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LINEAR INTEGRAL EQUATIONS AND NONLINEAR DIFFERENCE–DIFFERENCE EQUATIONS

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In this paper we present a systematic method to obtain various integrable nonlinear difference–difference equations and the associated linear integral equations from which their solutions can be inferred. It is argued that these difference–difference equations can be regarded as arising from Bianchi identities expressing the commutativity of Bäcklund transformations. Applying an appropriate continuum limit we first obtain integrable nonlinear differential–difference equations together with the associated linear integral equations and after a second continuum limit we can obtain the corresponding integrable nonlinear partial differential equations and their linear integral equations. As special cases we treat the difference–difference versions and the differential–difference versions of the Korteweg–de Vries equation, the modified Korteweg–de Vries equation, the nonlinear Schrödinger equation, the isotropic classical Heisenberg spin chain, and the complex and real sine–Gordon equation.

1. Introduction

In the last decade a lot of insight has been gained in the integrability of nonlinear partial differential equations (PDE's)$^{1-3}$). One of the most successful methods has been the inverse-scattering transform formalism, which provides an exact linearization in the sense that the initial value problem of the nonlinear PDE is reduced to the solution of only linear equations. In fact, for suitable boundary conditions at infinity, the time evolution of scattering data is governed by linear relations, so that the solutions of the PDE can be obtained with the help of a Gel'fand–Levitan equation.

More recently, Fokas and Ablowitz^{4}) have proposed a direct method of exact linearization of the Korteweg–de Vries (KdV) equation, based on a singular linear integral equation involving an arbitrary contour and measure. Since then, other singular linear integral equations have been studied for various other PDE's as well, including the nonlinear Schrödinger equation, the (complex and real) sine-Gordon equation, the Boussinesq equation, the equation of motion for the Heisenberg spin chain etc.$^{5-7}$), see also ref. 8 for a treatment of the Kadomtsev–Petviashvili equation and the Benjamin–Ono equation. It has also been shown that Bäcklund transformations (BT's) for PDE's can be generated by an appropriate singular transformation of the measure or by an equivalent transformation of the plane-wave factor occurring in the integral equation^{9,10}).

In this paper we study the problem of discretizing nonlinear PDE's to obtain difference–difference equations, while retaining their integrability. This problem has of late been addressed by several authors using different starting points. Ablowitz and Ladik^{11}) started from a discretized linear problem, Hirota^{12}) started from a discretized bilinear differential equation, and Date et al. used a discretized bilinear identity^{13}), cf. also ref. 14. The treatment in the present paper is based on the singular linear integral equations associated with the various PDE's mentioned above, and it will be shown that these same integral equations, after no more than a simple and straightforward discretization of the plane-wave factor, also yield a direct linearization of nonlinear difference–difference equations which may be regarded as the double-discrete analogues of the PDE's.

As corollaries the following two results will be given. First of all it is shown that the integrable nonlinear difference–difference equations we obtain are equivalent to Bianchi identities expressing the commutativity of Bäcklund transformations, cf. refs. 3, 15–21. Secondly we will see that after applying a suitable continuum limit to the difference–difference equations as well as to the associated wave factors in the singular integral equations, we obtain integrable differential–difference equations together with their direct linearizations. (A relationship between BT's and differential–difference equations was presented in refs. 22, 23.)

The outline of this paper is as follows. Sections 2–5 give a treatment of the difference–difference equation of the KdV type. In particular, section 2 summarizes the main results, and some details of the derivation are presented in section 3. The continuum limit, as well as the differential–difference equation that is obtained in this limit, are treated in section 4, along with some interesting special cases, and in section 5 the relation with Bäcklund transformations and Bianchi identities is discussed. The last two sections are devoted to a treatment of the difference–difference versions of the nonlinear Schrödinger equation, the sine-Gordon equation and the equation of motion for the Heisenberg spin chain, with a summary of the main results in section 6, and a discussion of the continuum limit and a discussion of the connection with Bäcklund transformations in section 7. (A preliminary account of the considerations in this paper was given in ref. 24.)

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2. The KdV class; results

In this section we present two linear integral equations involving an arbitrary contour and measure, which linearize a certain class of nonlinear difference–difference equations and differential–difference equations of the Korteweg–de Vries type.

Proposition. Let $u_k(n,m)$ be a solution of the linear integral equation

$$u_k(n,m)+\mathrm{i}\rho_k(n,m)\int_C \mathrm{d}\lambda(l)\,\frac{u_l(n,m)}{k+l}=\frac{\rho_k(n,m)}{k+\alpha},\quad n,m\in\mathbb{Z},\quad k,\alpha\in\mathbb{C}\,,\qquad(2.1)$$

where C and $\mathrm{d}\lambda(k)$ are an arbitrary contour and measure in the complex k-plane, and

$$\rho_k(n,m)=\left(\frac{p+k}{p-k}\right)^n\left(\frac{q+k}{q-k}\right)^m\rho_k(0,0),\quad p,q\in\mathbb{C}\,.\qquad(2.2)$$

Let the contour C and measure $\mathrm{d}\lambda(k)$ be such that the homogeneous integral equation corresponding to (2.1) has only the zero solution. Then the function

$$u(n,m)\equiv\mathrm{i}\int_C\frac{u_k(n,m)}{k+\beta}\,\mathrm{d}\lambda(k),\quad n,m\in\mathbb{Z},\beta\in\mathbb{C}\,,\qquad(2.3)$$

obeys the following nonlinear difference–difference equation:

$$\begin{aligned}&[(p-\alpha)u(n,m)-(p+\beta)u(n+1,m)+1]\\&\quad\times[(p-\beta)u(n,m+1)-(p+\alpha)u(n+1,m+1)+1]\\&=[(q-\alpha)u(n,m)-(q+\beta)u(n,m+1)+1]\\&\quad\times[(q-\beta)u(n+1,m)-(q+\alpha)u(n+1,m+1)+1]\,,\end{aligned}\qquad(2.4)$$

for fixed α, β, p and q.

Corollary. Let $u_k(n,t)$ be a solution of the linear integral equation

$$u_k(n,t)+\mathrm{i}\rho_k(n,t)\int_C\mathrm{d}\lambda(l)\,\frac{u_l(n,t)}{k+l}=\frac{\rho_k(n,t)}{k+\alpha},\quad n\in\mathbb{Z},t,k,\alpha\in\mathbb{C}\,,\qquad(2.5)$$

where C and $\mathrm{d}\lambda(k)$ are an arbitrary contour and measure in the complex k-plane, and

$$\rho_k(n,t)=\left(\frac{p+k}{p-k}\right)^n\exp\left(\frac{-2kt}{p^2-k^2}\right)\rho_k(0,0),\quad p\in\mathbb{C}\,.\qquad(2.6)$$

Let the contour C and measure $\mathrm{d}\lambda(k)$ be such that the homogeneous integral equation corresponding to (2.5) has only the zero solution. Then the function

$$u(n,t)\equiv\mathrm{i}\int_C\frac{u_k(n,t)}{k+\beta}\,\mathrm{d}\lambda(k),\quad n\in\mathbb{Z},t,\beta\in\mathbb{C}\,,\qquad(2.7)$$

obeys the following nonlinear differential–difference equation:

$$\begin{aligned}\partial_t u(n,t)=&-[2p-(p+\alpha)(p+\beta)u(n+1,t)+(p-\alpha)(p-\beta)u(n-1,t)]^{-1}\\&\times[u(n+1,t)-u(n-1,t)+2pu(n+1,t)u(n-1,t)\\&+2pu^2(n,t)-(2p+\alpha+\beta)u(n,t)u(n+1,t)\\&-(2p-\alpha-\beta)u(n,t)u(n-1,t)]\,,\end{aligned}\qquad(2.8)$$

for fixed α, β and p.

Remarks

i) The integral equations (2.1) and (2.5) may be regarded as the double-discrete and the single-discrete analogues of the integral equation

$$u_k(x,t)+\mathrm{i}\rho_k(x,t)\int_C\mathrm{d}\lambda(l)\,\frac{u_l(x,t)}{k+l}=\rho_k(x,t),\quad k,x,t\in\mathbb{C}\,,\qquad(2.9)$$

where

$$\rho_k(x,t)=\mathrm{e}^{\mathrm{i}(kx-k^3t)}\rho_k(0,0)\,,\qquad(2.10)$$

which was proposed by Fokas and Ablowitz[4] for the linearization of the Korteweg–de Vries equation, cf. also refs. 5 and 25.

ii) From the proposition and its corollary it is clear that the difference–difference equation (2.4) as well as the differential–difference equation (2.8) are both also completely integrable, since solutions can be obtained from the linear integral equations (2.1) and (2.5), respectively.

iii) In the continuum limit with $r\equiv n-t/p\to\infty$, $p\to\infty$, $2r/p\to\mathrm{i}x$, and using the time scaling $(4/3)t/p^4\to\mathrm{i}\tau$, we obtain from (2.6) the result

$$\begin{aligned}\rho_k(n,t)&=\exp\left[\left(r+\frac{t}{p}\right)\ln\frac{p+k}{p-k}-\frac{2kt}{p^2-k^2}\right]\rho_k(0,0)\\&\to\rho_k(x,\tau)=\mathrm{e}^{\mathrm{i}(kx-k^3\tau)}\rho_k(0,0)\,,\end{aligned}\qquad(2.11)$$

and from (2.5) with $u_k(n,t)\to u_k(x,\tau)$ we obtain the integral equation

$$u_k(x,\tau)+\mathrm{i}\rho_k(x,\tau)\int_C\mathrm{d}\lambda(l)\,\frac{u_l(x,\tau)}{k+l}=\frac{\rho_k(x,\tau)}{k+\alpha}\,,\qquad(2.12)$$

with $\rho_k(x,\tau)$, as given by (2.11). Defining $u(x,\tau)$ by

$$u(n,t)\to u(x,\tau) = \mathrm{i}\int_C \mathrm{d}\lambda(k)\,\frac{u_k(x,\tau)}{k+\beta}, \tag{2.13}$$

we also have with $n = r + t/p$, $2r/p \to \mathrm{i}x$ the relation

$$\partial_t u(n,t) = -\frac{4\mathrm{i}}{3p^4}\,\partial_\tau u(x,\tau) + \frac{2\mathrm{i}}{p^2}\,\partial_x u(x,\tau)\,. \tag{2.14}$$

From (2.8) and $2r/p \to \mathrm{i}x$ we have

$$\begin{aligned} u(n\pm 1,t) &\to u(x,\tau) \mp 2\mathrm{i}p^{-1}\partial_x u(x,\tau) - 2p^{-2}\partial_x^2 u(x,\tau) \\ &\quad \pm \frac{4}{3}\mathrm{i}p^{-3}\partial_x^3 u(x,\tau) + \frac{2}{3}p^{-4}\partial_x^4 u(x,\tau) + \cdots, \end{aligned} \tag{2.15}$$

and using (2.8), together with (2.14) and the scaling $(4/3)t/p^4 \to \mathrm{i}\tau$, it is straightforward to show that $u(x,\tau)$ satisfies the PDE

$$\partial_\tau u = \partial_x^3 u - 3\mathrm{i}\left[\frac{-\alpha\beta(\partial_x u)^2 + \mathrm{i}(\alpha+\beta)(\partial_x u)\partial_x^2 u + (\partial_x^2 u)^2}{1-(\alpha+\beta)u + 2\mathrm{i}\partial_x u}\right]. \tag{2.16}$$

This PDE contains a number of special cases which have been given before:

i) Defining $u_{0,0} \equiv -\mathrm{i}\alpha\beta u$ and taking the limit $\alpha,\beta\to\infty$, we obtain

$$\partial_\tau u_{0,0} = \partial_x^3 u_{0,0} - 3(\partial_x u_{0,0})^2\,, \tag{2.17}$$

which is the potential KdV, as given in eq. (2.13) of ref. 10.

ii) Defining $u_{0,1} \equiv -\mathrm{i}\alpha u$ and taking the limits $\alpha\to\infty$, $\beta\to 0$, we find

$$\partial_\tau u_{0,1} = \partial_x^3 u_{0,1} - \frac{3(\partial_x u_{0,1})\partial_x^2 u_{0,1}}{\mathrm{i} + u_{0,1}}, \tag{2.18}$$

which has been given in eq. (2.14) of ref. 10 and which is equivalent to the potential MKdV.

iii) Finally, for $\alpha = \beta = 0$, $u_{1,1} \equiv -\mathrm{i}u$ satisfies

$$\partial_\tau u_{1,1} = \partial_x^3 u_{1,1} + \frac{3(\partial_x^2 u_{1,1})^2}{1-2\partial_x u_{1,1}}, \tag{2.19}$$

cf. eq. (2.15) of ref. 10, which can be shown to be equivalent to the MKdV.

3. The KdV class; derivation

In this section we give the derivation of the proposition presented in section 2. For that purpose, consider the linear integral equation

$$u_k^i(n,m;\alpha) + \mathrm{i}\rho_k(n,m)\int_C \mathrm{d}\lambda(l)\,\frac{u_l^i(n,m;\alpha)}{k+l} = \frac{\rho_k(n,m)k^i}{k+\alpha}, \quad n,m,i\in\mathbb{Z},\ k,\alpha\in\mathbb{C}\,, \tag{3.1}$$

with $\rho_k(n,m)$ given by (2.2). Define

$$u^{i,j}(n,m;\alpha,\beta) \equiv \mathrm{i}\int_C \mathrm{d}\lambda(k)\,\frac{u_k^j(n,m;\alpha)k^i}{k+\beta}, \quad n,m,i,j\in\mathbb{Z},\ \alpha,\beta\in\mathbb{C}\,. \tag{3.2}$$

Then from (3.1) and (3.2), using the relation

$$\rho_k(n+1,m) = \left(\frac{p+k}{p-k}\right)\rho_k(n,m)\,, \tag{3.3}$$

we can derive

$$\begin{aligned} (p-k)u_k^i(n+1,m;\alpha) &+ \mathrm{i}\rho_k(n,m)\int_C \mathrm{d}\lambda(l)\,\frac{(p-l)u_l^i(n+1,m;\alpha)}{k+l} \\ &= (p+k)\,\frac{\rho_k(n,m)k^i}{k+\alpha} - \rho_k(n,m)u^{1,i}(n+1,m;\alpha,0)\,. \end{aligned} \tag{3.4}$$

Taking into account that the homogeneous integral equation corresponding to (3.1) has only the zero solution, we have

$$\begin{aligned} (p-k)u_k^i(n+1,m;\alpha) &= pu_k^i(n,m;\alpha) + u_k^{i+1}(n,m;\alpha) \\ &\quad - u^{1,i}(n+1,m;\alpha,0)u_k^1(n,m;0)\,. \end{aligned} \tag{3.5}$$

Using the relation

$$u_k^{i+1}(n,m;0) = u_k^{i+1}(n,m;\alpha) + \alpha u_k^i(n,m;\alpha)\,, \tag{3.6}$$

we have, taking $i=0$,

$$\begin{aligned} (p-k)u_k^0(n+1,m;\alpha) &= (p-\alpha)u_k^0(n,m;\alpha) \\ &\quad + [1-u^{1,0}(n+1,m;\alpha,0)]u_k^1(n,m;0)\,, \end{aligned} \tag{3.7}$$

and dividing by $(k+\beta)$ and integrating over C we obtain, using (3.2),

$$\begin{aligned} (p-\alpha)u^{0,0}(n,m;\alpha,\beta) &- (p+\beta)u^{0,0}(n+1,m;\alpha,\beta) + 1 \\ &= [1-u^{1,0}(n+1,m;\alpha,0)][1-u^{1,0}(n,m;\beta,0)]\,, \end{aligned} \tag{3.8}$$

where we have also used the symmetry property

$$u^{i,j}(n,m;\alpha,\beta) = u^{j,i}(n,m;\beta,\alpha)\,. \tag{3.9}$$

Because eq. (3.3) is invariant under $p \to -p$, $n \to n+1$, $n+1 \to n$, we also have a second relation which can be obtained from (3.8), replacing p by $-p$ and interchanging n and $n+1$, i.e.

$$(-p-\alpha)u^{0,0}(n+1,m;\alpha,\beta) - (-p+\beta)u^{0,0}(n,m;\alpha,\beta) + 1$$
$$= [1-u^{1,0}(n,m;\alpha,0)][1-u^{1,0}(n+1,m;\beta,0)] . \tag{3.10}$$

Finally, in view of (2.2), we also have two relations which can be found from (3.8) and (3.10), replacing p by q, and $(n+1,m)$ and (n,m) by $(n,m+1)$ and (n,m) respectively, i.e.

$$(q-\alpha)u^{0,0}(n,m;\alpha,\beta) - (q+\beta)u^{0,0}(n,m+1;\alpha,\beta) + 1$$
$$= [1-u^{1,0}(n,m+1;\alpha,0)][1-u^{1,0}(n,m;\beta,0)] , \tag{3.11}$$
$$(-q-\alpha)u^{0,0}(n,m+1;\alpha,\beta) - (-q+\beta)u^{0,0}(n,m;\alpha,\beta) + 1$$
$$= [1-u^{1,0}(n,m;\alpha,0)][1-u^{1,0}(n,m+1;\beta,0)] . \tag{3.12}$$

Eq. (2.4) can now be derived directly by eliminating $u^{1,0}$ from (3.8) and (3.10)–(3.12). In fact, dividing eq. (3.8) by (3.11), and eq. (3.12) with $n \to n+1$ by eq. (3.10) with $m \to m+1$, we obtain eq. (2.4).

