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BÄCKLUND TRANSFORMATIONS AND SINGULAR INTEGRAL EQUATIONS

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A systematic method for deriving Bäcklund transformations for singular (linear) integral equations is presented. The method leads in a natural way to the Bäcklund transformations for the corresponding (integrable) nonlinear partial differential equations. By repeated use of the method a hierarchy of singular integral equations and Bäcklund transformations can be derived, corresponding to so-called multi-modified partial differential equations. Specific examples include the Korteweg–de Vries equation and the first, second, and third modified Korteweg–de Vries equation; the Nonlinear Schrödinger equation and the Anisotropic Heisenberg Spin Chain; the sine-Gordon equation and the modified sine-Gordon equation.

1. Introduction

Bäcklund transformations^{1}) for integrable nonlinear partial differential equations (PDE's) were discovered in the investigation of the sine-Gordon equation in the context of differential geometry, a hundred years ago. For a review, see ref. 2. In the past decade many new Bäcklund transformations (BT's) for PDE's were discovered$^{3-14}$). In particular in refs. 5–7 BT's were derived from symmetry properties of the Lax representation. More recently BT's have been investigated using a singular transformation of the reflection coefficient in the Inverse Scattering Transform$^{8-11}$).

Very recently a connection between the Korteweg–de Vries equation (KdV) and a linear singular integral equation with arbitrary measure and contour was discovered by Fokas and Ablowitz^{15}). Extending their treatment we have investigated the singular integral equations corresponding to the Nonlinear Schrödinger equation (NLS), the Isotropic Heisenberg Spin Chain, the modified Korteweg–de Vries equation, the sine-Gordon equation, the Boussinesq equation, etc.$^{16-19}$).

In the present paper we give a systematic method to derive BT's connecting two solutions of a given integral equation related by a singular transformation of the measure, as introduced in ref. 20, see also ref. 21. From these BT's for the integral equations, it is straightforward to derive the BT's for the corresponding PDE's

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and also to derive new singular integral equations corresponding to so-called modified PDE's. Starting from the integral equation for the modified PDE, the procedure can be repeated again, in principle, to obtain multi-modified PDE's. (For the reflection coefficient in the Inverse Scattering Transform similar transformations have been used, but our treatment is more general, due to the fact that the integral equation contains an arbitrary measure and contour.) A different treatment in which the Bäcklund transformation is used to obtain the first and second modified KdV equation has been given within the context of Hirota's method[22]).

The outline of the present paper is as follows. In section 2 we derive the BT for the integral equation corresponding to the class of PDE's containing the KdV equation and we derive the singular integral equation for the modified Korteweg—de Vries equation (MKdV). In section 3 the BT for the MKdV, the modified modified KdV equation and the modified modified modified KdV equation are discussed. In section 4 we treat the class of the NLS equation and the integral equation and BT for the Anisotropic Heisenberg Spin Chain (AHSC). In section 5 the real and complex versions of the modified sine-Gordon equation are derived from the BT's for the real and complex sine-Gordon equation. Finally, in section 6 it is shown how we can derive BT's for the wave functions in the spectral problem, leading e.g. to an alternative form of the BT for the AHSC.

2. The KdV class

In this section we start from the integral equation defining the KdV class and derive a (matrix) Bäcklund transformation using a singular transformation of the measure in the integral equation. From the (matrix) BT we also obtain a (matrix) modified PDE with the associated singular integral equation and linear problem. The integral equation for the (matrix) modified PDE turns out to be a generalization of the integral equation of type II for the MKdV class proposed in ref. 18.

2.1. *Integral equation and constitutive relations*

The integral equation defining the KdV class is

$$\boldsymbol{u}_k(x,t) + \mathrm{i}\rho_k(x,t)\int_C \mathrm{d}\lambda(l)\,\frac{\boldsymbol{u}_l(x,t)}{k+l} = \rho_k(x,t)\boldsymbol{c}_k, \tag{2.1}$$

from which the vector function $\boldsymbol{u}_k(x,t)$ with components $u_k^{(n)}$, n integer, should be solved as a function of the complex variable k. In eq. (2.1) $\boldsymbol{c}_k$ is a vector with components $(\boldsymbol{c}_k)_n = 1/k^n$, n integer, C is an arbitrary contour in the complex

k-plane and $\mathrm{d}\lambda(l)$ is an arbitrary measure. $\rho_k(x, t)$ is a plane-wave factor satisfying the linear differential equations

$$-\,\mathrm{i}\partial_x\rho_k(x,t) = k\rho_k(x,t), \quad \mathrm{i}\partial_t\rho_k(x,t) = \omega(k)\rho_k(x,t)\,,$$
$$\omega(k) = \sum_r \lambda_r k^r, \quad \lambda_r = 0, \quad \text{for } r \text{ even}\,, \tag{2.2}$$

$\omega(k)$ being the dispersion. The measure and the contour are to be chosen in such a way that the solution $\boldsymbol{u}_k(x, t)$ of eq. (2.1) is unique, see ref. 18, cf. also ref. 15.

