$$\mathbf{H}-\mathbf{H}^0=\int_{\mathbf{C}}\boldsymbol{\Phi}_l\cdot \mathrm{d}\boldsymbol{\lambda}(l)\cdot(\boldsymbol{\Phi}_l^0)^{-1}\ . \tag{15}$$

Identifying the two BTs with the two primitive translations on a two-dimensional lattice, i.e. taking

$$\mathbf{A}\equiv\mathbf{A}(n,m)\ ,\quad \hat{\mathbf{A}}\equiv\mathbf{A}(n+1,m)\ ,\quad \tilde{\mathbf{A}}\equiv\mathbf{A}(n,m+1)\ ,$$

$$\tilde{\hat{\mathbf{A}}}\equiv\mathbf{A}(n+1,m+1)\ , \tag{16}$$

for an arbitrary $N\times N$ matrix $\mathbf{A}$, it is clear that (9), (11) and (13) are three integrable difference–difference equations for $N\times N$ matrices defined at the sites of a two-dimensional lattice. The corresponding spectral problem is given by (3) and (6) with (10) and (12) for the matrices $\mathbf{g}$ and $\mathbf{H}$, and the integral equation (1) provides a direct linearization. In fact, if we start from a reference solution $\boldsymbol{\Phi}_k^0$ of the spectral problem (3) and (6) with the associated solutions $\mathbf{Q}^0$, $\mathbf{Q}'^0$, $\mathbf{g}^0$, $\mathbf{H}^0$ of (9), (11) and (13), then the solution $\boldsymbol{\Phi}_k$ of the integral equation (1), together with $\mathbf{Q}$, $\mathbf{Q}'$, $\mathbf{g}$, $\mathbf{H}$ given by (4), (5), (14) and (15) is another solution of the spectral problem (3), (6) and the difference–difference equations (9), (11), (13). The integral equation (1) may be regarded as generalization of the reference-free integral equations studied previously [3] [‡1].

Finally in the case that $\mathbf{J}$ and $\mathbf{J}'$ are invertible one can take $\mathbf{J}=\mathbf{J}'=\mathbf{1}$ in eqs. (3)–(6), (9b) and (10)–(13) without loss of generality, as can be shown by obvious substitutions.

3. Superposition principle. From the linearizing integral transform (1) one may in principle generate a hierarchy of solutions $\boldsymbol{\Phi}_k^0\to\boldsymbol{\Phi}_k^1\to\boldsymbol{\Phi}_k^2\ ...$ of the spectral problem, as well as of the partial difference equations. All these solutions, however, may be obtained in one step, as follows from the following superposition principle. Let us define a transition operator $\mathbf{T}_k^{i+1,i}$ with

$$\boldsymbol{\Phi}_k^{i+1}\equiv\mathbf{T}_k^{i+1,i}\cdot\boldsymbol{\Phi}_k^i\ , \tag{17}$$

with

$$\mathbf{T}_k^{i+1,i}=1+\int_{\mathbf{C}_i}\frac{\boldsymbol{\Phi}_l^{i+1}\cdot \mathrm{d}\boldsymbol{\lambda}_i(l)\cdot(\boldsymbol{\Phi}_l^i)^{-1}}{k-l}\ , \tag{18}$$

then the superposition principle states that

$$\boldsymbol{\Phi}_k^j=\mathbf{T}_k^{j,i}\cdot\boldsymbol{\Phi}_k^i\quad (j>i)\ , \tag{19}$$

with

$$\mathbf{T}_k^{j,i}=1+\sum_{r=i}^{j-1}\int_{\mathbf{C}_r}\frac{\boldsymbol{\Phi}_l^{j}\cdot \mathrm{d}\boldsymbol{\lambda}_r(l)\cdot(\boldsymbol{\Phi}_l^i)^{-1}}{k-l}\ . \tag{20}$$