4. The KdV class; continuum limit

Eq. (2.4) is a difference–difference equation which may be regarded as a discrete analogue of a differential–difference equation. To obtain a corresponding differential–difference equation we consider a limit

$$m \to \infty , \quad b \to 0 , \quad mb = t , \tag{4.1}$$

in which b is a suitably chosen lattice parameter characterizing the distance between two successive time-points, i.e. two successive sites m and $m+1$, and in which t can be identified with the continuous time variable.

Rewriting the difference–difference equation (2.4) as follows:

$$[p-q-(p+\alpha)(p+\beta)u(n+1,m) + (p-\alpha)(p-\beta)u(n,m+1)]$$
$$\times [u(n+1,m+1) - u(n,m)]$$
$$+ (p+q)[u(n+1,m) - u(n,m+1) + (p-q)u(n,m+1)u(n+1,m)$$
$$+ (p-q)u(n,m)u(n+1,m+1) - (p-q+\alpha+\beta)u(n,m)u(n+1,m)$$
$$- (p-q-\alpha-\beta)u(n,m+1)u(n+1,m+1)] = 0 , \tag{4.2}$$

it is clear that we can take a continuum limit with $b = p+q$ as lattice parameter, provided that

$$u(n+1,m+1) - u(n,m) = \mathcal{O}(p+q) . \tag{4.3}$$

Eq. (4.3) can be satisfied by relabeling the sites (n,m) of the two-dimensional lattice as (n',m), $n' \equiv n-m$, and by using the relation

$$a(n',m+1) = a(n',t) + (p+q)\partial_t a(n',t) + \mathcal{O}([p+q]^2) , \tag{4.4}$$

for an arbitrary function $a(n',m)$, in the continuum limit. Up to the order $(p+q)$, we then have the relations

$$\begin{aligned} &u(n,m) \to u(n',t) , \\ &u(n+1,m) \to u(n'+1,t) , \\ &u(n,m+1) \to u(n'-1,t) + (p+q)\partial_t u(n'-1,t) , \\ &u(n+1,m+1) \to u(n',t) + (p+q)\partial_t u(n',t) . \end{aligned} \tag{4.5}$$

Eq. (2.8) can now be obtained inserting (4.5) in (4.2) and taking only the terms $\mathcal{O}(p+q)$, in the limit $(p+q) \to 0$.

To obtain eq. (2.5) we consider the continuum limit of the factor (2.2), which we rewrite as follows:

$$\rho_k(n,m) = \left(\frac{p+k}{p-k}\right)^{n'} \left(1 + \frac{2k(p+q)}{(p-k)(q-k)}\right)^m \rho_k(0,0), \quad n' \equiv n-m , \tag{4.6}$$

and in the limit (4.1) with $b = p+q$, we immediately obtain

$$\rho_k(n,m) \to \rho_k(n',t) = \left(\frac{p+k}{p-k}\right)^{n'} \exp\left(\frac{-2kt}{p^2-k^2}\right)\rho_k(0,0) . \tag{4.7}$$

If we now drop the primes, it is clear from the proposition given in section 2 that the function $u(n,t)$ defined by (2.7) and (2.5) satisfies the differential–difference equation (2.8), and thus the corollary of the proposition has been proved. (It is also possible to prove this corollary directly from (2.5)–(2.7) without using the proposition itself.)

Note that the integral equation (2.5) can also be formulated in terms of the variable $z \equiv (p+k)(p-k)^{-1}$, with the corresponding factor $\rho_z(n\,t) = z^n \exp[(-1/2p)(z-z^{-1})t]$, cf. ref. 26.

We shall now consider some special cases of eq. (2.8). For that purpose we first rewrite (2.8) using the substitutions

$$\begin{aligned} &u(n,t) \to \frac{2pu(n,t)}{(p+\alpha)(p+\beta)}, \quad \frac{t}{2p} \to t , \\ &a \equiv \frac{p-\alpha}{p+\alpha}, \quad b \equiv \frac{p-\beta}{p+\beta} . \end{aligned} \tag{4.8}$$

Eq. (2.8) then becomes

$$\begin{aligned}\partial_t u(n,t) = {}&[1-u(n+1,t)+abu(n-1,t)]^{-1}[u(n-1,t)-u(n+1,t)\\&+(2ab+a+b)u(n,t)u(n-1,t)+(2+a+b)u(n,t)u(n+1,t)\\&-(1+a)(1+b)u(n-1,t)u(n+1,t)-(1+a)(1+b)u^2(n,t)]\,.\end{aligned}\tag{4.9}$$

Some special cases of eq. (4.9) are

i) $a=0$, $b=0$, $A(n,t)\equiv -\ln[1-u(n,t)]$:

$$\partial_t A(n,t) = 2 - e^{A(n,t)-A(n-1,t)} - e^{A(n+1,t)-A(n,t)}\,,\tag{4.10}$$

and $B(n,t)\equiv A(n-2,t)-A(n,t)$ obeys the equation of motion for the Toda lattice[27,28]), i.e.

$$\partial_t^2 B(n,t) = 2e^{-B(n,t)} - e^{-B(n+2,t)} - e^{-B(n-2,t)}\,.\tag{4.11}$$

Note that in passing from (4.10) to (4.11) the first-order differential equation (4.10) involving all lattices sites, is decomposed into two identical second-order differential equations (4.11) for the even and odd sites respectively.

ii) $au(n,t)\to u(n,t)$, $a\to\infty$, $b=0$:

$$\partial_t u(n,t) = [1-u(n,t)+u(n-1,t)][1-u(n+1,t)+u(n,t)]-1\,.\tag{4.12}$$

Under the substitution $C(n,t)\equiv u(n,t)-u(n-1,t)$, eq. (4.12) reduces to the discrete Korteweg–de Vries equation, cf. refs. 29 and 30,

$$\partial_t C(n,t) = [1-C(n,t)][C(n-1,t)-C(n+1,t)]\,.\tag{4.13}$$

Eq. (4.13) can also be derived from (4.10) using the substitution

$$A(n,t)-A(n-1,t) = \ln[1-C(n,t)]\,.$$

iii) $a=-1$, $b=1$, $D(n,t)=\frac{1}{2}\ln[2u(n,t)-1]$:

$$\partial_t D(n,t) = \tanh[D(n-1,t)-D(n+1,t)]\,.\tag{4.14}$$

iv) $au(n,t)\to -u(n,t)$, $a\to\infty$, $b=1$,
$E(n,t)\equiv[u(n,t)-u(n-1,t)]\,[u(n-1,t)-1]^{-1}$:

$$\partial_t E(n,t) = [1-E^2(n,t)][E(n-1,t)-E(n+1,t)]\,,\tag{4.15}$$

the discrete modified Korteweg–de Vries equation[31]).

From the considerations given above it is clear that all differential–difference equations (4.10)–(4.15) are integrable, since their solutions can be obtained from the linear integral equation (2.5).

5. The KdV class; connection with Bäcklund transformations

From appendix A of ref. 10 it is clear that the transformation

$$\rho_k\to\tilde\rho_k = \frac{q+k}{q-k}\rho_k\tag{5.1}$$

induces a Bäcklund transformation of the singular integral equation

$$u_k + \mathrm{i}\rho_k\int_C \mathrm{d}\lambda(l)\,\frac{u_l}{k+l} = \frac{\rho_k}{k+\alpha}\,,\tag{5.2}$$

with a corresponding transformation

$$u=\int_C \mathrm{d}\lambda(k)\,\frac{u_k}{k+\beta}\to\tilde u=\int_C \mathrm{d}\lambda(k)\,\frac{\tilde u_k}{k+\beta}\,,\tag{5.3}$$

where $\tilde u_k$ is the solution of (5.2) with $\tilde\rho_k$ instead of ρ_k. (Note that it is also possible to obtain the function $\tilde u$, defined in (5.3), by a singular transformation of the measure $\mathrm{d}\lambda(k)\to\mathrm{d}\tilde\lambda(k)=(q+k)(q-k)^{-1}\,\mathrm{d}\lambda(k)$ as $\tilde u=\int_C \mathrm{d}\tilde\lambda(k)\tilde u_k(k+\beta)^{-1}$, where now $\tilde u_k$ is the solution of (5.2) with the measure $\mathrm{d}\tilde\lambda(k)$ instead of $\mathrm{d}\lambda(k)$, see ref. 10.)

Comparing eq. (5.1) with the relation

$$\rho_k(n,m+1)=\rho_k(n,m)\,\frac{q+k}{q-k}\,,\tag{5.4}$$

which follows from (2.2), it is clear that eq. (2.4) with $u(n,m)\to u(n',t)$, $u(n+1,m)\to u(n'+1,t)$, $u(n,m+1)\to\tilde u(n'-1,t)$, $u(n+1,m+1)\to\tilde u(n',t)$, i.e.

$$\begin{aligned}&[(p+\beta)u(n'+1,t)-(p-\alpha)u(n',t)-1]\\&\quad\times[(p-\beta)\tilde u(n'-1,t)-(p+\alpha)\tilde u(n',t)+1]\\&=[(q+\beta)\tilde u(n'-1,t)-(q-\alpha)u(n',t)-1]\\&\quad\times[(q-\beta)u(n'+1,t)-(q+\alpha)\tilde u(n',t)+1]\,,\end{aligned}\tag{5.5}$$

provides a Bäcklund transformation of the differential–difference equation (2.8).

Furthermore, introducing a second Bäcklund transformation

$$\rho_k\to\hat\rho_k=\frac{p+k}{p-k}\rho_k\,,\tag{5.6}$$

leading to

$$u \to \hat{u} = \int_C \mathrm{d}\lambda(k)\,\frac{\hat{u}_k}{k+\beta}, \tag{5.7}$$

where $\hat{u}_k$ is the solution of (5.2) with $\hat{\rho}_k$ instead of ρ_k, it is clear that eq. (2.4) with $u(n, m) \to u$, $u(n+1, m) \to \hat{u}$, $u(n, m+1) \to \tilde{u}$, $u(n+1, m+1) \to \hat{\tilde{u}}$, i.e.

$$\begin{aligned}[](p+\beta)\hat{u} - (p-\alpha)u - 1][(p-\beta)\tilde{u} - (p+\alpha)\hat{\tilde{u}} + 1] \\ = [(q+\beta)\tilde{u} - (q-\alpha)u - 1][(q-\beta)\hat{u} - (q+\alpha)\hat{\tilde{u}} + 1]\,, \end{aligned} \tag{5.8}$$

is a Bianchi-identity (see refs. 15, 17 and 18) expressing the commutativity of the BT's (5.1) and (5.6). This holds independently of the specific dependence of the factor ρ_k on variables or lattice sites. In fact, (5.5) is a Bianchi-identity for any partial differential equation which can be derived from the integral equation (5.2) with factor $\rho_k = \rho_k(x, t)$, for any differential–difference equation which follows from (5.2) with factor $\rho_k(n, t) = (s+k)^n (s-k)^{-n} \rho_k(0, t)$, as well as for any difference–difference equation which can be derived from (5.2) with arbitrary $\rho_k(n, m)$.

On the other hand, as we have shown in the previous sections, a Bianchi-identity involving u, $\hat{u}$, $\tilde{u}$, $\hat{\tilde{u}}$, following from two BT's, as given by (5.1) and (5.6) of the integral equation (5.2) with the replacements $u \to u(n, m)$, $\hat{u} \to u(n+1, m)$, $\tilde{u} \to u(n, m+1)$ and $\hat{\tilde{u}} \to u(n+1, m+1)$ leads in a natural way to an integrable difference–difference equation (2.4) associated with the integral equation (2.1), (2.2).

The above procedure can be applied to other linear integral equations as well (see e.g. refs. 5 and 6). As a first step one derives the Bianchi-identity expressing the commutativity of Bäcklund transformations of the factor ρ_k in the integral equation. Secondly the Bianchi-identity is interpreted as an (integrable) difference–difference equation which can be derived from the integral equation with a factor ρ_k such as specified in (2.2). Furthermore, choosing a small parameter for which one may take a continuum limit of the type (4.1), one can derive an (integrable) differential–difference equation, the solutions of which can be obtained from the integral equation with a factor $\rho_k(n, t) = (p+k)^n (p-k)^{-n} \times \varrho_k(0, t)$ and the Bianchi-identity immediately leads to a BT for the differential–difference equation. In the following sections we shall work out the procedure mentioned above for the integral equation of the NLS type.

6. The NLS class; results

In this section we present two linear integral equations involving an arbitrary contour and measure, which linearize certain nonlinear difference–difference equations and differential–difference equations of the nonlinear Schrödinger type.