From eqs. (2.1) and (2.2) one can derive the constitutive relations[18])

$$2\mathrm{i}\partial_x\boldsymbol{u}_k = -\,k\boldsymbol{u}_k - \mathbf{J}^{\mathrm{T}}\cdot\boldsymbol{u}_k + \mathrm{i}\mathbf{U}\cdot\mathbf{O}\cdot\boldsymbol{u}_k\,, \tag{2.3}$$

$$2\mathrm{i}\partial_t\,\boldsymbol{u}_k = \omega(k)\boldsymbol{u}_k + \omega(\mathbf{J}^{\mathrm{T}})\cdot\boldsymbol{u}_k - \mathrm{i}\mathbf{U}\cdot\mathbf{R}\cdot\boldsymbol{u}_k\,, \tag{2.4}$$

$$k^p\boldsymbol{u}_k = \mathbf{J}^{\mathrm{T}\,p}\cdot\boldsymbol{u}_k + \mathrm{i}\mathbf{U}\cdot\mathbf{R}_p\cdot\boldsymbol{u}_k \qquad (p\ \text{even})\,, \tag{2.5}$$

in which the (symmetric) matrix $\mathbf{U}$ can be obtained from the dyadic $\boldsymbol{u}_k\boldsymbol{c}_k$ by an integration over the same contour that occurs in (2.1),

$$\mathbf{U} \equiv \int_C \mathrm{d}\lambda(k)\boldsymbol{u}_k\boldsymbol{c}_k\,, \tag{2.6}$$

and in which we have used the following notations:

$$(\mathbf{J})_{n,m} \equiv \delta_{m,n+1}, \quad (\mathbf{J}^{\mathrm{T}})_{n,m} \equiv \delta_{n,m+1}, \qquad (\mathbf{O})_{n,m} \equiv \delta_{n,0}\delta_{m,0}\,, \tag{2.7}$$

$$\mathbf{R}_r \equiv (\operatorname{sgn} r)\sum_{j=0}^{|r|-1}\mathbf{J}^{j+r/2-|r|/2}\cdot\mathbf{O}\cdot(-\,\mathbf{J}^{\mathrm{T}})^{-j-1+r/2+|r|/2}, \qquad \mathbf{R} \equiv \sum_{r\ \text{odd}}\lambda_r\mathbf{R}_r\,. \tag{2.8}$$

2.2. *Matrix PDE*

In this subsection we recapitulate some results from ref. 18. Differentiating (2.3) with respect to x and using (2.5) we can derive

$$(k + \mathrm{i}\partial_x)\,\mathrm{i}\partial_x\boldsymbol{u}_k = -\,(\partial_x\mathbf{U})\cdot\mathbf{O}\cdot\boldsymbol{u}_k\,. \tag{2.9}$$

Integrating (2.3) over the contour C, cf. (2.6), we obtain

$$2\mathrm{i}\partial_x\mathbf{U} = -\,\mathbf{U}\cdot\mathbf{J} - \mathbf{J}^{\mathrm{T}}\cdot\mathbf{U} + \mathrm{i}\mathbf{U}\cdot\mathbf{O}\cdot\mathbf{U}\,, \tag{2.10}$$

which will be used in subsection 2.4.

From (2.3)–(2.5) one may derive various PDE's for different choices of $\omega(k)$. Taking as an example $\omega(k) = k^3$, we have[18])

$$(\partial_t - \partial_x^3)\boldsymbol{u}_k = -\,3(\partial_x\mathbf{U})\cdot\mathbf{O}\cdot\partial_x\boldsymbol{u}_k\,, \tag{2.11}$$

which upon integration over the contour C, cf. eq. (2.6), yields the following matrix PDE:

$$(\partial_t - \partial_x^3)\mathbf{U} = -3(\partial_x \mathbf{U})\cdot \mathbf{O} \cdot (\partial_x \mathbf{U})\,. \tag{2.12}$$