Note that eq. (20) does not contain the intermediate $\boldsymbol{\Phi}_k^r$, $(i<r<j)$. Furthermore it is understood that the contours C_r for different r do not intersect. The proof of (20) is straightforward using the decomposition into partial fractions, i.e.

$$[(k-l)(k-l')]^{-1}$$
$$=[(k-l)(l-l')]^{-1}+[(k-l')(l'-l)]^{-1}\ .$$

As a consequence of eq. (20) two integral transformation operators of the type (18) with different measures and nonintersecting contours commute with each other, as well as with the BTs in eqs. (3) and (6), as has been indicated schematically in fig. 1 [‡2]. In or-

$$\begin{tikzcd}
 & \hat{\Phi}^0_k \arrow[dr, "(C_1{,}d\lambda_1)"] & \\
\Phi^0_k \arrow[ur, "BT"] \arrow[dr, "(C_1{,}d\lambda_1)"'] & & \hat{\Phi}^1_k \\
 & \Phi^1_k \arrow[ur, "BT"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \Phi^1_k \arrow[dr, "(C_2{,}d\lambda_2)"] & \\
\Phi^0_k \arrow[ur, "(C_1{,}d\lambda_1)"] \arrow[dr, "(C_2{,}d\lambda_2)"'] & & \Phi^2_k \\
 & \bar{\Phi}^1_k \arrow[ur, "(C_1{,}d\lambda_1)"'] &
\end{tikzcd}$$

Fig. 1. Commutation properties of the integral-transformation operators and Bäcklund transformations.

[‡1] A connection with the reference-free systems can be obtained by taking $\mathbf{Q}^0$, $\mathbf{Q}'^0$ to be constant matrices, independent of n and m, and commuting with each other and with $\mathbf{J}$ and $\mathbf{J}'$.

[‡2] Noncommutativity of the integral transforms can only occur when the contours have intersection points, leading to contributions from poles in the superposition formulae. An example of this feature arises in the usual Riemann–Hilbert transform leading to the infinite-dimensional symmetry algebras [12].

der to obtain with the integral equation (1) more general solutions than with the corresponding reference-free integral equation, one should start from a reference wave function $\boldsymbol{\Phi}_k^0$ satisfying the spectral problem, such that $\boldsymbol{\Phi}_k^0$ cannot be found from the reference-free integral equation.

4. Reduction problem. Eqs. (9), (11) and (13) are very general matrix equations and without well-defined symmetry requirements on the integrations in (1), the above scheme cannot be applied to specific subclasses of the general equations, cf. ref. [6]. Therefore it is of interest to investigate under which conditions for measures and contours one may obtain a reduction. A systematic investigation of the possible reductions based on symmetry requirements may be carried out with the help of the theory of the reduction group [11], but here we restrict ourselves to the example that the measure with contour is antihermitean, i.e.

$$\int_{\mathbf{C}} \mathbf{A}_l \cdot \mathrm{d}\boldsymbol{\lambda}(l) \cdot \mathbf{B}_l + \int_{\mathbf{C}^*} \mathbf{A}_{l'} \cdot \mathrm{d}\boldsymbol{\lambda}^\dagger(l') \cdot \mathbf{B}_{l'} = 0 \tag{21}$$

for arbitrary matrices $\mathbf{A}_l$ and $\mathbf{B}_l$. Assuming also that $\boldsymbol{\Phi}_k^0(n,m) \cdot \boldsymbol{\Phi}_k^{0\dagger}(n,m) = f_k(n,m)\,\mathbf{1}$, it can be shown that the operator $\mathbf{T}_k \equiv \mathbf{T}_k^{1,0}$, $(\boldsymbol{\Phi}_k \equiv \boldsymbol{\Phi}_k^1)$, defined by (18) is a unitary operator, i.e.