Proposition. Let $\phi_k(n, m; \alpha)$ be a solution of the linear integral equation

$$\begin{aligned} &\phi_k(n, m; \alpha) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l')\,\frac{\rho_k(n, m)\rho_{l'}^*(n, m)}{(k-l')(l'-l)}\,\phi_l(n, m; \alpha) \\ &\quad = \frac{\rho_k(n, m)}{k+\alpha}, \quad n, m \in \mathbb{Z},\ k, \alpha \in \mathbb{C}\,, \end{aligned} \tag{6.1}$$

where C and $\mathrm{d}\lambda(k)$ are an arbitrary contour and measure in the complex k-plane, and

$$\rho_k(n, m) = \left(\frac{p-k}{p+k}\right)^n \left(\theta'\,\frac{q-k}{q^*-k}\right)^m \rho_k(0, 0), \quad p, q, \theta' \in \mathbb{C},\ p = -p^*,\ |\theta'| = 1\,. \tag{6.2}$$

Let the contour C and the measure $\mathrm{d}\lambda(k)$ be such that the homogeneous integral equation corresponding to (6.1) has only the zero solution, and define

$$\phi(n, m; \alpha, \beta) \equiv \int_C \frac{\phi_k(n, m; \alpha)}{k+\beta}\,\mathrm{d}\lambda(k), \quad n, m \in \mathbb{Z},\ \beta \in \mathbb{C}\,. \tag{6.3}$$

Then the following results hold, for special choices of p, q, α, β and θ':
1) The function

$$\phi(n, m) \equiv 2p\phi(n, m; -p, p) \tag{6.4}$$

obeys the double-discrete nonlinear Schrödinger equation (ddNLS), i.e.

$$\begin{aligned} &2|p|^2 + 2|q|^2 + \theta'(q+p)(q-p)\phi(n, m)\phi^*(n, m+1) \\ &\qquad + \theta'^*(q^*+p^*)(q^*-p^*)\phi^*(n, m)\phi(n, m+1) \\ &\quad = |q+p|^2(1+|\phi(n, m+1)|^2) \\ &\qquad \times \frac{(q^*-p^*)\phi(n+1, m+1) - \theta'(q-p)\phi(n, m)}{\theta'(q+p)\phi(n+1, m) - (q^*+p^*)\phi(n, m+1)} \\ &\qquad + |q-p|^2(1+|\phi(n, m)|^2)\,\frac{\theta'(q+p)\phi(n+1, m) - (q^*+p^*)\phi(n, m+1)}{(q^*-p^*)\phi(n+1, m+1) - \theta'(q-p)\phi(n, m)}\,. \end{aligned} \tag{6.5}$$

2) If $\lambda \equiv -\mathrm{i}p(q+q^*)|q|^{-2}$, $\theta' = q^* q^{-1}$, $|q-q^*|^2|p|^2 = 4|q|^4$, then the vector function

$$\boldsymbol{S}(n,m) = \left(\frac{S^+(n,m)+S^{+*}(n,m)}{2}, \frac{S^+(n,m)-S^{+*}(n,m)}{2\mathrm{i}}, S^z(n,m)\right), \quad (6.6)$$

where

$$\begin{aligned} S^+(n,m) &= p\phi(n+1,m;0,0) - p\phi(n,m;0,0)\,, \\ S^z(n,m) &= -1 - p\psi(n+1,m;\,0,0) + p\psi(n,m;\,0,0), \end{aligned} \quad (6.7)$$

obeys the double-discrete isotropic Heisenberg spin chain (ddIHSC), i.e.

$$\begin{aligned} &\boldsymbol{S}(n,m+1) - \boldsymbol{S}(n,m) + \lambda \frac{\boldsymbol{S}(n,m)\times \boldsymbol{S}(n,m+1)}{1+\boldsymbol{S}(n,m)\cdot \boldsymbol{S}(n,m+1)} \\ &\quad - [\boldsymbol{S}(n,m)+\boldsymbol{S}(n,m+1)]\left[1 - \tfrac{1}{2}\lambda^2 \frac{\{1-\boldsymbol{S}(n,m)\cdot \boldsymbol{S}(n,m+1)\}}{\{1+\boldsymbol{S}(n,m)\cdot \boldsymbol{S}(n,m+1)\}^2}\right]^{1/2} \\ &= \boldsymbol{S}(n+1,m) - \boldsymbol{S}(n+1,m+1) + \lambda \frac{\boldsymbol{S}(n+1,m)\times \boldsymbol{S}(n+1,m+1)}{1+\boldsymbol{S}(n+1,m)\cdot \boldsymbol{S}(n+1,m+1)} \\ &\quad - [\boldsymbol{S}(n+1,m)+\boldsymbol{S}(n+1,m+1)] \\ &\quad \times \left[1 - \tfrac{1}{2}\lambda^2 \frac{\{1-\boldsymbol{S}(n+1,m)\cdot \boldsymbol{S}(n+1,m+1)\}}{\{1+\boldsymbol{S}(n+1,m)\cdot \boldsymbol{S}(n+1,m+1)\}^2}\right]^{1/2}, \\ &\boldsymbol{S}(n,m)\cdot \boldsymbol{S}(n,m) = 1\,. \end{aligned} \quad (6.8)$$

3) If $\lambda \equiv qp^{-1}$, $\theta' = 1$, $q^* = -q$, then the function

$$s(n,m) = \tfrac{1}{2}p\phi(n,m;0,0) \quad (6.9)$$

obeys the double-discrete complex sine-Gordon equation (ddCSG), i.e.

$$\begin{aligned} &\lambda[s(n,m)+s(n,m+1)][1-4|s(n,m+1)-s(n+1,m+1)|^2]^{1/2} \\ &\quad + \lambda[s(n+1,m)+s(n+1,m+1)][1-4|s(n,m)-s(n+1,m)|^2]^{1/2} \\ &\quad + [s(n,m+1)-s(n+1,m+1)][1-4\lambda^2|s(n,m)+s(n,m+1)|^2]^{1/2} \\ &\quad - [s(n,m)-s(n+1,m)][1-4\lambda^2|s(n+1,m)+s(n+1,m+1)|^2]^{1/2} = 0\,. \end{aligned} \quad (6.10)$$

Remarks

i) Eqs. (6.5), (6.8) and (6.10), which form the content of the proposition, are obtained as special reductions of the two coupled equations

$$\begin{aligned} &\theta[1-(p+\beta)\hat{\tilde{\psi}}^* + (p+\alpha)\tilde{\psi}^*][\theta'(q+\beta)\phi - (q^*+\alpha)\tilde{\phi}] \\ &\quad + [1+(q^*+\alpha)\tilde{\psi}^* - (q^*+\beta)\psi^*][\theta(p+\alpha)\tilde{\phi} - (p^*+\beta)\hat{\tilde{\phi}}] \\ &= \theta'[1-(q+\beta)\hat{\tilde{\psi}}^* + (q+\alpha)\hat{\psi}^*][\theta(p+\beta)\phi - (p^*+\alpha)\hat{\phi}] \\ &\quad + [1+(p^*+\alpha)\hat{\psi}^* - (p^*+\beta)\psi^*][\theta'(q+\alpha)\hat{\phi} - (q^*+\beta)\hat{\tilde{\phi}}]\,, \end{aligned} \quad (6.11)$$

$$\begin{aligned} &[\theta(p+\beta)\hat{\tilde{\phi}}^* - (p^*+\alpha)\tilde{\phi}^*][\theta'(q+\beta)\phi - (q^*+\alpha)\tilde{\phi}] \\ &\quad + [1-(p^*+\beta)\hat{\tilde{\psi}} + (p^*+\alpha)\tilde{\psi}][1+(q^*+\alpha)\tilde{\psi}^* - (q^*+\beta)\psi^*] \\ &= [\theta'(q+\beta)\hat{\tilde{\phi}}^* - (q^*+\alpha)\hat{\phi}^*][\theta(p+\beta)\phi - (p^*+\alpha)\hat{\phi}] \\ &\quad + [1-(q^*+\beta)\hat{\tilde{\psi}} + (q^*+\alpha)\hat{\psi}][1+(p^*+\alpha)\hat{\psi}^* - (p^*+\beta)\psi^*]\,, \end{aligned} \quad (6.12)$$

where we have used the abbreviations

$$\begin{aligned} &F = F(n,m;\alpha,\beta), \quad \hat{F} = F(n+1,m;\alpha,\beta)\,, \\ &\tilde{F} = F(n,m+1;\alpha,\beta), \quad \hat{\tilde{F}} = F(n+1,m+1;\alpha,\beta)\,, \\ &F^* \equiv [F(n,m;\alpha^*,\beta^*)]^*, \quad \text{etc.,} \quad \text{for } F = \phi, \psi\,, \end{aligned}$$

$$\psi(n,m;\alpha,\beta) \equiv \int_C \mathrm{d}\lambda(k) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \phi_{l'}^*(n,m;\alpha)}{(k+\beta)(k-l')}, \quad (6.13)$$

in combination with the relations

$$\begin{aligned} 1 &= [1-(p+\beta)\hat{\psi}^* + (p+\alpha)\psi^*][1-(p^*+\beta)\hat{\psi} + (p^*+\alpha)\psi] \\ &\quad + [\theta(p+\alpha)\phi - (p^*+\beta)\hat{\phi}][\theta^*(p^*+\alpha)\phi^* - (p+\beta)\hat{\phi}^*]\,, \end{aligned} \quad (6.14)$$

$$\begin{aligned} 1 &= [1-(q+\beta)\tilde{\psi}^* + (q+\alpha)\psi^*][1-(q^*+\beta)\tilde{\psi} + (q^*+\alpha)\psi] \\ &\quad + [\theta'(q+\alpha)\phi - (q^*+\beta)\tilde{\phi}][\theta'^*(q^*+\alpha)\phi^* - (q+\beta)\tilde{\phi}^*]\,, \end{aligned} \quad (6.15)$$

and

$$[1+(\alpha-\beta)\psi^*][1+(\alpha-\beta)\psi] + (\alpha-\beta)^2\phi\phi^* = 1\,. \quad (6.16)$$

The result, as given in eqs. (6.11), (6.12), (6.14)–(6.16), which is the counterpart for the NLS class of the difference–difference equation (2.4), is derived in appendix A from the integral equation (6.1) with

$$\rho_k(n,m) = \left(\theta\,\frac{p-k}{p^*-k}\right)^n \left(\theta'\,\frac{q-k}{q^*-k}\right)^m \rho_k(0,0), \quad p,q,\theta,\theta' \in \mathbb{C}, |\theta| = |\theta'| = 1\,. \quad (6.17)$$

The derivation of eqs. (6.5), (6.8) and (6.10) from the general result (6.11), (6.12) and (6.14)–(6.16) is given in appendix B.

ii) Eqs. (6.11) and (6.12) may be regarded as Bianchi-identities expressing the commutativity of the two Bäcklund transformations induced by

$$\rho_k \to \hat{\rho}_k = \theta\left(\frac{p-k}{p^*-k}\right)\rho_k, \quad |\theta| = 1\,, \tag{6.18}$$

$$\rho_k \to \tilde{\rho}_k = \theta'\left(\frac{q-k}{q^*-k}\right)\rho_k, \quad |\theta'| = 1\,, \tag{6.19}$$

cf. (6.17) and (6.13) and (6.3).

iii) Various double-discrete versions of the NLS and the IHSC have been obtained in the literature with corresponding Gel'fand–Levitan equation[11]), or bilinearization[13]). The Lax representations corresponding to (6.8), (6.10) and (6.5) are given in appendix C.

iv) Eq. (6.10) is a double-discrete version of the complex sine-Gordon equation. A double-discrete version of the sine-Gordon equation (cf. refs. 12 and 32) may be obtained from the integral equation (2.1) in the special case that

$$\rho_k(n,m) = \left(\frac{p+k}{p-k}\right)^n \left(\frac{q+k^{-1}}{q-k^{-1}}\right)^m \rho_k(0,0), \quad n, m \in \mathbb{Z},\ p, q, k \in \mathbb{C}\,. \tag{6.20}$$

In fact, in ref. 24 it has been shown that the function

$$w(n,m) = \frac{1}{2\mathrm{i}} \ln\left[-1 + \mathrm{i}\int_C \mathrm{d}\lambda(k)\, u_k(n,m)\right], \tag{6.21}$$

where $u_k(n,m)$ is the solution of (2.1), with (6.20), and with $\alpha = 0$, satisfies

$$\begin{aligned}&\sin[w(n,m) + w(n+1,m) + w(n,m+1) + w(n+1,m+1)]\\ &\quad - pq\,\sin[w(n,m) + w(n+1,m+1) - w(n+1,m) - w(n,m+1)] = 0\,,\end{aligned} \tag{6.22}$$

which can be regarded as the double-discrete sine-Gordon equation. In appendix D it will be shown how eq. (6.22) can be obtained from a Lax representation.

Corollary. Let $\phi_k(n,t;\alpha)$ be a solution of the linear integral equation

$$\begin{aligned}&\phi_k(n,t;\alpha) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l')\, \frac{\rho_k(n,t)\rho^*_{l'}(n,t)}{(k-l')(l'-l)}\, \phi_l(n,t;\alpha)\\ &\quad = \frac{\rho_k(n,t)}{k+\alpha}, \quad n \in \mathbb{Z},\ t \in \mathbb{R},\ k, \alpha \in \mathbb{C}\,,\end{aligned} \tag{6.23}$$

where C and $\mathrm{d}\lambda(k)$ are an arbitrary contour and measure in the complex k-plane. Let the contour C and the measure $\mathrm{d}\lambda(k)$ be such that the homogeneous integral equation corresponding to (6.23) has only the zero solution, and define

$$\phi(n,t;\alpha,\beta) \equiv \int_C \frac{\phi_k(n,t;\alpha)}{k+\beta}\, \mathrm{d}\lambda(k), \quad n \in \mathbb{Z},\ t \in \mathbb{R},\ \alpha, \beta \in \mathbb{C}\,. \tag{6.24}$$

Then the following results hold for special choices of $\rho_k(n,t)$, p, α and β:

1) If

$$\begin{aligned}&\rho_k(n,t) = \left(\frac{p-k}{p+k}\right)^n \exp\left[\mathrm{i}t\, \frac{[2pk(f^*-f) + 2k^2(f+f^*)]}{p^2-k^2}\right]\rho_k(0,0),\\ &\qquad p, f \in \mathbb{C},\ p = -p^*,\ |f| = 1\,,\end{aligned} \tag{6.25}$$

then the function

$$\phi(n,t) \equiv 2p\phi(n,t;-p,p) \tag{6.26}$$

obeys the discrete nonlinear Schrödinger equation (dNLS), i.e.

$$\mathrm{i}\partial_t\phi(n,t) = (f+f^*)\phi(n,t) - [1 + |\phi(n,t)|^2][f\phi(n+1,t) + f^*\phi(n-1,t)]\,. \tag{6.27}$$

2) If $\rho_k(n,t)$ is given by (6.25), then the vector function

$$\boldsymbol{S}(n,t) = \left(\frac{S^+(n,t) + S^{+*}(n,t)}{2}, \frac{S^+(n,t) - S^{+*}(n,t)}{2\mathrm{i}}, \quad [1 - |S^+(n,t)|^2]^{1/2}\right), \tag{6.28}$$

where

$$S^+(n,t) = p\phi(n+1,t;0,0) - p\phi(n,t;0,0)\,, \tag{6.29}$$

obeys the discrete isotropic Heisenberg spin chain (dIHSC), i.e.

$$\begin{aligned}&\partial_t \boldsymbol{S}(n,t) = (f+f^*)\left[\frac{\boldsymbol{S}(n,t) \times \boldsymbol{S}(n+1,t)}{1 + \boldsymbol{S}(n,t)\cdot \boldsymbol{S}(n+1,t)} - \frac{\boldsymbol{S}(n-1,t) \times \boldsymbol{S}(n,t)}{1 + \boldsymbol{S}(n-1,t)\cdot \boldsymbol{S}(n,t)}\right],\\ &\qquad \boldsymbol{S}(n,t)\cdot \boldsymbol{S}(n,t) = 1\,.\end{aligned} \tag{6.30}$$

3) If

$$\rho_k(n,t) = \left(\frac{p-k}{p+k}\right)^n \exp\left(-\frac{pt}{k}\right)\rho_k(0,0), \quad p \in \mathbb{C},\ p = -p^*\,, \tag{6.31}$$

then the function

$$s(n,t) \equiv \tfrac{1}{2}p\phi(n,t;0,0) \tag{6.32}$$

obeys the discrete complex sine-Gordon equation (dCSG)

$$\partial_t[s(n,t)-s(n-1,t)] = [s(n,t)+s(n-1,t)] \times [1-4|s(n,t)-s(n-1,t)|^2]^{1/2}\,. \tag{6.33}$$

Remarks

i) The integral equations (6.1) and (6.22) may be regarded as double-discrete and single-discrete analogues of the integral equation

$$\phi_k(x,t) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l')\, \frac{\rho_k(x,t)\rho^*_{l'}(x,t)}{(k-l')(l'-l)}\, \phi_l(x,t) = \frac{\rho_k(x,t)}{k+\alpha}\,,$$

$$\rho_k(x,t) = \mathrm{e}^{\mathrm{i}[kx-\omega(k)t]}\rho_k(0,0), \quad x,t\in\mathbb{R}, k\in\mathbb{C}\,, \tag{6.34}$$

which gives a direct linearization of e.g. the nonlinear Schrödinger equation, the equation of motion for the isotropic Heisenberg spin chain, and the complex sine-Gordon equation, cf. ref. 5.

ii) From the proposition given in this section and from its corollary it is clear that the difference–difference equations (6.5), (6.8) and (6.10), and the differential–difference equations (6.27), (6.30) and (6.33) are also completely integrable, since solutions can be obtained from the linear integral equations (6.1) and (6.23).