The (0, 0) (1, 0) and (1, 1) elements of $\mathbf{U}$ obey respectively

$$(\partial_t - \partial_x^3)u_{0,0} = -3(\partial_x u_{0,0})^2\,, \tag{2.13}$$

$$(\partial_t - \partial_x^3)u_{1,0} = -3\frac{(\partial_x u_{1,0})\partial_x^2 u_{1,0}}{\mathrm{i} + u_{1,0}}\,, \tag{2.14}$$

$$(\partial_t - \partial_x^3)u_{1,1} = 3\frac{(\partial_x^2 u_{1,1})^2}{1 - 2\partial_x u_{1,1}}\,, \tag{2.15}$$

and the relations between the different elements are

$$\partial_x u_{0,0} = (\mathrm{i} + u_{1,0})^{-1}\partial_x^2 u_{1,0}\,, \tag{2.16}$$

$$u_{1,0} = -\mathrm{i} \pm [2\partial_x u_{1,1} - 1]^{1/2}\,. \tag{2.17}$$

Eq. (2.13) is the potential Korteweg–de Vries equation, i.e. $\partial_x u_{0,0}$ satisfies the KdV; eq. (2.14) is equivalent to the potential modified Korteweg–de Vries equation, i.e. $v = \partial_x \ln(\mathrm{i} + u_{1,0})$ satisfies $\partial_t v - \partial_x^3 v + 6v^2\partial_x v = 0$, and (2.15) is equivalent to the MKdV.

The special case $\omega(k) = k^{-1}$ will be discussed in section 5.

2.3. *Singular transformation of the measure*

We introduce the singular transformation of the measure, cf. ref. 23,

$$\mathrm{d}\lambda(k) \to \mathrm{d}\tilde{\lambda}(k) = \frac{p-k}{p+k}\,\mathrm{d}\lambda(k)\,, \tag{2.18}$$

where p is a complex parameter, and consider the corresponding solution $\tilde{\boldsymbol{u}}_k\,(x, t)$ of the integral equation (2.1) with $\mathrm{d}\lambda(l)$ replaced by $\mathrm{d}\tilde{\lambda}(l)$, i.e.

$$\tilde{\boldsymbol{u}}_k + \mathrm{i}\rho_k \int_C \mathrm{d}\tilde{\lambda}(l)\,\frac{\tilde{\boldsymbol{u}}_l}{k+l} = \rho_k \boldsymbol{c}_k\,, \tag{2.19}$$

$$\tilde{\mathbf{U}} \equiv \int_C \mathrm{d}\tilde{\lambda}(l)\tilde{\boldsymbol{u}}_l\,\boldsymbol{c}_l\,. \tag{2.20}$$

In eq. (2.18) it is understood that $\mathrm{d}\tilde{\lambda}(k)$ is such that the solution of the integral equation (2.19) is also unique, and the contour should not pass through p and $-p$. In appendix A it will be argued that (2.18) increases the number of solitons by one, and another way of getting the BT will be presented.

Starting from (2.18) one can derive a relation between $\tilde{\mathbf{U}}$ and $\mathbf{U}$ which is a matrix generalization of the well-known BT for the KdV. In fact, using (2.18), (2.19) and the decomposition into partial fractions,

$$\frac{p-k}{(p+l)(k+l)}=\frac{1}{k+l}-\frac{1}{p+l}, \tag{2.21}$$

we obtain

$$\begin{aligned}(p-k)\tilde{\boldsymbol{u}}_k+\mathrm{i}\rho_k\int\limits_C \mathrm{d}\lambda(l)\frac{(p-l)\tilde{\boldsymbol{u}}_l}{k+l}&=(p-k)\rho_k\boldsymbol{c}_k+\mathrm{i}\rho_k\int\limits_C \mathrm{d}\lambda(l)\frac{(p-l)\tilde{\boldsymbol{u}}_l}{p+l}\\ &=(p-k)\rho_k\boldsymbol{c}_k+\mathrm{i}\rho_k\tilde{\mathbf{U}}\cdot\mathbf{O}\cdot\boldsymbol{c}_k\,. \end{aligned}\tag{2.22}$$

Taking into account that the homogeneous integral equation (2.1) has only the zero solution, we obtain the relation

$$(p-k)\tilde{\boldsymbol{u}}_k=p\boldsymbol{u}_k-\mathbf{J}^{\mathrm{T}}\cdot\boldsymbol{u}_k+\mathrm{i}\tilde{\mathbf{U}}\cdot\mathbf{O}\cdot\boldsymbol{u}_k\,, \tag{2.23}$$

which may be regarded as the basic relation for the BT, and which will be used in the following subsections. Note that from (2.23) $\tilde{\boldsymbol{u}}_k$ can be expressed in terms of the vector $\boldsymbol{u}_k$, which is the solution of (2.1) with the measure $\mathrm{d}\lambda(k)$, and various integrals of $\boldsymbol{u}_k$.