$$\mathbf{T}_k \cdot \mathbf{T}_k^\dagger = \mathbf{1}\,, \quad \boldsymbol{\Phi}_k(n,m) \cdot \boldsymbol{\Phi}_k^\dagger(n,m) = f_k(n,m)\,\mathbf{1}\,. \tag{22}$$

Taking for convenience $\mathbf{J} = \mathbf{J}' = \mathbf{1}$ and using (22), (3) and (6), it follows that the set of properties

$$\mathbf{Q} + \mathbf{Q}^\dagger = 2b\mathbf{1}\,, \quad \mathbf{Q}' + \mathbf{Q}'^\dagger = 2b'\mathbf{1}\,,$$
$$\mathbf{Q} \cdot \mathbf{Q}^\dagger = q\mathbf{1}\,, \quad \mathbf{Q}' \cdot \mathbf{Q}'^\dagger = q'\mathbf{1}\,, \tag{23}$$

with b, b', q, q' independent of n and m, is invariant under the linearizing integral transform (1). In fact, assuming (23) to hold for $\mathbf{Q}^0$, $\mathbf{Q}'^0$, we have (22) with $f_k(n+1,m)/f_k(n,m) = k^2 + 2kb + q$ and $f_k(n,m+1)/f_k(n,m) = k^2 + 2kb' + q'$ and this together with (22) yields (23). Furthermore, considering (12) with $\mathbf{J} = \mathbf{J}' = \mathbf{1}$, (15) and (23) one can show that the set of properties

$$\mathbf{H} + \mathbf{H}^\dagger = 0\,, \quad (\tfrac{1}{2}\mathbf{I} - \tfrac{1}{2}\mathbf{I}^\dagger + \hat{\mathbf{H}} - \mathbf{H})^2 = (b^2 - q)\mathbf{1}\,,$$
$$(\tfrac{1}{2}\mathbf{I}' - \tfrac{1}{2}\mathbf{I}'^\dagger + \tilde{\mathbf{H}} - \mathbf{H})^2 = (b'^2 - q')\mathbf{1}\,, \tag{24}$$

is also invariant under the integral transform (1).

In the special case $N = 2$, eq. (23) yields

$$\mathbf{Q} = b\mathbf{1} + \mathrm{i} \sum_\nu S_\nu \boldsymbol{\sigma}^\nu\,, \quad \mathbf{Q}' = b'\mathbf{1} + \mathrm{i} \sum_\nu S'_\nu \boldsymbol{\sigma}^\nu$$
$$(\nu = x, y, z)\,, \tag{25}$$

in which $\boldsymbol{S}$ and $\boldsymbol{S}'$ are real vectors satisfying the normalization $\boldsymbol{S} \cdot \boldsymbol{S} \equiv s = q - b^2$, $\boldsymbol{S}' \cdot \boldsymbol{S}' \equiv s' = q' - b'^2$. Inserting (25) into (9) and using the notation (16) we obtain

$$\tfrac{1}{2}(\tilde{\boldsymbol{S}} - \boldsymbol{S}) + \frac{(b' - b)(\boldsymbol{S} \times \tilde{\boldsymbol{S}}) - \tfrac{1}{2}(\boldsymbol{S} + \tilde{\boldsymbol{S}})D}{1 + \boldsymbol{S} \cdot \tilde{\boldsymbol{S}}}$$
$$= -\tfrac{1}{2}(\hat{\tilde{\boldsymbol{S}}} - \hat{\boldsymbol{S}}) + \frac{(b' - b)(\hat{\boldsymbol{S}} \times \hat{\tilde{\boldsymbol{S}}}) - \tfrac{1}{2}(\hat{\boldsymbol{S}} + \hat{\tilde{\boldsymbol{S}}})\hat{D}}{1 + \hat{\boldsymbol{S}} \cdot \hat{\tilde{\boldsymbol{S}}}}\,,$$
$$D^2 \equiv 2ss' - 2s(b' - b)^2 - s^2 + 2[s' + (b' - b)^2]\,\boldsymbol{S} \cdot \tilde{\boldsymbol{S}}$$
$$+ (\boldsymbol{S} \cdot \tilde{\boldsymbol{S}})^2\,, \tag{26}$$