7. The NLS class; continuum limit

In section 6 we have given some difference–difference equations of the NLS class, i.e. the ddNLS (6.5), the ddIHSC (6.8) and the ddCSG (6.10). To obtain the corresponding differential–difference equations we consider the limit (4.1), in which b is a suitably chosen lattice parameter.

From eq. (6.5) it is clear that we can take

$$q \to -p\,, \tag{7.1}$$

$$\theta'(q-p) - (q^*-p^*) \to 0\,, \tag{7.2}$$

$$\phi(n+1, m+1; -p, p) \to \phi(n, m; -p, p)\,. \tag{7.3}$$

Eq. (7.1) suggests that we can choose $p+q$ as a small parameter and eq. (7.2) with $p^* = -p$ is automatically satisfied for $\theta' = q^*/q$.

In eq. (6.8) we can take a limit

$$\lambda \to 0, \quad |q| \to |p|\,, \tag{7.4}$$

$$\boldsymbol{S}(n+1, m+1) \to \boldsymbol{S}(n,m)\,. \tag{7.5}$$

Therefore, in both cases, $p+q$ can be chosen as a small parameter, provided that (7.3) and (7.5) respectively are satisfied. Accordingly, we take

$$p+q \to -2\mathrm{i}pft/m, \tag{7.6}$$

where f is a phase factor expressing the phase of $p+q$. Up to the order $(p+q)$ we then have the following relations:

$$\begin{aligned} &\phi(n,m) \to \phi(n',t), \quad n' \equiv n-m\,,\\ &\phi(n+1, m) \to \phi(n'+1, t)\,,\\ &\phi(n, m+1) \to \phi(n'-1, t) + \tfrac{1}{2}\mathrm{i}\frac{p+q}{p} f^*\partial_t\phi(n'-1,t)\,,\\ &\phi(n+1, m+1) \to \phi(n', t) + \tfrac{1}{2}\mathrm{i}\frac{p+q}{p} f^*\partial_t\phi(n',t)\,, \end{aligned} \tag{7.7}$$

and also

$$\begin{aligned} &\boldsymbol{S}(n,m) \to \boldsymbol{S}(n',t)\,,\\ &\boldsymbol{S}(n+1, m) \to \boldsymbol{S}(n'+1, t)\,,\\ &\boldsymbol{S}(n, m+1) \to \boldsymbol{S}(n'-1, t) + \tfrac{1}{2}\mathrm{i}\frac{p+q}{p} f^*\partial_t\boldsymbol{S}(n'-1,t)\,,\\ &\boldsymbol{S}(n+1, m+1) \to \boldsymbol{S}(n', t) + \tfrac{1}{2}\mathrm{i}\frac{p+q}{p} f^*\partial_t\boldsymbol{S}(n',t)\,. \end{aligned} \tag{7.8}$$

Taking (6.5) with $p^* = -p$, $\theta' = q^*/q$ and (7.1) and neglecting all terms $\mathcal{O}(p+q)$, we have

$$1 = [1+|\phi(n,m)|^2] \times \frac{(p+q)\phi(n+1,m) + (f^*/f)(p+q)\phi(n,m+1)}{(p+q)(1+f^*/f)\phi(n,m) - 2p[\phi(n+1,m+1) - \phi(n,m)]}\,, \tag{7.9}$$

and (6.27) follows immediately inserting (7.7) in (7.9). Analogously, eq. (6.30) follows from (6.8) inserting (7.8)and considering only terms $\mathcal{O}(\lambda)$.

The integral equation (6.23), together with eq. (6.25), can be derived from (6.1) taking the continuum limit of the factor $\rho_k(n,m)$ given in (6.2). We first rewrite $\rho_k(n,m)$ as follows,

$$\rho_k(n,m) = \left(\frac{p-k}{p+k}\right)^{n'} \times \left[1 - \frac{(\theta' pq + p^*q^*) - k(p^*+q^*+p\theta'+q\theta') + k^2(1+\theta')}{(p^*-k)(q^*-k)}\right]^m,$$

$$n' = n-m\,, \tag{7.10}$$

and from $p^* = -p$, $\theta' = q^*/q$ and eq. (7.5) we have

$$\frac{(\theta' pq + p^*q^*) - k(p^* + q^* + p\theta' + q\theta') + k^2(1+\theta')}{(p^*-k)(q^*-k)}$$
$$\rightarrow \frac{2pk\mathrm{i}(f^*-f)t/m + 2k^2\mathrm{i}(f^*+f)t/m}{|p|^2+k^2}, \tag{7.11}$$

and from (7.10) we immediately obtain

$$\rho_k(n,m) \rightarrow \rho_k(n',t), \tag{7.12}$$

in which $\rho_k(n',t)$ is given by (6.25).

Finally we can take a limit of eq. (6.10) with $\lambda \rightarrow 0$, provided that

$$s(n,m+1) - s(n+1,m+1) - s(n,m) + s(n+1,m) \rightarrow 0\,. \tag{7.13}$$

Therefore we take

$$\begin{aligned} &s(n,m) \rightarrow s(n,t)\,, \\ &s(n+1,m) \rightarrow s(n+1,t)\,, \\ &s(n,m+1) \rightarrow s(n,t) + 2\lambda\partial_t s(n,t)\,, \\ &s(n+1,m+1) \rightarrow s(n+1,t) + 2\lambda\partial_t s(n+1,t)\,, \end{aligned} \tag{7.14}$$

and keeping only the terms of order λ, eq. (6.10) reduces to eq. (6.33). The integral equation (6.23), together with eq. (6.31), follows by considering the continuum limit of (6.2) with $q = \lambda p = \frac{1}{2}pt/m$, $\theta' = 1$. We have

$$\frac{\rho_k(n,t)}{\rho_k(0,0)} = \left(\frac{p-k}{p+k}\right)^n \left(\frac{\frac{1}{2}pt/m - k}{-\frac{1}{2}pt/m - k}\right)^m \rightarrow \left(\frac{p-k}{p+k}\right)^n \exp\left(\frac{-pt}{k}\right), \tag{7.15}$$

in agreement with (6.31).

Remarks

i) In this section the corollary of the proposition given in section 6 has been proved by taking a suitably chosen continuum limit of the integral equation, as well as of the difference–difference equation given in the proposition. Of course eqs. (6.27), (6.30) and (6.33) can also be derived directly from (6.23) with (6.25) and (6.31). In fact in ref. 33 the dNLS, dIHSC and dCSG have been obtained from an (equivalent) integral equation in terms of the complex variable $z = (p-k)/(p+k)$ and $\rho_k(n,m) \rightarrow z^{n'} \mathrm{e}^{\mathrm{i}\omega(z)t}$, where the dispersion $\omega(z)$ is given by $\omega(z) = fz + f^*z^{-1} - (f + f^*)$ in the case of the dNLS and the dIHSC and by $\omega(z) = -\mathrm{i}(z+1)(z-1)^{-1}$ in the case of the dCSG.

ii) In the dNLS (6.27) we can take $f = f^* = 1$ without loss of generality, as can be seen introducing a new function $\Phi(n,t) = f^n\phi(n,t)\exp[\mathrm{i}(f + f^* - 2)t]$. Taking a continuum limit, however, eq. (6.27) after some obvious transformations changes into the NLS for $f = 1$ and into the complex modified Korteweg–de Vries equation for $f = -\mathrm{i}$. The dIHSC, which was given in refs. 34 and 35 for $f = 1$, reduces in the continuum limit to $\partial_t \boldsymbol{S} = \boldsymbol{S} \times \partial_x^2 \boldsymbol{S}$, which is the well-known IHSC. For $f = \mathrm{i}$, eq. (6.30) is trivial and a higher-order expansion in powers of λ is necessary to obtain a meaningful difference–difference equation. Some details on the continuum limit of the dNLS, as well as of the dIHSC and the dCSG, will be given in appendix E.

iii) In section 5 we discussed the relation between the difference–difference equations for the KdV class on the one hand, and the BT for the differential–difference equations, as well as the Bianchi-identities expressing the commutativity of BT's on the other hand. Such relations exist also in connection with the difference–difference equations of the NLS class.

In fact, eq. (B.5) of appendix B, with $p^* = -p$, $\theta = -1$, $2p\phi(-p,p) \rightarrow \phi(n',t)$, $2p\hat{\phi}(-p,p) \rightarrow \phi(n'+1,t)$, $2p\tilde{\phi}(-p,p) \rightarrow \tilde{\phi}(n'-1,t)$ and $2p\hat{\tilde{\phi}}(-p,p) \rightarrow \tilde{\phi}(n',t)$ is the BT for the dNLS (6.27) associated with the transformation

$$\rho_k(n,t) \rightarrow \tilde{\rho}_k(n,t)\theta'\left(\frac{q-k}{q^*-k}\right), \tag{7.16}$$

whereas eq. (B.5) with parameters $r = -r^*$ and s, instead of $p = -p^*$ and q, expresses the commutativity of the BT's

$$\rho_k \rightarrow \tilde{\rho}_k = \rho_k\theta' \frac{s-k}{s^*-k}, \quad \rho_k \rightarrow \hat{\rho}_k = -\rho_k\theta\frac{r-k}{r+k}, \tag{7.17}$$

for the NLS, the dNLS and ddNLS, independent of the specific dependence of ρ_k on variables or lattice sites.

Furthermore, eq. (B.18) with $\boldsymbol{S} \rightarrow \boldsymbol{S}(n',t)$, $\hat{\boldsymbol{S}} \rightarrow \boldsymbol{S}(n'+1,t)$, $\tilde{\boldsymbol{S}} \rightarrow \tilde{\boldsymbol{S}}(n'-1,t)$, $\hat{\tilde{\boldsymbol{S}}} \rightarrow \tilde{\boldsymbol{S}}(n',t)$ provides a BT for the dIHSC (6.30) under the transformation (7.16) with $\theta' = q^*/q$ and eq. (B.18) with $\frac{1}{2}p\phi \rightarrow s(n,t)$, $\frac{1}{2}p\hat{\phi} \rightarrow s(n+1,t)$, $\frac{1}{2}p\tilde{\phi} \rightarrow \tilde{s}(n,t)$, $\frac{1}{2}p\hat{\tilde{\phi}} \rightarrow \tilde{s}(n+1,t)$ is the BT for the dCSG (6.33) corresponding to (7.16) with $\theta' = 1$. Eqs. (B.18) and (B.19) can also be regarded as expressing the commutativity of Bäcklund transformations.

iv) From eqs. (B.6), (B.7) and (6.11) one obtains for the function $\chi(n,m) = \alpha^2\phi(n,m)$, in the limit $\alpha = \beta = \alpha^* = \beta^*$, $\alpha \rightarrow \infty$ the equation

$$\begin{aligned} &-\theta[(p-p^*) \pm \mathrm{i}\{|p-p^*|^2 - 4|\theta\tilde{\chi} - \hat{\tilde{\chi}}|^2\}^{1/2}](\theta'\chi - \tilde{\chi}) \\ &\quad + [(q-q^*) \mp \mathrm{i}\{|q-q^*|^2 - 4|\theta'\chi - \tilde{\chi}|^2\}^{1/2}](\theta\tilde{\chi} - \hat{\tilde{\chi}}) \\ &= -\theta'[(q-q^*) \mp \mathrm{i}\{|q-q^*|^2 - 4|\theta'\hat{\chi} - \hat{\tilde{\chi}}|^2\}^{1/2}](\theta\chi - \hat{\chi}) \\ &\quad + [(p-p^*) \pm \mathrm{i}\{|p-p^*|^2 - 4|\theta\chi - \hat{\chi}|^2\}^{1/2}](\theta'\hat{\chi} - \hat{\tilde{\chi}})\,, \end{aligned} \tag{7.18}$$

in which we have used the notations (6.13). Eq. (7.18) can also be regarded as a double discrete version of the NLS, as well as of the CSG. By taking a suitable continuum limit, i.e. eqs. (7.6) and (7.7) in the case of the NLS, and eq. (7.14) together with $\theta' = 1$, $q = \lambda p = \frac{1}{2}pt/m$, in the case of the CSG, one may derive differential–difference equations which are different from (6.27) and (6.33). These differential–difference equations may be regarded as discrete versions of the NLS and the CSG, since in the limit $p \to 0$, $n \to \infty$, $2n/p = -\mathrm{i}x$, these equations go over into the NLS and the CSG respectively. We shall not write down the explicit results for the differential–difference equations for the NLS and the CSG, since the expressions are more complicated than the ones given in (6.27) and (6.33).