2.4. *Matrix Bäcklund transformation for* $\mathbf{U}$

The inverse transformation of (2.18) can be obtained interchanging $p\leftrightarrow -p$, $\mathrm{d}\lambda(k)\leftrightarrow\mathrm{d}\tilde{\lambda}(k)$. From (2.23) we thus obtain the inverse relation

$$(-p-k)\boldsymbol{u}_k=-p\tilde{\boldsymbol{u}}_k-\mathbf{J}^{\mathrm{T}}\cdot\tilde{\boldsymbol{u}}_k+\mathrm{i}\mathbf{U}\cdot\mathbf{O}\cdot\tilde{\boldsymbol{u}}_k\,, \tag{2.24}$$

cf. (2.23) with $p\leftrightarrow -p$, $\boldsymbol{u}_k\leftrightarrow\tilde{\boldsymbol{u}}_k$, $\mathbf{U}\leftrightarrow\tilde{\mathbf{U}}$. Multiplying (2.23) by the vector $\boldsymbol{c}_k$ and integrating over the contour C with the measure $\mathrm{d}\lambda(k)$, we obtain, taking into account that $(p-k)\,\mathrm{d}\lambda(k)=(p+k)\,\mathrm{d}\tilde{\lambda}(k)$,

$$p(\tilde{\mathbf{U}}-\mathbf{U})=-\tilde{\mathbf{U}}\cdot\mathbf{J}-\mathbf{J}^{\mathrm{T}}\cdot\mathbf{U}+\mathrm{i}\tilde{\mathbf{U}}\cdot\mathbf{O}\cdot\mathbf{U}\,. \tag{2.25}$$

Adding (2.25) and its inverse with $p\leftrightarrow -p$, $\mathbf{U}\leftrightarrow\tilde{\mathbf{U}}$, we find, after eliminating $\mathbf{J}$ and $\mathbf{J}^{\mathrm{T}}$ with eq. (2.10),

$$2\mathrm{i}\partial_x(\tilde{\mathbf{U}}+\mathbf{U})=2p(\tilde{\mathbf{U}}-\mathbf{U})+\mathrm{i}(\tilde{\mathbf{U}}-\mathbf{U})\cdot\mathbf{O}\cdot(\tilde{\mathbf{U}}-\mathbf{U})\,. \tag{2.26}$$

Eq. (2.26) is the spatial part of the matrix BT associated with the integral equation (2.1), and is independent of the dispersion $\omega(k)$. The time-dependent part of the matrix BT can be inferred from the matrix PDE and (2.26). In the special case

$\omega(k) = k^3$, cf. (2.12),

$$\partial_t(\tilde{\mathbf{U}} + \mathbf{U}) = \partial_x^2[-\mathrm{i}p(\tilde{\mathbf{U}} - \mathbf{U}) + \tfrac{1}{2}(\tilde{\mathbf{U}} - \mathbf{U}) \cdot \mathbf{O} \cdot (\tilde{\mathbf{U}} - \mathbf{U})] - 3(\partial_x \mathbf{U}) \cdot \mathbf{O} \cdot \partial_x \mathbf{U} - 3(\partial_x \tilde{\mathbf{U}}) \cdot \mathbf{O} \cdot \partial_x \tilde{\mathbf{U}} \,. \tag{2.27}$$

The (0, 0) element of (2.26) and (2.27) reduces to the well-known BT for the (potential) KdV[3-5]).

2.5. *Modified matrix PDE*

Introducing

$$\mathbf{U}^- \equiv \tilde{\mathbf{U}} - \mathbf{U} \,, \tag{2.28}$$

we have from (2.26)

$$\partial_x \mathbf{U} = -\tfrac{1}{2}\partial_x \mathbf{U}^- - \tfrac{1}{2}\mathrm{i}p\mathbf{U}^- + \tfrac{1}{4}\mathbf{U}^- \cdot \mathbf{O} \cdot \mathbf{U}^- \,, \tag{2.29}$$

and the PDE for the matrix $\mathbf{U}^-$ can be derived inserting (2.28) and (2.29) in eq. (2.12) and its counterpart with $\mathbf{U} \to \tilde{\mathbf{U}}$,

$$(\partial_t - \partial_x^3)\mathbf{U}^- = [(3\mathrm{i}p\mathbf{U}^- - \tfrac{3}{2}\mathbf{U}^- \cdot \mathbf{O} \cdot \mathbf{U}^-) \cdot \mathbf{O} \cdot \partial_x \mathbf{U}^-]^{\mathrm{s}} \,, \tag{2.30}$$

in which the superscript s denotes the symmetrical part of a matrix, i.e. $B^{\mathrm{s}}_{n,m} = \tfrac{1}{2}(B_{n,m} + B_{m,n})$ for an arbitrary matrix $\mathbf{B}$. Eq. (2.30) is a completely integrable matrix PDE, which we call the modified matrix PDE of (2.12). In this paper the term modified PDE will be used to denote a PDE, the solutions of which are obtained by combining a solution of another PDE and its Bäcklund transform. The relation mapping a solution of the modified PDE on a solution of the original PDE will be called a Miura transformation[24]). In the case under consideration eq. (2.29) is a matrix Miura transformation, mapping a solution $\mathbf{U}^-$ of (2.30) on a solution $\mathbf{S} \equiv \partial_x \mathbf{U}$ of the matrix KdV $(\partial_t - \partial_x^3)\mathbf{S} = -6(\mathbf{S} \cdot \mathbf{O} \cdot \partial_x \mathbf{S})^{\mathrm{s}}$. In this connection it may be noted that the Miura transformation (2.29), mapping a solution of (2.30) on the matrix KdV, remains valid if all the matrices $\mathbf{O}$ are replaced by arbitrary constant symmetric matrices $\mathbf{P}$, as can be checked by explicit calculation.