which is the lattice version of the equation of motion for the isotropic Heisenberg spin chain. Eq. (13) for $\mathbf{J} = \mathbf{J}' = \mathbf{1}$, together with (24) for $N = 2$ yields a closed equation in terms of 1, 2 elements H_{12}, $\hat{H}_{12}$, $\tilde{H}_{12}$ and $\hat{\tilde{H}}_{12}$. This equation can be regarded as a lattice version of the complex sine-Gordon equation, as well as the nonlinear Schrödinger equation and the complex modified Korteweg–de Vries equation. Furthermore, a different reduction leading to lattice versions of the sine-Gordon equation and the modified Korteweg–de Vries equation can be obtained using a symmetric measure, i.e. (21) with $\mathrm{d}\boldsymbol{\lambda}^\dagger(l')$ replaced by $-\mathrm{d}\boldsymbol{\lambda}^{\mathrm{T}}(-l')$, where T denotes the transposed matrix and $-l'$ is on the contour C. These equations together with their continuum limits will be discussed in a forthcoming paper [13].

5. Gauge equivalence. We finally consider a gauge equivalence introducing new wave functions $\boldsymbol{\Psi}_k$ by

$$\boldsymbol{\Phi}_k = Y_k \mathbf{g} \cdot \boldsymbol{\Psi}_k\,. \tag{27}$$

From (3) and (6) we then obtain the linear problem

$$\hat{\boldsymbol{\Psi}}_k = (\mathbf{1} + k\mathbf{S}) \cdot \boldsymbol{\Psi}_k\,, \quad \tilde{\boldsymbol{\Psi}}_k = (\mathbf{1} + k\mathbf{S}') \cdot \boldsymbol{\Psi}_k\,, \tag{28}$$

with $\mathbf{S} = \hat{\mathbf{g}}^{-1} \cdot \mathbf{J} \cdot \mathbf{g}$, $\mathbf{S}' = \tilde{\mathbf{g}}^{-1} \cdot \mathbf{J}' \cdot \mathbf{g}$. Eq. (28) implies that $\mathbf{S}$ satisfies (9) with $\mathbf{J} = \mathbf{J}' = \mathbf{1}$. Furthermore, if the reference wave function $\mathbf{\Psi}_k^0$ satisfies $\mathbf{\Phi}_k^0 = Y_k^0 \mathbf{g}^0 \cdot \mathbf{\Psi}_k^0$, then the solutions of the integral equation

$$\mathbf{\Psi}_k - \int_{\mathbf{C}} \mathbf{\Psi}_l \cdot \mathrm{d}\lambda(l) \cdot \frac{(\mathbf{\Psi}_l^0)^{-1} \cdot \mathbf{\Psi}_k^0}{k-l} = \frac{\mathbf{\Psi}_k^0}{k} \tag{29}$$

satisfies (28) with

$$Y_k = k Y_k^0 ,$$
$$\mathbf{g} = \mathbf{g}^0 \cdot \left(\mathbf{1} + \int_{\mathbf{C}} \mathbf{\Psi}_l \cdot \mathrm{d}\lambda(l) \cdot (\mathbf{\Psi}_l^0)^{-1} \right)^{-1} . \tag{30}$$

This implies that the gauge equivalence (27), as well as the spectral problem (28) and eq. (9) with $\mathbf{J} = \mathbf{J}' = \mathbf{1}$, $\mathbf{Q} = \mathbf{S}$, $\mathbf{Q}' = \mathbf{S}'$ are invariant under the linear integral transform (29). Furthermore the relation (30) between $\mathbf{g}$ and $\mathbf{g}^0$ is equivalent to (14) and a superposition principle for the integral transform (29) can be inferred from the superposition principle for (1) together with (27).

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