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Appendix A

In this appendix we derive the coupled equations for the NLS class (6.11), (6.12) and (6.14)–(6.16) starting from the following two integral equations:

$$\phi^i_k(\alpha) + \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k}{k-l'} \psi^{i*}_{l'}(\alpha) = \rho_k \frac{k^i}{k+\alpha}, \tag{A.1}$$

$$\psi^i_k(\alpha) - \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k}{k-l'} \phi^{i*}_{l'}(\alpha) = 0, \tag{A.2}$$

or, equivalently,

$$\phi^i_k(\alpha) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho^*_{l'}}{(k-l')(l'-l)} \phi^i_l(\alpha) = \rho_k \frac{k^i}{k+\alpha}, \tag{A.3}$$

which for $i = 0$ with $\rho_k = \rho_k(n, m)$ reduces to (6.1), and

$$\psi^i_k(\alpha) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho^*_{l'}}{(k-l')(l'-l)} \psi^i_l(\alpha) = \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho^*_{l'}}{k-l'} \frac{l'^i}{l'+\alpha}. \tag{A.4}$$

Define

$$\phi^{i,j}(\alpha, \beta) \equiv \int_C \mathrm{d}\lambda(k) \frac{\phi^j_k(\alpha) k^i}{k+\beta}, \tag{A.5}$$

$$\psi^{i,j}(\alpha, \beta) \equiv \int_C \mathrm{d}\lambda(k) \frac{\psi^j_k(\alpha) k^i}{k+\beta}. \tag{A.6}$$

It is easy to show that

$$\phi^{i,j}(\alpha, \beta) = \phi^{j,i}(\beta, \alpha), \tag{A.7}$$

$$\psi^{i,j}(\alpha, \beta) = -\psi^{j,i*}(\beta, \alpha), \tag{A.8}$$

$$\phi^{i+1}_k(0) = \phi^{i+1}_k(\alpha) + \alpha\phi^i_k(\alpha), \tag{A.9}$$

$$\psi^{i+1}_k(0) = \psi^{i+1}_k(\alpha) + \alpha\psi^i_k(\alpha). \tag{A.10}$$

Using (A.1) and (A.2) we have

$$k\phi^i_k(\alpha) + \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k}{k-l'} l'\psi^{i*}_{l'}(\alpha) = \rho_k \frac{k^{i+1}}{k+\alpha} - \psi^{1,i*}(\alpha, 0)\rho_k, \tag{A.11}$$

$$k\psi^i_k(\alpha) - \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k}{k-l'} l'\phi^{i*}_{l'}(\alpha) = \phi^{1,i*}(\alpha, 0)\rho_k, \tag{A.12}$$

leading to

$$\begin{aligned} &k\phi^i_k(\alpha) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho^*_{l'}}{(k-l')(l'-l)} l\phi^i_l(\alpha) \\ &\quad = \rho_k \frac{k^{i+1}}{k+\alpha} - \psi^{1,i*}(\alpha, 0)\rho_k - \phi^{1,i}(\alpha, 0) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho^*_{l'}}{k-l'}, \end{aligned} \tag{A.13}$$

and

$$\begin{aligned} &k\psi^i_k(\alpha) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho^*_{l'}}{(k-l')(l'-l)} l\psi^i_l(\alpha) \\ &\quad = \phi^{1,i*}(\alpha, 0)\rho_k + \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho^*_{l'}}{k-l'} \frac{l'^{i+1}}{l'+\alpha} - \psi^{1,i}(\alpha, 0) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho^*_{l'}}{k-l'}. \end{aligned} \tag{A.14}$$

Taking into account that the solutions of the integral equations (A.3) and (A.4) are unique and using also (A.9) and (A.10), we have

$$(k+\alpha)\phi_k^i(\alpha) = \phi_k^{i+1}(0) - \psi^{1,i*}(\alpha,0)\phi_k^1(0) - \phi^{1,i}(\alpha,0)\psi_k^1(0)\,, \tag{A.15}$$

$$(k+\alpha)\psi_k^i(\alpha) = \psi_k^{i+1}(0) - \psi^{1,i}(\alpha,0)\psi_k^1(0) + \phi^{1,i*}(\alpha,0)\phi_k^1(0)\,. \tag{A.16}$$

Dividing (A.15) and (A.16) for $i=0$ by $k+\beta$, integrating over the contour C, and using (A.7) and (A.8), we have

$$(\alpha-\beta)\phi^{0,0}(\alpha,\beta) = (1-\psi^{1,0*}(\alpha,0))\phi^{1,0}(\beta,0) - (1-\psi^{1,0*}(\beta,0))\phi^{1,0}(\alpha,0)\,, \tag{A.17}$$

$$1+(\alpha-\beta)\psi^{0.0}(\alpha,\beta) = (1-\psi^{1,0}(\alpha,0))(1-\psi^{1,0*}(\beta,0)) + \phi^{1,0*}(\alpha,0)\phi^{1,0}(\beta,0)\,, \tag{A.18}$$

which for $\beta=\alpha$ reduces to the identity

$$(1-\psi^{1,0}(\alpha,0))(1-\psi^{1,0*}(\alpha,0)) + \phi^{1,0*}(\alpha,0)\phi^{1,0}(\alpha,0) = 1\,. \tag{A.19}$$

Let us now consider the Bäcklund transformation (6.18) of the integral equation (A.3). It is then straightforward to show that the solutions $\hat\phi_k^i(\alpha)$ and $\hat\psi_k^i(\alpha)$ of (A.1) and (A.2) with ρ_k replaced by $\hat\rho_k$ satisfy the integral relations

$$\begin{aligned}(p^*-k)\hat\phi_k^i(\alpha) &+ \int_C \mathrm{d}\lambda(l)\int_{C^*}\mathrm{d}\lambda^*(l')\,\frac{\rho_k\rho_{l'}^*}{(k-l')(l'-l)}\,(p^*-l)\hat\phi_l^i(\alpha)\\ &= \theta(p-k)\rho_k\frac{k^i}{k+\alpha} + \theta\hat\psi^{1,i*}(\alpha,0)\rho_k + \hat\phi^{1,i}(\alpha,0)\int_{C^*}\mathrm{d}\lambda^*(l')\,\frac{\rho_k\rho_{l'}^*}{k-l'}\,,\end{aligned} \tag{A.20}$$

and

$$\begin{aligned}(p^*-k)\hat\psi_k^i(\alpha) &+ \int_C \mathrm{d}\lambda(l)\int_{C^*}\mathrm{d}\lambda^*(l')\,\frac{\rho_k\rho_{l'}^*}{(k-l')(l'-l)}\,(p^*-l)\hat\psi_l^i(\alpha)\\ &= -\theta\hat\phi^{1,i*}(\alpha,0)\rho_k + \int_{C^*}\mathrm{d}\lambda^*(l')\,\frac{\rho_k\rho_{l'}^*}{k-l'}\,\frac{(p^*-l')l'^i}{l'+\alpha}\\ &\quad + \hat\psi^{1,i}(\alpha,0)\int_{C^*}\mathrm{d}\lambda^*(l')\,\frac{\rho_k\rho_{l'}^*}{k-l'}\,,\end{aligned} \tag{A.21}$$

where $\hat\psi^{i,j}(\alpha,\beta)$ and $\hat\phi^{i,j}(\alpha,\beta)$ are defined by (A.5) and (A.6) with $\hat\psi_k^j(\alpha)$ and $\hat\phi_k^j(\alpha)$ instead of $\psi_k^j(\alpha)$ and $\phi_k^j(\alpha)$.

Comparing with the integral equations (A.3) and (A.4) and using also (A.9) and (A.10) we immediately have

$$\begin{aligned}(p^*-k)\hat\phi_k^i(\alpha) &= \theta(p+\alpha)\phi_k^i(\alpha) - \theta\phi_k^{i+1}(0)\\ &\quad + \theta\hat\psi^{1,i*}(\alpha,0)\phi_k^1(0) + \hat\phi^{1,i}(\alpha,0)\psi_k^1(0)\,,\end{aligned} \tag{A.22}$$

$$\begin{aligned}(p^*-k)\hat\psi_k^i(\alpha) &= (p^*+\alpha)\psi_k^i(\alpha) - \psi_k^{i+1}(0)\\ &\quad + \hat\psi^{1,i}(\alpha,0)\psi_k^1(0) - \theta\hat\phi^{1,i*}(\alpha,0)\phi_k^1(0)\,.\end{aligned} \tag{A.23}$$

Hence, after dividing (A.22) and (A.23) for $i=0$ by $k+\beta$, and integrating over the contour C, we have

$$\begin{aligned}(p^*+\beta)\hat\phi(\alpha,\beta) - \theta(p+\alpha)\phi(\alpha,\beta) &= (1-\psi^{1,0*}(\beta,0))\hat\phi^{1,0}(\alpha,0)\\ &\quad - \theta(1-\hat\psi^{1,0*}(\alpha,0))\phi^{1,0}(\beta,0)\,,\end{aligned} \tag{A.24}$$

$$\begin{aligned}-1+(p^*+\beta)\hat\psi(\alpha,\beta) - (p^*+\alpha)\psi(\alpha,\beta) &= -(1-\hat\psi^{1,0}(\alpha,0))(1-\psi^{1,0*}(\beta,0))\\ &\quad - \theta\hat\phi^{1,0*}(\alpha,0)\phi^{1,0}(\beta,0)\,,\end{aligned} \tag{A.25}$$

where we have omitted the superscripts 0,0 on the left-hand side.

Eqs. (6.11), (6.12) and (6.14)–(6.16) are most easily derived introducing the matrix

$$\mathbf{g}(\alpha) \equiv \begin{pmatrix} 1-\psi^{1,0}(\alpha,0) & \phi^{1,0}(\alpha,0)\\ -\phi^{1,0*}(\alpha,0) & 1-\psi^{1,0*}(\alpha,0)\end{pmatrix}. \tag{A.26}$$

Then (A.24) and (A.25) can be cast in matrix notation

$$\begin{aligned}&\begin{pmatrix} \theta(1-(p+\beta)\hat\psi^*(\alpha,\beta)+(p+\alpha)\psi^*(\alpha,\beta)) & -(p^*+\beta)\hat\phi(\alpha,\beta)+\theta(p+\alpha)\phi(\alpha,\beta)\\ \theta(p+\beta)\hat\phi^*(\alpha,\beta)-(p^*+\alpha)\phi^*(\alpha,\beta) & 1-(p^*+\beta)\hat\psi(\alpha,\beta)+(p^*+\alpha)\psi(\alpha,\beta)\end{pmatrix}\\ &\qquad = \hat{\mathbf{g}}^{-1}(\alpha)\cdot\boldsymbol{\theta}\cdot\mathbf{g}(\beta),\quad \boldsymbol{\theta}\equiv\begin{pmatrix}\theta & 0\\ 0 & 1\end{pmatrix},\end{aligned} \tag{A.27}$$

where $\hat{\mathbf{g}}(\alpha)$ can be obtained from $\mathbf{g}(\alpha)$ replacing $\phi^{1,0}$ and $\psi^{1,0}$ by $\hat\phi^{1,0}$ and $\hat\psi^{1,0}$.

Considering the Bäcklund transformation (6.19), we have in a similar way, omitting the superscripts 0,0 on the left-hand side

$$\begin{aligned}&\begin{pmatrix} \theta'(1-(q+\beta)\tilde\psi^*(\alpha,\beta)+(q+\alpha)\psi^*(\alpha,\beta)) & -(q^*+\beta)\tilde\phi(\alpha,\beta)+\theta'(q+\alpha)\phi(\alpha,\beta)\\ \theta'(q+\beta)\tilde\phi^*(\alpha,\beta)-(q^*+\alpha)\phi^*(\alpha,\beta) & 1-(q^*+\beta)\tilde\psi(\alpha,\beta)+(q^*+\alpha)\psi(\alpha,\beta)\end{pmatrix}\\ &\qquad = \tilde{\mathbf{g}}^{-1}(\alpha)\cdot\boldsymbol{\theta}'\cdot\mathbf{g}(\beta),\quad \boldsymbol{\theta}'\equiv\begin{pmatrix}\theta' & 0\\ 0 & 1\end{pmatrix},\end{aligned} \tag{A.28}$$

where $\tilde\psi(\alpha,\beta)=\tilde\psi^{0,0}(\alpha,\beta)$ and $\tilde\phi(\alpha,\beta)=\tilde\phi^{0,0}(\alpha,\beta)$ are defined by (A.5) and (A.6) with $\psi_k^j(\alpha)$ and $\phi_k^j(\alpha)$ replaced by the solutions $\tilde\psi_k^j(\alpha)$ and $\tilde\phi_k^j(\alpha)$ of (A.1) and (A.2) with $\tilde{\rho}_k$ instead of ρ_k, and $\tilde{\mathbf{g}}(\alpha)$ can be obtained from $\mathbf{g}(\alpha)$ replacing $\phi^{1,0}$ and $\psi^{1,0}$ by $\tilde{\phi}^{1,0}$ and $\tilde{\psi}^{1,0}$.

Furthermore eq. (A.19) implies that

$$\mathbf{g}(\alpha)\cdot\mathbf{g}^{\dagger}(\alpha)=\mathbb{1}\,, \tag{A.29}$$

in which the 2×2 matrix $\mathbf{g}^{\dagger}(\alpha)$ is defined by

$$g^{\dagger}_{pq}(\alpha)=g^{*}_{qp}(\alpha)=(g_{qp}(\alpha^{*}))^{*}\,. \tag{A.30}$$

For the matrices $\mathbf{g}(\alpha)$, $\hat{\mathbf{g}}(\alpha)$, $\tilde{\mathbf{g}}(\alpha)$ and the matrix $\hat{\tilde{\mathbf{g}}}(\alpha)$, which can be obtained from (A.26) replacing $\phi^{1,0}$ and $\psi^{1,0}$ by $\hat{\tilde{\phi}}^{1,0}$ and $\hat{\tilde{\psi}}^{1,0}$ respectively, we have the obvious relations

$$\hat{\tilde{\mathbf{g}}}^{-1}(\alpha)\cdot\boldsymbol{\theta}\cdot\tilde{\mathbf{g}}(\beta)\cdot\tilde{\mathbf{g}}^{-1}(\beta)\cdot\boldsymbol{\theta}'\cdot\mathbf{g}(\alpha)=\hat{\tilde{\mathbf{g}}}^{-1}(\alpha)\cdot\boldsymbol{\theta}'\cdot\hat{\mathbf{g}}(\beta)\cdot\hat{\mathbf{g}}^{-1}(\beta)\cdot\boldsymbol{\theta}\cdot\mathbf{g}(\alpha)\,, \tag{A.31}$$

$$\begin{aligned}\mathbb{1}&=\hat{\mathbf{g}}^{-1}(\alpha)\cdot\boldsymbol{\theta}\cdot\mathbf{g}(\beta)\cdot(\hat{\mathbf{g}}^{-1}(\alpha)\cdot\boldsymbol{\theta}\cdot\mathbf{g}(\beta))^{\dagger}\\&=\tilde{\mathbf{g}}^{-1}(\alpha)\cdot\boldsymbol{\theta}'\cdot\mathbf{g}(\beta)\cdot(\tilde{\mathbf{g}}^{-1}(\alpha)\cdot\boldsymbol{\theta}'\cdot\mathbf{g}(\beta))^{\dagger}\,,\end{aligned} \tag{A.32}$$

$$\mathbb{1}=\mathbf{g}^{-1}(\alpha)\cdot\mathbf{g}(\beta)\cdot\mathbf{g}^{-1}(\beta)\cdot\mathbf{g}(\alpha)\,. \tag{A.33}$$

Inserting the expressions (A.26)–(A.28) we obtain eqs. (6.11) and (6.12) from eq. (A.31), eqs. (6.14) and (6.15) from eq. (A.32) and eq. (6.16) from eq. (A.33). Eq. (6.16) is an algebraic identity relating $\phi(\alpha,\beta)$ and $\psi(\alpha,\beta)$, eqs. (6.14) and (6.15) are relations between $\phi(\alpha,\beta)$ and $\psi(\alpha,\beta)$ and their Bäcklund transforms under the transformations (6.18) and (6.19), and (6.11) and (6.12) are Bianchi-identities expressing the commutativity of both BT's.