Taking the (0, 0) element of (2.30) we have immediately

$$(\partial_t - \partial_x^3)u^-_{0,0} = 3\mathrm{i}pu^-_{0,0}\partial_x u^-_{0,0} - \tfrac{3}{2}(u^-_{0,0})^2\partial_x u^-_{0,0} \,, \tag{2.31}$$

which is equivalent to the MKdV, since the term with p can be transformed away. For the (1, 1) element we have obtained the PDE

$$\begin{aligned} (\partial_t - \partial_x^3)z &= -\tfrac{1}{2}(\partial_x z)^3 + \tfrac{3}{2}p^2(\sinh^2 z)\partial_x z \,, \\ z &\equiv \operatorname{arsinh}[-\mathrm{i}p^{-1}\partial_x \ln(u^-_{1,1} - \mathrm{i}p^{-1})] \,, \end{aligned} \tag{2.32}$$

cf. appendix B for some details. Eq. (2.32) has been given in ref. 5 and is a special case of the second modified KdV, cf. eq. (3.21) and refs. 22, 25 and 26.

2.6. *Integral equation and constitutive relations for the matrix modified PDE*

In subsection 2.5 we have shown that the matrix $\mathbf{U}^- = \tilde{\mathbf{U}} - \mathbf{U}$ obeys a PDE. In this subsection we will derive a linear integral equation for this modified PDE. (This integral equation will be used as a starting point in section 3 to repeat the procedure and to derive the BT for the modified PDE, as well as the second and third modified PDE.) For this purpose we need a wave function $\boldsymbol{u}_k^-$, which upon integration yields the potential $\mathbf{U}^-$.

Defining

$$\boldsymbol{u}_k^{\pm} = (p-k)\tilde{\boldsymbol{u}}_k \pm (p+k)\boldsymbol{u}_k\,, \tag{2.33}$$

with inverse

$$\boldsymbol{u}_k = \frac{\boldsymbol{u}_k^+ - \boldsymbol{u}_k^-}{2(p+k)}, \quad \tilde{\boldsymbol{u}}_k = \frac{\boldsymbol{u}_k^+ + \boldsymbol{u}_k^-}{2(p-k)}, \tag{2.34}$$

and defining the new measure

$$\mathrm{d}\lambda_1(k) = \frac{\mathrm{d}\tilde{\lambda}(k)}{p-k} = \frac{\mathrm{d}\lambda(k)}{p+k}, \tag{2.35}$$

we have

$$\mathbf{U}^{\pm} \equiv \int_C \mathrm{d}\lambda_1(k)\, \boldsymbol{u}_k^{\pm} \boldsymbol{c}_k = \tilde{\mathbf{U}} \pm \mathbf{U}\,, \tag{2.36}$$

in agreement with (2.28) for $\mathbf{U}^-$. Two integral equations for $\boldsymbol{u}_k^+ + \epsilon \boldsymbol{u}_k^-$, $\epsilon = \pm 1$, follow quite directly from (2.1) and its counterpart with $u_k \to \tilde{u}_k$, $\mathrm{d}\lambda(k) \to \mathrm{d}\tilde{\lambda}(k)$, expressing the measures $\mathrm{d}\lambda(k)$ and $\mathrm{d}\tilde{\lambda}(k)$ in terms of $\mathrm{d}\lambda_1(k)$. The result is

$$(\boldsymbol{u}_k^+ + \epsilon \boldsymbol{u}_k^-) + \mathrm{i}(p - \epsilon k)\rho_k \int_C \mathrm{d}\lambda_1(l) \frac{\boldsymbol{u}_l^+ + \epsilon \boldsymbol{u}_l^-}{k+l} = 2(p - \epsilon k)\rho_k \boldsymbol{c}_k, \quad \epsilon = \pm 1\,. \tag{2.37}$$

Eq. (2.37) can be rewritten as

$$\boldsymbol{u}_k^- + \frac{\mathrm{i}(p^2 - k^2)}{k} \rho_k \int_C \mathrm{d}\lambda_1(l) \frac{\boldsymbol{u}_l^+}{k+l} = \frac{2(p^2-k^2)}{k} \rho_k \boldsymbol{c}_k - \frac{p}{k} \boldsymbol{u}_k^+\,, \tag{2.38}$$

$$\boldsymbol{u}_k^+ + \frac{\mathrm{i}(p^2 - k^2)}{k} \rho_k \int_C \mathrm{d}\lambda_1(l) \frac{\boldsymbol{u}_l^-}{k+l} = -\frac{p}{k} \boldsymbol{u}_k^-\,. \tag{2.39}$$

From (2.38) and (2.39) one can also obtain one single integral equation containing only $\boldsymbol{u}_k^-$ which reads

$$\boldsymbol{u}_k^- + \int\limits_C \mathrm{d}\lambda_1(l) \int\limits_C \mathrm{d}\lambda_1(l') \frac{(p^2 + kl')}{(k + l')(l' + l)} \rho_k \rho_{l'} \boldsymbol{u}_l^-$$

$$+ \rho_k \left[\mathrm{i}p \int\limits_C \mathrm{d}\lambda_1(l) \frac{\boldsymbol{u}_l^-}{l} - p^2 \int\limits_C \mathrm{d}\lambda_1(l) \int\limits_C \mathrm{d}\lambda_1(l') \frac{\rho_{l'} \boldsymbol{u}_l^-}{l'(l' + l)} \right] = -2k\rho_k \boldsymbol{c}_k \,. \tag{2.40}$$

The linear integral equation (2.40), together with eq. (2.36), provides in a direct way solutions of the matrix PDE given by (2.30).

It is also straightforward to derive the constitutive relations in terms of the vectors $\boldsymbol{u}_k^+$ and $\boldsymbol{u}_k^-$. In order to do so, one could start from the integral equations for $\boldsymbol{u}_k^+$ and $\boldsymbol{u}_k^-$ with an appropriate uniqueness condition, but it is less laborious to use the constitutive relations corresponding to eq. (2.1), cf. (2.3)–(2.5), and the Bäcklund relation (2.23), in combination with (2.33).

Multiplying (2.23) by $(p + k)$ and the inverse relation (2.24) by $(p - k)$, and adding and subtracting the result, with the use of (2.33) one obtains

$$k\boldsymbol{u}_k^+ = -2p\boldsymbol{u}_k^- - \mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{u}_k^+ + \tfrac{1}{2}\mathrm{i}\mathbf{U}^+ \cdot \mathbf{O} \cdot \boldsymbol{u}_k^+ - \tfrac{1}{2}\mathrm{i}\mathbf{U}^- \cdot \mathbf{O} \cdot \boldsymbol{u}_k^- \,, \tag{2.41}$$

$$k\boldsymbol{u}_k^- = \mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{u}_k^- + \tfrac{1}{2}\mathrm{i}\mathbf{U}^- \cdot \mathbf{O} \cdot \boldsymbol{u}_k^+ - \tfrac{1}{2}\mathrm{i}\mathbf{U}^+ \cdot \mathbf{O} \cdot \boldsymbol{u}_k^- \,, \tag{2.42}$$

which can be generalized using the algebraic relation

$$k^b \boldsymbol{u}_k = \mathbf{J}^{\mathrm{T}b} \cdot \boldsymbol{u}_k + \mathrm{i}\mathbf{U} \cdot \mathbf{R}_b \cdot \boldsymbol{u}_k \qquad (b \text{ even}) \,, \tag{2.43}$$

given in eq. (7.3) of ref. 18 and in eq. (2.5) of the present paper, and eqs. (2.33), (2.41) and (2.42). The result is

$$\begin{aligned} k^b \boldsymbol{u}_k^+ = {} & (-\mathbf{J}^{\mathrm{T}})^b \cdot \boldsymbol{u}_k^+ + \tfrac{1}{2}\mathrm{i}\mathbf{U}^+ \cdot \mathbf{R}_b \cdot \boldsymbol{u}_k^+ + \tfrac{1}{2}i(-1)^b \mathbf{U}^- \cdot \mathbf{R}_b \cdot \boldsymbol{u}_k^- \\ & - p(1 - (-1)^b)(k^{b-1} \boldsymbol{u}_k^- - \tfrac{1}{2}\mathrm{i}\mathbf{U}^- \cdot \mathbf{R}_{b-1} \cdot \boldsymbol{u}_k^+) \,, \end{aligned} \tag{2.44}$$

$$\begin{aligned} k^b \boldsymbol{u}_k^- = {} & \mathbf{J}^{\mathrm{T}b} \cdot \boldsymbol{u}_k^- + \tfrac{1}{2}\mathrm{i}(-1)^b \mathbf{U}^+ \cdot \mathbf{R}_b \cdot \boldsymbol{u}_k^- + \tfrac{1}{2}\mathrm{i}\mathbf{U}^- \cdot \mathbf{R}_b \cdot \boldsymbol{u}_k^+ \\ & - \tfrac{1}{2}\mathrm{i}p(1 - (-1)^b)\mathbf{U}^- \cdot \mathbf{R}_{b-1} \cdot \boldsymbol{u}_k^- \qquad (b \text{ integer}) \,. \end{aligned} \tag{2.45}$$