Identifying $\phi(\alpha,\beta)$, $\hat{\phi}(\alpha,\beta)$, $\tilde{\phi}(\alpha,\beta)$, $\hat{\tilde{\phi}}(\alpha,\beta)$, $\psi(\alpha,\beta)$, $\hat{\psi}(\alpha,\beta)$, $\tilde{\psi}(\alpha,\beta)$, $\hat{\tilde{\psi}}(\alpha,\beta)$ with $\phi(n,m;\alpha,\beta)$, $\phi(n+1,m;\alpha,\beta)$, $\phi(n,m+1;\alpha,\beta)$, $\phi(n+1,m+1;\alpha,\beta)$, $\psi(n,m;\alpha,\beta)$, $\psi(n+1,m;\alpha,\beta)$, $\psi(n,m+1;\alpha,\beta)$ and $\psi(n+1,m+1;\alpha,\beta)$, respectively, we obtain from (6.11), (6.12) and (6.14)–(6.16) a set of difference–difference equations for the four coupled fields ϕ, ψ, ϕ^{*}, ψ^{*}, defined on a two-dimensional lattice. (Note that for general α and β, $\phi^{*}(n,m;\alpha,\beta)=[\phi(n,m;\alpha^{*},\beta^{*})]^{*}$.) This set of equations is completely integrable, since the solutions follow from (6.1) with the factor $\rho_k(n,m)$ as given in (6.17). In the general case the set of equations is rather complicated, but in appendix B we shall work out the special cases mentioned in the proposition given in section 6.

Appendix B

In this appendix we derive the ddNLS, the ddIHSC and the ddCSG, as given by eqs. (6.5), (6.8) and (6.10).

(1) For the ddNLS we take $\alpha=-p$, $\beta=p$, $p^{*}=-p$. From (6.14) and (6.16), with $\psi\equiv\psi(-p,p)$; $\phi\equiv\phi(-p,p)$ etc., using also (A.7) and (A.8), one can derive

$$1=(1+2p\hat{\psi}(p,-p))(1-2p\psi(-p,p))\,, \tag{B.1}$$

and

$$\frac{1+2p\psi(p,-p)}{1+2p\hat{\psi}(p,-p)}=1+4|p|^2|\phi(-p,p)|^2\,, \tag{B.2}$$

and from (6.11) we have

$$\frac{1+2p\hat{\tilde{\psi}}(p,-p)}{1+2p\hat{\psi}(p,-p)}=\frac{\theta\theta'(q-p)\phi(-p,p)+(q^{*}-p^{*})\hat{\tilde{\phi}}(-p,p)}{\theta(q^{*}+p^{*})\tilde{\phi}(-p,p)+\theta'(q+p)\hat{\phi}(-p,p)}\,. \tag{B.3}$$

From (6.15) we obtain

$$\begin{aligned}&\frac{1-(\theta'(q-p)\phi(-p,p)-(q^{*}-p^{*})\tilde{\phi}(-p,p))}{1+4|p|^2|\phi(-p,p)|^2}\\&\qquad\times(\theta'^{*}(q^{*}+p^{*})\phi^{*}(-p,p)-(q+p)\tilde{\phi}^{*}(-p,p))\\&\quad=\left(\frac{q+p}{2p}\,\frac{1+2p\tilde{\psi}(p,-p)}{1+2p\psi(p,-p)}-\frac{q-p}{2p}\right)\left(\frac{q^{*}+p}{2p}\,\frac{1-2p\tilde{\psi}(-p,p)}{1-2p\psi(-p,p)}-\frac{q^{*}-p}{2p}\right).\end{aligned} \tag{B.4}$$

Using (B.1)–(B.3) to eliminate the ψ's we have

$$\begin{aligned}1&+\frac{|q|^2-|p|^2}{2|p|^2}+\theta'(q+p)(q-p)\phi(-p,p)\tilde{\phi}^{*}(-p,p)\\&+\theta'^{*}(q^{*}+p^{*})(q^{*}-p^{*})\phi^{*}(-p,p)\tilde{\phi}(-p,p)\\&=\frac{|q+p|^2}{4|p|^2}(1+4|p|^2|\tilde{\phi}(-p,p)|^2)\\&\quad\times\frac{\theta\theta'(q-p)\phi(-p,p)+(q^{*}-p^{*})\hat{\tilde{\phi}}(-p,p)}{\theta(q^{*}+p^{*})\tilde{\phi}(-p,p)+\theta'(q+p)\hat{\phi}(-p,p)}\\&\quad+\frac{|q-p|^2}{4|p|^2}(1+4|p|^2|\phi(-p,p)|^2)\\&\quad\times\frac{\theta(q^{*}+p^{*})\tilde{\phi}(-p,p)+\theta'(q+p)\hat{\phi}(-p,p)}{\theta\theta'(q-p)\phi(-p,p)+(q^{*}-p^{*})\hat{\tilde{\phi}}(-p,p)}\,,\end{aligned} \tag{B.5}$$

and eq. (6.5) follows from (B.5) using (6.13), (6.4), and $\theta=-1$.

(2) In the case that $\alpha=\beta=\alpha^{*}=\beta^{*}$, one can solve $\hat{\psi}-\psi\equiv\hat{\psi}(\alpha,\alpha)-\psi(\alpha,\alpha)$ from (6.14) as a function of $\phi\equiv\phi(\alpha,\alpha)$. We have

$$\hat{\psi}(\alpha,\alpha)-\psi(\alpha,\alpha)=\frac{p-p^*}{2|p+\alpha|^2}\pm\frac{\mathrm{i}}{2|p+\alpha|^2}$$
$$\times\,[|p^*-p|^2-4|p+\alpha|^2|\theta(p+\alpha)\phi(\alpha,\alpha)-(p^*+\alpha)\hat{\phi}(\alpha,\alpha)|^2]^{1/2}\,, \tag{B.6}$$

and in a similar way we have from (6.15)

$$\tilde{\psi}(\alpha,\alpha)-\psi(\alpha,\alpha)=\frac{q-q^*}{2|q+\alpha|^2}\pm\frac{\mathrm{i}}{2|q+\alpha|^2}$$
$$\times\,[|q^*-q|^2-4|q+\alpha|^2|\theta'(q+\alpha)\phi(\alpha,\alpha)-(q^*+\alpha)\tilde{\phi}(\alpha,\alpha)|^2]^{1/2}\,. \tag{B.7}$$

Inserting (B.6) and (B.7), eqs. (6.11) and (6.12) can be expressed in terms of the ϕ's. We now restrict ourselves to the special case $p^*=-p$, $\theta=-1$, $\alpha=\beta=0$. In that case we obtain from (6.11) and (6.12), with $\phi\equiv\phi(0,0)$, $\psi\equiv\psi(0,0)$, the following equations:

$$\pm\frac{\mathrm{i}|p|}{p}(\theta' q\phi(0,0)-q^*\tilde{\phi}(0,0))[1-|p|^2|\tilde{\phi}(0,0)-\tilde{\hat{\phi}}(0,0)|^2]^{1/2}$$
$$\pm\frac{\mathrm{i}|p|}{p}(\theta' q\hat{\phi}(0,0)-q^*\tilde{\hat{\phi}}(0,0))[1-|p|^2|\phi(0,0)-\hat{\phi}(0,0)|^2]^{1/2}$$
$$-p(\tilde{\phi}(0,0)-\tilde{\hat{\phi}}(0,0))$$
$$\times\left\{\frac{q+q^*}{2q}\mp\frac{\mathrm{i}|q|}{q}\left[\frac{|q^*-q|^2}{4|q|^2}-|\theta' q\phi(0,0)-q^*\tilde{\phi}(0,0)|^2\right]^{1/2}\right\}$$
$$+\theta' p(\phi(0,0)-\hat{\phi}(0,0))$$
$$\times\left\{\frac{q+q^*}{2q^*}\pm\frac{\mathrm{i}|q|}{q^*}\left[\frac{|q^*-q|^2}{4|q|^2}-|\theta' q\hat{\phi}(0,0)-q^*\tilde{\hat{\phi}}(0,0)|^2\right]^{1/2}\right\}=0\,, \tag{B.8}$$

and

$$p(\tilde{\phi}^*(0,0)-\tilde{\hat{\phi}}^*(0,0))(\theta' q\phi(0,0)-q^*\tilde{\phi}(0,0))$$
$$+p(\phi(0,0)-\hat{\phi}(0,0))(\theta' q\tilde{\hat{\phi}}^*(0,0)-q^*\hat{\phi}^*(0,0))$$
$$\mp\frac{\mathrm{i}|p|}{p}[1-|p|^2|\tilde{\phi}(0,0)-\tilde{\hat{\phi}}(0,0)|^2]^{1/2}$$
$$\times\left\{\frac{q+q^*}{2q}\mp\frac{\mathrm{i}|q|}{q}\left[\frac{|q^*-q|^2}{4|q|^2}-|\theta' q\phi(0,0)-q^*\tilde{\phi}(0,0)|^2\right]^{1/2}\right\}$$
$$\pm\frac{\mathrm{i}|p|}{p}[1-|p|^2|\phi(0,0)-\hat{\phi}(0,0)|^2]^{1/2}$$
$$\times\left\{\frac{q+q^*}{2q}\mp\frac{\mathrm{i}|q|}{q}\left[\frac{|q^*-q|^2}{4|q|^2}-|\theta' q\hat{\phi}(0,0)-q^*\tilde{\hat{\phi}}(0,0)|^2\right]^{1/2}\right\}=0\,. \tag{B.9}$$

For the IHSC we consider the special case $\theta'=q^*/q$, in addition to $p^*=-p$, $\theta=-1$, $\alpha=\beta=0$. Introducing real vectors $\boldsymbol{S}$ and $\boldsymbol{U}$ by

$$\boldsymbol{S}\equiv\left(\frac{S^++S^{+*}}{2},\frac{S^+-S^{+*}}{2\mathrm{i}},S^z\right),$$
$$S^+\equiv p\hat{\phi}(0,0)-p\phi(0,0)\,, \tag{B.10}$$
$$S^z\equiv-1-p\hat{\psi}(0,0)+p\psi(0,0)=\mp\mathrm{i}p|p|^{-1}(1-|S^+|^2)^{1/2},\quad \boldsymbol{S}\cdot\boldsymbol{S}=1\,,$$

and

$$\boldsymbol{U}\equiv\left(\frac{U^++U^{+*}}{2},\frac{U^+-U^{+*}}{2\mathrm{i}},U^z\right),$$
$$U^+\equiv p\tilde{\phi}(0,0)-p\phi(0,0)\,,$$
$$U^z\equiv\tfrac{1}{2}(q-q^*)p|q|^{-2}-p\tilde{\psi}(0,0)+p\psi(0,0)=\mp\mathrm{i}p|p|^{-1}(\mu-|U^+|^2)^{1/2}\,, \tag{B.11}$$
$$\boldsymbol{U}\cdot\boldsymbol{U}=\mu\equiv\tfrac{1}{4}|q-q^*|^2|p|^2/|q|^4\,,$$

eqs. (6.11) and (6.12), or eqs. (B.8) and (B.9) can be combined to give the equations

$$\tilde{\boldsymbol{S}}\times\boldsymbol{U}+\boldsymbol{S}\times\hat{\boldsymbol{U}}+\tfrac{1}{2}\lambda(\tilde{\boldsymbol{S}}-\boldsymbol{S})=0 \tag{B.12}$$

and

$$\boldsymbol{U}\cdot\tilde{\boldsymbol{S}}-\hat{\boldsymbol{U}}\cdot\boldsymbol{S}=0\,, \tag{B.13}$$

where

$$\lambda=-\mathrm{i}p\frac{q+q^*}{|q|^2}\,. \tag{B.14}$$

From the definitions of $\boldsymbol{S}$ and $\boldsymbol{U}$ we have the obvious relation

$$\boldsymbol{U}+\tilde{\boldsymbol{S}}=\boldsymbol{S}+\hat{\boldsymbol{U}}\,. \tag{B.15}$$

Using (B.15) to eliminate $\boldsymbol{U}$ from (B.12), we obtain

$$(\boldsymbol{S}+\tilde{\boldsymbol{S}})\times\hat{\boldsymbol{U}}=\boldsymbol{S}\times\tilde{\boldsymbol{S}}-\tfrac{1}{2}\lambda(\tilde{\boldsymbol{S}}-\boldsymbol{S})\,, \tag{B.16}$$

and from (B.12) and (B.15) with $\boldsymbol{S}\to\hat{\boldsymbol{S}}$, $\tilde{\boldsymbol{S}}\to\hat{\tilde{\boldsymbol{S}}}$, $\boldsymbol{U}\to\hat{\boldsymbol{U}}$, $\hat{\boldsymbol{U}}\to\hat{\hat{\boldsymbol{U}}}$, eliminating $\hat{\hat{\boldsymbol{U}}}$, we also obtain a second relation,

$$(\hat{\boldsymbol{S}}+\hat{\tilde{\boldsymbol{S}}})\times\hat{\boldsymbol{U}}=-\hat{\boldsymbol{S}}\times\hat{\tilde{\boldsymbol{S}}}-\tfrac{1}{2}\lambda(\hat{\tilde{\boldsymbol{S}}}-\hat{\boldsymbol{S}})\,. \tag{B.17}$$

Taking into account that $\boldsymbol{U}\cdot\boldsymbol{U}=\mu$, one can solve $\hat{\boldsymbol{U}}$ from (B.16) as well as from (B.17) to obtain the equation

$$\mp\frac{\boldsymbol{S}+\tilde{\boldsymbol{S}}}{1+\boldsymbol{S}\cdot\tilde{\boldsymbol{S}}}[2\mu-\tfrac{1}{2}\lambda^2-1+(2\mu+\tfrac{1}{2}\lambda^2)\boldsymbol{S}\cdot\tilde{\boldsymbol{S}}+(\boldsymbol{S}\cdot\tilde{\boldsymbol{S}})^2]^{1/2}$$

$$+\tilde{\boldsymbol{S}}-\boldsymbol{S}+\lambda\frac{\boldsymbol{S}\times\tilde{\boldsymbol{S}}}{1+\boldsymbol{S}\cdot\tilde{\boldsymbol{S}}}$$

$$=\mp\frac{\hat{\boldsymbol{S}}+\hat{\tilde{\boldsymbol{S}}}}{1+\hat{\boldsymbol{S}}\cdot\hat{\tilde{\boldsymbol{S}}}}[2\mu-\tfrac{1}{2}\lambda^2-1+(2\mu+\tfrac{1}{2}\lambda^2)\hat{\boldsymbol{S}}\cdot\hat{\tilde{\boldsymbol{S}}}+(\hat{\boldsymbol{S}}\cdot\hat{\tilde{\boldsymbol{S}}})^2]^{1/2}$$

$$+\hat{\boldsymbol{S}}-\hat{\tilde{\boldsymbol{S}}}+\lambda\frac{\hat{\boldsymbol{S}}\times\hat{\tilde{\boldsymbol{S}}}}{1+\hat{\boldsymbol{S}}\cdot\hat{\tilde{\boldsymbol{S}}}}\,. \tag{B.18}$$

In the special case $\mu=1$, taking the upper signs and using the identifications $\boldsymbol{S}\to\boldsymbol{S}(n,m)$, $\hat{\boldsymbol{S}}\to\boldsymbol{S}(n+1,m)$, $\tilde{\boldsymbol{S}}\to\boldsymbol{S}(n,m+1)$ and $\hat{\tilde{\boldsymbol{S}}}\to\boldsymbol{S}(n+1,m+1)$, eq. (B.18) immediately reduces to the ddIHSC given in (6.8).