From (2.3), (2.23), (2.33) and (2.36) one finds

$$2\mathrm{i}\partial_x \boldsymbol{u}_k = \boldsymbol{u}_k^- - \mathrm{i}\mathbf{U}^- \cdot \mathbf{O} \cdot \boldsymbol{u}_k \,. \tag{2.46}$$

Multiplying (2.46) by $(\mathrm{p} + k)$, and the inverse relation with $\boldsymbol{u}_k \rightarrow \tilde{\boldsymbol{u}}_k$, $\mathbf{U}^- \rightarrow -\mathbf{U}^-$, $\boldsymbol{u}_k^- \rightarrow \boldsymbol{u}_k^-$ by $(p - k)$, and adding and subtracting one obtains

$$\mathrm{i}\partial_x \boldsymbol{u}_k^+ = p\boldsymbol{u}_k^- + \tfrac{1}{2}\mathrm{i}\mathbf{U}^- \cdot \mathbf{O} \cdot \boldsymbol{u}_k^- \,, \tag{2.47}$$

$$\mathrm{i}\partial_x \boldsymbol{u}_k^- = -ku_k^- + \tfrac{1}{2}\mathrm{i}\mathbf{U}^- \cdot \mathbf{O} \cdot \boldsymbol{u}_k^+ \,. \tag{2.48}$$

In an analogous way from (2.4) one derives the relations for the time derivatives, i.e.

$$\mathrm{i}\partial_t \boldsymbol{u}_k^{\pm} = \tfrac{1}{2}(\omega(k) + \omega(\mathbf{J}^{\mathrm{T}})) \cdot \boldsymbol{u}_k^{\pm} - \tfrac{1}{4}\mathrm{i}(\mathbf{U}^{+} \cdot \mathbf{R} \cdot \boldsymbol{u}_k^{\pm} + \mathbf{U}^{-} \cdot \mathbf{R} \cdot \boldsymbol{u}_k^{\mp})\,, \tag{2.49}$$

which in the special case $\omega(k) = k^3$ reduce to, cf. (2.11), (2.33) and (2.29),

$$(\partial_t - \partial_x^3)\boldsymbol{u}_k^{\pm} = \tfrac{3}{2}(\mathrm{i}p\mathbf{U}^{-} - \tfrac{1}{2}\mathbf{U}^{-} \cdot \mathbf{O} \cdot \mathbf{U}^{-}) \cdot \mathbf{O} \cdot \partial_x \boldsymbol{u}_k^{\pm} - \tfrac{3}{2}(\partial_x \mathbf{U}^{-}) \cdot \mathbf{O} \cdot \partial_x \boldsymbol{u}_k^{\mp}\,. \tag{2.50}$$

Eqs. (2.44), (2.45), (2.47) and (2.48), which are independent of $\omega(k)$, together with the two eqs. (2.50), in the case $\omega(k) = k^3$, form the constitutive relations corresponding to the modified matrix PDE (2.30).

Remark. For $p = 0$, eqs. (2.38), (2.39) with (2.35) are equivalent to eqs. (3.1a) and (3.1b) of ref. 18, with $v_k^{(n)} = -\tfrac{1}{2}u_k^{-(n)}/k$, $w_k^{(n)} = +\tfrac{1}{2}\mathrm{i}u_k^{+(n)}/k$, which define the integral equation of type II, describing the MKdV class. The constitutive relations (2.44), (2.45), (2.47)–(2.49) are generalizations to the case $p \neq 0$ of the relations (3.15), (3.19) and (3.25a)–(3.25d) in ref. 18. (The matrices $\mathbf{V}$ and $\mathbf{W}$ in ref. 18 correspond to $-\tfrac{1}{2}\mathbf{U}^{-}$ and $\tfrac{1}{2}\mathrm{i}\mathbf{U}^{+}$, respectively, in the special case $p = 0$.)

3. The generalized MKdV class and beyond

In the preceding section we applied a singular transformation of the measure (2.18) to the linear integral equation associated with the matrix PDE (2.14) for the KdV class. We have shown that the transformation of the measure leads in a natural way to a Bäcklund transformation for the matrix PDE, as well as to a *modified* matrix PDE (2.30) with corresponding linear integral equation (2.40) and constitutive relations (2.44), (2.45), (2.47)–(2.49). This integral equation, which is equivalent to (2.37) defines the generalized MKdV class. In the present section the procedure will be applied again, to derive the BT for the modified matrix PDE, as well as a *second modified* matrix PDE with its linear integral equation and linear problem. At the end of the section we shall apply the scheme for the third time to derive the BT for the (0, 0) element of the second modified matrix PDE, as well as the PDE for the (0, 0) element of the third modified matrix PDE.