(3) Finally, for the ddCSG we consider the special case $p=\mp\mathrm{i}|p|$, $\theta'=-q^*/q$, $\lambda=q/p$, in addition to $\theta=-1$, $\alpha=\beta=0$. From (B.8) we obtain

$$\lambda^*(\phi(0,0)+\tilde{\phi}(0,0))[1-|p|^2|\tilde{\phi}(0,0)-\hat{\tilde{\phi}}(0,0)|^2]^{1/2}$$

$$+\lambda^*(\hat{\phi}(0,0)+\hat{\tilde{\phi}}(0,0))[1-|p|^2|\phi(0,0)-\hat{\phi}(0,0)|^2]^{1/2}$$

$$+(\tilde{\phi}(0,0)-\hat{\tilde{\phi}}(0,0))$$

$$\times\left\{\frac{\lambda-\lambda^*}{2\lambda}+\frac{|\lambda|}{\lambda}\left[\frac{(\lambda^*+\lambda)^2}{4|\lambda|^2}-|\lambda|^2|p|^2|\phi(0,0)+\tilde{\phi}(0,0)|^2\right]^{1/2}\right\}$$

$$+(\phi(0,0)-\hat{\phi}(0,0))$$

$$\times\left\{\frac{\lambda-\lambda^*}{2\lambda}-\frac{|\lambda|}{\lambda}\left[\frac{(\lambda^*+\lambda)^2}{4|\lambda|^2}-|\lambda|^2|p|^2|\hat{\phi}(0,0)+\hat{\tilde{\phi}}(0,0)|^2\right]^{1/2}\right\}=0\,, \tag{B.19}$$

which for λ real, i.e. $q^*=-q$, $\theta'=1$, together with (6.9) and (6.13) reduces to (6.10).

Appendix C

In this appendix we will give some results concerning Lax representations for the ddIHSC and the ddCSG, and also for the ddNLS. From eqs. (A.22) and (A.23) of appendix A we have for $i=0$

$$(p^*-k)\widehat{\phi}^0_k(\alpha)=\theta(p+\alpha)\phi^0_k(\alpha)-\theta(1-\widehat{\psi}^{1,0*}(\alpha,0))\phi^1_k(0)+\widehat{\phi}^{1,0}(\alpha,0)\psi^1_k(0)\,, \tag{C.1}$$

and

$$(p^*-k)\widehat{\psi}^0_k(\alpha)=(p^*+\alpha)\psi^0_k(\alpha)-(1-\widehat{\psi}^{1,0}(\alpha,0))\psi^1_k(0)-\theta\widehat{\phi}^{1,0*}(\alpha,0)\phi^1_k(0)\,. \tag{C.2}$$

Using the identifications (6.13), and defining the 2-component vector

$$\chi_k(n,m;\alpha)\equiv(\phi^0_k(n,m;\alpha),\psi^0_k(n,m;\alpha))\,, \tag{C.3}$$

we can cast (C.1) and (C.2) in matrix form as follows:

$$(p^*-k)\chi_k(n+1,m;\alpha)=\begin{pmatrix}\theta(p+\alpha) & 0\\ 0 & p^*+\alpha\end{pmatrix}\cdot\chi_k(n,m;\alpha)$$

$$-\mathbf{g}^{-1}(n+1,m;\alpha)\cdot\boldsymbol{\theta}\cdot\chi^1_k(n,m;0)\,, \tag{C.4}$$

cf. also eqs. (A.26) and (A.29) with $\hat{\mathbf{g}}(\alpha)\equiv\mathbf{g}(n+1,m;\alpha)$.

Eliminating $\chi^1_k(n,m;0)\equiv(\phi^1_k(0),\psi^1_k(0))$ from (C.4) with the relation

$$\chi^1_k(n,m;0)=(k+\beta)\mathbf{g}(n,m;\beta)\cdot\chi_k(n,m;\beta)\,, \tag{C.5}$$

cf. (A.15) (A.16) and (A.26), we obtain the expression

$$(p^*-k)\chi_k(n+1,m;\alpha)=\begin{pmatrix}\theta(p+\alpha) & 0\\ 0 & p^*+\alpha\end{pmatrix}\cdot\chi_k(n,m;\alpha)$$

$$-(k+\beta)\mathbf{g}^{-1}(n+1,m;\alpha)\cdot\boldsymbol{\theta}\cdot\mathbf{g}(n,m;\beta)\cdot\chi_k(n,m;\beta)\,. \tag{C.6}$$

In a similar way one has

$$(q^*-k)\chi_k(n,m+1;\alpha)=\begin{pmatrix}\theta'(q+\alpha) & 0\\ 0 & q^*+\alpha\end{pmatrix}\cdot\chi_k(n,m;\alpha)$$

$$-(k+\beta)\mathbf{g}^{-1}(n,m+1;\alpha)\cdot\boldsymbol{\theta}'\cdot\mathbf{g}(n,m;\beta)\cdot\chi_k(n,m;\beta)\,. \tag{C.7}$$

The combinations $\mathbf{g}^{-1}(n+1,m;\alpha)\cdot\boldsymbol{\theta}\cdot\mathbf{g}(n,m;\beta)$ and $\mathbf{g}^{-1}(n,m+1;\alpha)\cdot\boldsymbol{\theta}'\cdot\mathbf{g}(n,m;\beta)$ in eqs. (C.6) and (C.7) are given in (A.27) and (A.28).

For special choices of α, β, θ, θ', p and q we find from (C.6) and (C.7) the Lax representations of the ddIHSC, the ddCSG and also of the ddNLS.

(i) In the case $\alpha=\beta=0$, $p=-p^*$, $\theta=-1$, $\theta'q=q^*$, eq. (C.6) can be rewritten in the form, using also (A.27),

$$(p+k)\chi_k(n+1,m;0)=(p\mathbb{1}-k\mathbf{S}(n,m))\cdot\chi_k(n,m;0)\,, \tag{C.8}$$

where the spin matrix $\mathbf{S}$ is given by

$$\mathbf{S}(n,m)=\begin{pmatrix}S^z(n,m) & S^+(n,m)\\ S^-(n,m) & -S^z(n,m)\end{pmatrix},\quad \mathbf{S}=\mathbf{S}^\dagger,\ \mathbf{S}^2=\mathbb{1}\,, \tag{C.9}$$

and where $S^+(n,m)$, $S^-(n,m)=S^+(n,m)^*$, and $S^z(n,m)$ are given in (6.7). Eq. (C.7) can be rewritten introducing another spin vector $\boldsymbol{U}$, defined by (B.11), in terms of which we have, cf. (A.28),

$$(q^* - k)\chi_k(n, m+1; 0) = \left(q^* \mathbb{1} - \tfrac{1}{2}(\theta' + 1)k\mathbb{1} - k\frac{q^*}{p^*}\mathbf{U}(n,m) \right) \cdot \chi_k(n, m; 0)\,, \tag{C.10}$$

where the alternative spin matrix $\mathbf{U}$ is given by

$$\mathbf{U}(n,m) = \begin{pmatrix} U^z(n,m) & U^+(n,m) \\ U^-(n,m) & -U^z(n,m) \end{pmatrix}, \quad \mathbf{U} = \mathbf{U}^\dagger,\, \mathbf{U}^2 = \tfrac{1}{4}|p|^2|q^* - q|^2|q|^{-4}\mathbb{1}\,. \tag{C.11}$$

From the compatibility relations of (C.8) and (C.10) we can derive eq. (6.8) by choosing $\frac{1}{4}|p|^2|q^* - q|^2|q|^{-4} = 1$, and eliminating $\mathbf{U}$ in a similar fashion as was done in appendix B. Therefore (C.8) and (C.10) can be considered to be the Lax representation for the ddIHSC.

(ii) The Lax representation for eq. (6.10), the ddCSG, is immediately obtained for $\alpha = \beta = 0$, $p = -p^*$, $\theta = -1$, $q = -q^*$, $\theta' = 1$ from (C.6) and (C.7) by inserting (A.27) and (A.28) (with the identifications (6.13)) and using (B.6) and (B.7) for $\hat{\psi} - \psi$ and $\tilde{\psi} - \psi$.

(iii) Finally, we derive the Lax representation for the ddNLS as given in eq. (6.5). From (C.5) one has immediately for $\alpha \neq \beta$

$$\begin{aligned} &(k + \alpha)\chi_k(n, m; \alpha) = (k + \beta)\mathbf{g}^{-1}(n, m; \alpha) \cdot \mathbf{g}(n, m; \beta) \cdot \chi_k(n, m; \beta)\,, \\ &\mathbf{g}^{-1}(n, m; \alpha) \cdot \mathbf{g}(n, m; \beta) = \begin{pmatrix} 1 + (\beta - \alpha)\psi(\beta, \alpha) & (\alpha - \beta)\phi(\alpha, \beta) \\ (\beta - \alpha)\phi^*(\alpha, \beta) & 1 + (\alpha - \beta)\psi(\alpha, \beta) \end{pmatrix}, \end{aligned} \tag{C.12}$$

cf. (A.17), (A.18) and (A.26). For the ddNLS with $p^* = -p$, $\theta = -1$ we derive the Lax representation in terms of a two-component vector

$$\boldsymbol{\lambda}_k \equiv (\chi_{k,1}(n, m; p), \chi_{k,2}(n, m; -p))\,. \tag{C.13}$$

Using (C.12) to eliminate $\chi_{k,1}(n, m; -p)$ and $\chi_{k,2}(n, m; p)$ one derives from (C.6) the relation with $\phi(n,m) \equiv 2p\phi(n, m; -p, p)$

$$\begin{aligned} &(p^* - k)\boldsymbol{\lambda}_k(n+1, m) = \mathscr{L}_k(n,m) \cdot \boldsymbol{\lambda}_k(n,m)\,, \\ &\mathscr{L}_k(n,m) = \begin{pmatrix} (p-k)\theta & (p-k)\phi(n+1, m) \\ (p+k)\theta\phi^*(n+1, m) & p^* - k \end{pmatrix}, \end{aligned} \tag{C.14}$$

where use has been made of eq. (A.27).

From (C.7), using also (A.28) and (B.1), (B.3) one can also derive the other member of the Lax representation. The result is

$$\begin{aligned} &(q^* - k)\boldsymbol{\lambda}_k(n, m+1) = \mathscr{M}_k(n,m)\boldsymbol{\lambda}_k(n,m)\,, \\ &\mathscr{M}_k(n,m) = \begin{pmatrix} M_{k,11}(n,m) & M_{k,12}(n,m) \\ M_{k,21}(n,m) & M_{k,22}(n,m) \end{pmatrix}, \end{aligned} \tag{C.15}$$

where

$$\begin{aligned} M_{k,11}(n,m) &\equiv \frac{\theta'}{2p}\left((p-k)(q+p) + (p+k)(q-p)R(n,m)\right), \\ M_{k,12}(n,m) &\equiv \frac{k-p}{2p}\left((q^*-p)\phi(n, m+1) - \theta'(q-p)\phi(n,m)R(n,m)\right), \\ M_{k,21}(n,m) &\equiv \frac{k+p}{2p}\left(\theta'(q+p)\phi^*(n, m+1) - (q^*+p)\phi^*(n,m)R(n,m)\right), \\ M_{k,22}(n,m) &\equiv \frac{1}{2p}\left((p+k)(q^*-p) + (p-k)(q^*+p)R(n,m)\right), \end{aligned} \tag{C.16}$$

with

$$\begin{aligned} R(n,m) &\equiv \frac{1 - 2p\tilde{\psi}(-p, p)}{1 - 2p\psi(-p, p)} \\ &= \frac{\theta(q^*-p)\phi(n, m+1) + \theta'(q+p)\phi(n+1, m)}{\theta\theta'(q-p)\phi(n,m) + (q^*-p^*)\phi(n+1, m+1)}\,. \end{aligned} \tag{C.17}$$

Appendix D

To derive the lattice sine-Gordon equation (6.22) we start from the integral equation (3.1), in combination with eq. (6.20) for the factor $\rho_k(n,m)$. Introducing the notations

$$\begin{aligned} &v_{k,1}(n,m) \equiv u_k^0(n, m; 0), \quad v_{k,2}(n,m) \equiv u_k^1(n, m; 0)\,, \\ &v(n,m) \equiv u^{1,0}(n, m; 0, 0) = \mathrm{i}\int_C \mathrm{d}\lambda(k) u_k^0(n, m; 0)\,, \end{aligned} \tag{D.1}$$

we have from (3.5) with $i = 0$

$$(p-k)v_{k,1}(n+1, m) = pv_{k,1}(n,m) + (1 - v(n+1, m))v_{k,2}(n,m), \tag{D.2}$$

and the inverse relation is obtained by replacing $p \rightarrow -p$, $(n+1, m) \rightarrow (n,m)$, $(n,m) \rightarrow (n+1, m)$, i.e.

$$(p+k)v_{k,1}(n,m) = pv_{k,1}(n+1, m) - (1 - v(n,m))v_{k,2}(n+1, m)\,. \tag{D.3}$$

From (6.20) and (3.1) we have furthermore

$$
\begin{aligned}
&(q-k^{-1})u_k^i(n,m+1;\alpha)+\mathrm{i}\rho_k(n,m)\int_C \mathrm{d}\lambda(l)\,\frac{(q-l^{-1})u_l^i(n,m+1;\alpha)}{k+l}\\
&\quad=(q+k^{-1})\,\frac{\rho_k(n,m)k^i}{k+\alpha}-u^{0,i}(n,m+1;\alpha,0)\,\frac{1}{k}\,\rho_k(n,m)\,,
\end{aligned}
\tag{D.4}
$$

cf. (3.2), leading to

$$
\begin{aligned}
(q-k^{-1})u_k^i(n,m+1;\alpha)&=qu_k^{i-1}(n,m;\alpha)+u_k^{i-1}(n,m;\alpha)\\
&\quad-u^{0,i}(n,m+1;\alpha,0)u_k^0(n,m;0)\,.
\end{aligned}
\tag{D.5}
$$

For $\alpha=0$ and $i=1$, eq. (D.5) yields the relation

$$
(q-k^{-1})v_{k,2}(n,m+1)=qv_{k,2}(n,m)+(1-v(n,m+1))v_{k,1}(n,m)\,,
\tag{D.6}
$$

and the inverse relation is obtained by replacing $q\to-q$, $(n,m+1)\to(n,m)$, $(n,m)\to(n,m+1)$, i.e.

$$
(q+k^{-1})v_{k,2}(n,m)=qv_{k,2}(n,m+1)-(1-v(n,m))v_{k,1}(n,m+1)\,.
\tag{D.7}
$$

Eqs. (D.2), (D.3) and (D.6), (D.7), respectively, provide the two members of the following Lax representation in terms of the 2-component vector $\boldsymbol{v}_k\equiv(v_{k,1},v_{k,2})$, namely

$$
(p-k)\boldsymbol{v}_k(n+1,m)=\mathscr{L}_k(n,m)\cdot\boldsymbol{v}_k(n,m)\,,
\tag{D.8a}
$$

$$
\left(q-\frac{1}{k}\right)\boldsymbol{v}_k(n,m+1)=\mathscr{M}_k(n,m)\cdot\boldsymbol{v}_k(n,m)\,,
\tag{D.8b}
$$

where

$$
\mathscr{L}_k(n,m)=\begin{pmatrix} p & 1-v(n+1,m)\\ k^2(1-v(n,m))^{-1} & p\,\dfrac{1-v(n+1,m)}{1-v(n,m)}\end{pmatrix},
\tag{D.9a}
$$

$$
\mathscr{M}_k(n,m)=\begin{pmatrix} q\,\dfrac{1-v(n,m+1)}{1-v(n,m)} & \dfrac{1}{k^2}(1-v(n,m))^{-1}\\ 1-v(n,m+1) & q\end{pmatrix}.
\tag{D.9b}
$$

Substituting

$$
-1+v(n,m)=\exp 2\mathrm{i}\,w(n,m)\,,
\tag{D.10}
$$

cf. (6.21), the compatibility relation of (D.8a) and (D.8b), with (D.9), yields the lattice sine-Gordon equation (6.22) in terms of $w(n,m)$.

Remark

From (D.8) and (D.9) one can also derive a modified lattice equation in terms of the wave functions. Defining

$$
z_k(n,m)\equiv-\mathrm{e}^{-2\mathrm{i}w(n,m)}v_{k,1}(n,m)/v_{k,2}(n,m)\,,
\tag{D.11}
$$

we have from (D.8a) and (D.9a) the relation

$$
\begin{aligned}
&z_k(n+1,m)[k^2z_k(n,m)+p\,\mathrm{e}^{-2\mathrm{i}(w(n,m)-w(n+1,m))}]\\
&\quad=1+pz_k(n,m)\,\mathrm{e}^{2\mathrm{i}(w(n,m)-w(n+1,m))}\,,
\end{aligned}
\tag{D.12}
$$

and from (D.8b) and (D.9b) the relation

$$
\begin{aligned}
&z_k(n,m+1)[q+z_k(n,m)\,\mathrm{e}^{2\mathrm{i}(w(n,m+1)+w(n,m))}]\\
&\quad=qz_k(n,m)+\frac{1}{k^2}\,\mathrm{e}^{-2\mathrm{i}(w(n,m+1)+w(n,m))}\,.
\end{aligned}
\tag{D.13}
$$

From (D.12) we can solve $A(n,m)\equiv\exp 2\mathrm{i}(w(n+1,m)-w(n,m))$ in terms of z_k, and similarly from (D.13) $B(n,m)\equiv\exp 2\mathrm{i}(w(n,m+1)+w(n,m))$, and then use the relation

$$
A(n,m+1)B(n,m)=B(n+1,m)A(n,m)^{-1}\,,
\tag{D.14}
$$

in order to find the modified equation in terms of $z_k(n,m)$.

Appendix E

In this appendix we present some details on the continuum limit of the dNLS, the dIHSC, and the dCSG, as given by eqs. (6.27), (6.30) and (6.33).

1) We first consider the continuum limit of the dNLS in the two special cases: a) $f=1$, b) $f=-i$.

a) In the case $f=1$, we can take a continuum limit with $p\to\infty$, $n\to\infty$, $2n/p\to-\mathrm{i}x$. From (6.25), using also the time scaling $4t/p^2\to-\tau$, it is then clear that

$$
\begin{aligned}
\rho_k(n,t)&=\exp\left[n\ln\frac{p-k}{p+k}+\frac{4\mathrm{i}tk^2}{p^2-k^2}\right]\rho_k(0,0)\\
&\to\rho_k(x,\tau)=\exp[\mathrm{i}(kx-k^2\tau)]\rho_k(0,0)\,.
\end{aligned}
\tag{E.1}
$$

The function $\phi_k(x,\tau)$ defined by

$$
-p\phi_k(n,t;-p)\to\phi_k(x,\tau)
\tag{E.2}
$$

is a solution of the integral equation

$$
\phi_k(x,\tau)+\int_C\mathrm{d}\lambda(l)\int_{C^*}\mathrm{d}\lambda^*(l')\,\frac{\rho_k(x,\tau)\rho_{l'}^*(x,\tau)}{(k-l')(l'-l)}\,\phi_l(x,\tau)=\rho_k(x,\tau)\,,
\tag{E.3}
$$

with $\rho_k(x,\tau)$ given by (E.1). The function $\phi(x,\tau)$ defined by

$$-\tfrac{1}{2}p\phi(n,t)\rightarrow\phi(x,\tau) \tag{E.4}$$

can be found from the relation

$$\phi(x,\tau)=\int\limits_C \mathrm{d}\lambda(k)\phi_k(x,\tau)\,, \tag{E.5}$$

as follows from (6.24) and (6.26).

The PDE for $\phi(x,\tau)$ can be inferred from (6.27) with $f=1$ using the relation

$$-\tfrac{1}{2}p\phi(n\pm 1,t)\rightarrow\phi(x,\tau)\pm\frac{2\mathrm{i}}{p}\partial_x\phi(x,\tau)-\frac{2}{p^2}\partial_x^2\phi(x,\tau)\mp\frac{4}{3}\frac{\mathrm{i}}{p^3}\partial_x^3\phi(x,\tau)\ldots, \tag{E.6}$$

in combination with (E.4) and $4t/p^2\rightarrow-\tau$. The result is

$$(\mathrm{i}\partial_\tau+\partial_x^2)\phi(x,\tau)=-2|\phi(x,\tau)|^2\phi(x,\tau)\,, \tag{E.7}$$

which is the NLS, cf. also eq. (4.24) of ref. 5 with $\lambda_2=1$, $\lambda_3=0$.

b) In the case $f=-i$, we can take a continuum limit with $r\equiv n+2t\rightarrow\infty$, $p\rightarrow\infty$, $2r/p\rightarrow-\mathrm{i}x$. From (6.25), using also the time scaling $(8/3)\mathrm{i}t/p^3\rightarrow-\tau$, we have

$$\begin{aligned}\rho_k(n,t)&=\exp\left[(r-2t)\ln\frac{p-k}{p+k}-\frac{4pkt}{p^2-k^2}\right]\rho_k(0,0)\\&\rightarrow\rho_k(x,\tau)=\exp[\mathrm{i}(kx-k^3\tau)]\rho_k(0,0)\,.\end{aligned} \tag{E.8}$$

Defining $\phi_k(x,\tau)$ and $\phi(x,\tau)$ by (E.2) and (E.4), it is clear that $\phi_k(x,\tau)$ satisfies (E.3) with $\rho_k(x,\tau)$ given by (E.8).

The PDE for $\phi(x,\tau)$ can be inferred from (6.27) with $f=-i$. Using the relation

$$-\tfrac{1}{2}p\partial_t\phi(n,t)\rightarrow-(8/3)\mathrm{i}p^{-3}\partial_\tau\phi(x,\tau)+4\mathrm{i}p^{-1}\partial_x\phi(x,\tau)\,, \tag{E.9}$$

which follows from (E.4) with $n=r-2t$, $2r/p\rightarrow-\mathrm{i}x$, and using also (E.6) and (6.27) with $f=-i$, and again the time scaling $(8/3)\mathrm{i}t/p^3\rightarrow-\tau$, we obtain

$$(\partial_\tau-\partial_x^3)\phi(x,\tau)=6|\phi(x,\tau)|^2\partial_x\phi(x,\tau)\,, \tag{E.10}$$

which is the complex modified Korteweg–de Vries equation, cf. eq. (4.24) of ref. 5 with $\lambda_2=0$, $\lambda_3=1$. (Note that in eq. (4.24) of ref. 5 for $\lambda_3\neq0$, the terms with λ_2 can be eliminated after an obvious transformation.)

2) For the dIHSC with $f=1$, we again take the continuum limit $p\rightarrow\infty$, $n\rightarrow\infty$, $2n/p\rightarrow-\mathrm{i}x$, and use the time scaling $4t/p^2\rightarrow-\tau$. We have

$$\phi_k(n,t;0)\rightarrow\phi_k(x,\tau),\quad \phi(n,t;0,0)\rightarrow\phi(x,\tau)\,, \tag{E.11}$$

in which $\phi_k(x,\tau)$ satisfies the integral equation

$$\phi_k(x,\tau)+\int\limits_C \mathrm{d}\lambda(l)\int\limits_{C^*}\mathrm{d}\lambda^*(l')\frac{\rho_k(x,\tau)\rho_{l'}^*(x,\tau)}{(k-l')(l'-l)}\phi_l(x,\tau)=\frac{\rho_k(x,\tau)}{k}\,, \tag{E.12}$$

with $\rho_k(x,\tau)$ given by (E.1), and

$$\phi(x,\tau)=\int\limits_C \mathrm{d}\lambda(k)\phi_k(x,\tau)/k\,. \tag{E.13}$$

Furthermore, cf. (6.29),

$$S^+(n,t)=p\phi(n+1,t;0,0)-p\phi(n,t;0,0)\rightarrow2\mathrm{i}\partial_x\phi(x,\tau)\,. \tag{E.14}$$

The PDE for $\boldsymbol{S}(x,\tau)$, defined by (6.28) and $\boldsymbol{S}(n,t)\rightarrow\boldsymbol{S}(x,\tau)$ can be derived from (6.30), using

$$\boldsymbol{S}(n+1,t)+\boldsymbol{S}(n-1,t)-2\boldsymbol{S}(n,t)\rightarrow-\frac{4}{p^2}\partial_x^2\boldsymbol{S}(x,\tau) \tag{E.15}$$

and the time scaling $4t/p^2\rightarrow-\tau$. We find

$$\partial_\tau\boldsymbol{S}(x,\tau)=\boldsymbol{S}(x,\tau)\times\partial_x^2\boldsymbol{S}(x,\tau)\,, \tag{E.16}$$

which is the IHSC, cf. eq. (5.22) of ref. 5 with $\lambda_2=1$, $\lambda_3=0$.

3) For the dCSG we can take a continuum limit with $n\rightarrow\infty$, $p\rightarrow\infty$, $2n/p\rightarrow-\mathrm{i}x$, and the time scaling $-pt\rightarrow-\mathrm{i}\tau$. We have, cf. (6.31),

$$\begin{aligned}\rho_k(n,t)&=\exp\left[n\ln\frac{p-k}{p+k}-\frac{pt}{k}\right]\rho_k(0,0)\\&\rightarrow\rho_k(x,\tau)=\exp[\mathrm{i}(kx-k^{-1}\tau)]\rho_k(0,0)\,.\end{aligned} \tag{E.17}$$

Furthermore,

$$\phi_k(n,t;0)\rightarrow\phi_k(x,\tau),\quad \phi(x,t;0,0)\rightarrow\phi(x,\tau)\,, \tag{E.18}$$

where $\phi_k(x,\tau)$ satisfies (E.12) with $\rho_k(x,\tau)$ given by (E.17), and $\phi(x,\tau)$ can be obtained using (E.13). From (6.32) we have

$$S(n,t)-S(n-1,t)\rightarrow\mathrm{i}\partial_x\phi(x,\tau)\,, \tag{E.19}$$

and eq. (6.33) with eq. (E.19) and the time scaling $-pt\rightarrow-\mathrm{i}\tau$ leads to the complex sine-Gordon equation

$$\partial_x\partial_\tau\phi(x,\tau)=\phi(x,\tau)\{1-4|\partial_x\phi(x,\tau)|^2\}^{1/2}\,, \tag{E.20}$$

cf. eq. (6.10) of ref. 5.

References

1) M.J. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform (SIAM, Philadelphia, 1981).
2) R.K. Bullough and P.J. Caudrey, Selected Topics in Current Physics **17** (Springer, Berlin, 1980).
3) F. Calogero and A. Degasperis, Spectral Transform and Solitons, vol. 1 (North-Holland, Amsterdam, 1982).
4) A.S. Fokas and M.J. Ablowitz, Phys. Rev. Lett. **47** (1981) 1096.
5) F.W. Nijhoff, G.R.W. Quispel, J. van der Linden and H.W. Capel, Physica **119A** (1983) 101.
6) G.R.W. Quispel, F.W. Nijhoff and H.W. Capel, Phys. Lett. **91A** (1982) 143.
7) F.W. Nijhoff, H.W. Capel, G.R.W. Quispel and J. van der Linden, Phys. Lett. **93A** (1983) 455.
8) A.S. Fokas and M.J. Ablowitz, INS preprint **28** (1982).
9) F.W. Nijhoff and H.W. Capel, Phys. Lett. **91A** (1982) 431.
10) G.R.W. Quispel, F.W. Nijhoff, H.W. Capel and J. van der Linden, Physica A to be published.
11) M.J. Ablowitz and J.F. Ladik, Stud. Appl. Math. **55** (1976) 213; **57** (1977) 1.
12) R. Hirota, J. Phys. Soc. Japan **43** (1977) 1424, 2074, 2079; **45** (1978) 321; **46** (1977) 312.
13) E. Date, M. Jimbo and T. Miwa, J. Phys. Soc. Japan **51** (1982) 4116, 4125; **52** (1983) 388, 761, 766.
14) M. Jimbo and T. Miwa, Physica **2D** (1981) 407.
15) L. Bianchi, Vorlesungen über Differenzialgeometrie (Teubner, Leipzig, 1899) p. 461.
16) H.D. Wahlquist and F.B. Estabrook, Phys. Rev. Lett. **31** (1973) 1386.
17) G.L. Lamb Jr., J. Math. Phys. **15** (1974) 2157.
18) G.L. Lamb Jr., in: Bäcklund Transformations, R.M. Miura, ed., Lect. Notes in Math. **515** (Springer, Berlin, 1976) p. 69.
19) M. Boiti, C. Laddomada and F. Pempinelli, Phys. Lett. **83A** (1981) 188.
20) B.G. Konopelchenko, Phys. Lett. **87A** (1982) 445.
21) H. Steudel, Ann. Phys. Lpz. **32** (1975) 205, 445.
22) D. Levi and R. Benguria, Proc. Natl. Acad. Sci. USA **77** (1980) 5025.
23) D. Levi, J. Phys. **A14** (1981) 1083.
24) F.W. Nijhoff, G.R.W. Quispel and H.W. Capel, Phys. Lett. **97A** (1983) 125.
25) F.W. Nijhoff, J. van der Linden, G.R.W. Quispel and H.W. Capel, Phys. Let. **89A** (1982) 106.
26) M.J. Ablowitz and A.S. Fokas, INS preprint **25** (1983).
27) M. Toda, Phys. Rep. **18C** (1975) 1.
28) H. Flaschka, Progr. Theor. Phys. **51** (1974) 703.
29) S.V. Manakov, Zh. Eksp, Teor. Fiz. **67** (1974) 543.
30) M. Kac and P. van Moerbeke, Adv. Math. **16** (1975) 160; Proc. Natl. Acad. Sci. USA **72** (1975) 1627, 2879.
31) M.J. Ablowitz and J.F. Ladik, J. Math. Phys. **17** (1976) 1011.
32) S.J. Orfanidis, Phys. Rev. **D21** (1980) 1507.
33) F.W. Nijhoff, G.R.W. Quispel and H.W. Capel, Phys. Lett. **95A** (1983) 273.
34) L.D. Faddeev, Lectures delivered at Les Houches Summer School, preprint CEN-Saclay S.Ph.T/82/76.
35) Y. Ishimori, J. Phys. Soc. Japan **51** (1982) 3417.