3.1. *Bäcklund transformation for the modified matrix PDE*

We introduce the singular transformation of the measure

$$\mathrm{d}\lambda_1(k) \rightarrow \mathrm{d}\tilde{\lambda}_1(k) = \frac{q-k}{q+k}\,\mathrm{d}\lambda_1(k)\,, \tag{3.1}$$

and consider the corresponding solutions $\tilde{\boldsymbol{u}}_k^{+} + \epsilon\tilde{\boldsymbol{u}}_k^{-}$ of (2.37) with $\mathrm{d}\lambda_1(l)$ replaced by $\mathrm{d}\tilde{\lambda}_1(l)$, i.e.

$$(\tilde{\boldsymbol{u}}_k^+ + \epsilon\tilde{\boldsymbol{u}}_k^-) + \mathrm{i}(p-\epsilon k)\rho_k \int_C \mathrm{d}\tilde{\lambda}_1(l)\,\frac{\tilde{\boldsymbol{u}}_l^+ + \epsilon\tilde{\boldsymbol{u}}_l^-}{k+l} = 2(p-\epsilon k)\rho_k \boldsymbol{c}_k\,, \tag{3.2}$$

leading to

$$\tilde{\mathbf{U}}^{\pm} = \int_C \mathrm{d}\tilde{\lambda}_1(k)\,\tilde{\boldsymbol{u}}_k^{\pm}\boldsymbol{c}_k\,. \tag{3.3}$$

Eq. (3.2) has exactly the same form as eq. (2.19) with $\tilde{\boldsymbol{u}}_k \to \frac{1}{2}(\tilde{\boldsymbol{u}}_k^+ + \epsilon\tilde{\boldsymbol{u}}_k^-)$ and $\rho_k \to (p-\epsilon k)\rho_k$. This means that the BT for $\mathbf{U}^+$ and $\mathbf{U}^-$ can be obtained immediately from (2.23) for the KdV with $p \to q$. We thus have

$$(q-k)\left(\frac{\tilde{\boldsymbol{u}}_k^+ + \epsilon\tilde{\boldsymbol{u}}_k^-}{2}\right) = q\left(\frac{\boldsymbol{u}_k^+ + \epsilon\boldsymbol{u}_k^-}{2}\right) - \mathbf{J}^{\mathrm{T}}\cdot\left(\frac{\boldsymbol{u}_k^+ + \epsilon\boldsymbol{u}_k^-}{2}\right) + \mathrm{i}\left(\frac{\tilde{\mathbf{U}}^+ + \epsilon\tilde{\mathbf{U}}^-}{2}\right)\cdot\mathbf{O}\cdot\left(\frac{\boldsymbol{u}_k^+ + \epsilon\boldsymbol{u}_k^-}{2}\right), \tag{3.4}$$

which can be expressed as

$$(q-k)\tilde{\boldsymbol{u}}_k^{\mp} = q\boldsymbol{u}_k^{\mp} - \mathbf{J}^{\mathrm{T}}\cdot\boldsymbol{u}_k^{\mp} + \tfrac{1}{2}\mathrm{i}\tilde{\mathbf{U}}^+\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{\mp} + \tfrac{1}{2}\mathrm{i}\tilde{\mathbf{U}}^-\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{\pm}\,. \tag{3.5}$$

Eq. (3.5), which is independent of $\omega(k)$, may be regarded as the basic relation of the BT of the modified matrix PDE. Multiplying (3.5) by $\boldsymbol{c}_k$ and integrating over $\mathrm{d}\lambda_1(k)$, with the use of (3.1) to evaluate the left-hand side, we obtain

$$q(\tilde{\mathbf{U}}^{\mp} - \mathbf{U}^{\mp}) = -(\mathbf{J}^{\mathrm{T}}\cdot\mathbf{U}^{\mp} + \tilde{\mathbf{U}}^{\mp}\cdot\mathbf{J}) + \tfrac{1}{2}\mathrm{i}\tilde{\mathbf{U}}^+\cdot\mathbf{O}\cdot\mathbf{U}^{\mp} + \tfrac{1}{2}\mathrm{i}\tilde{\mathbf{U}}^-\cdot\mathbf{O}\cdot\mathbf{U}^{\pm}. \tag{3.6}$$

From (3.1)–(3.3), cf. (2.37) and (2.36), it is clear that the inverse Bäcklund transformation can be obtained substituting $q \leftrightarrow -q$, $\boldsymbol{u}_k^{\pm} \leftrightarrow \tilde{\boldsymbol{u}}_k^{\pm}$, $\mathbf{U}^{\pm} \leftrightarrow \tilde{\mathbf{U}}^{\pm}$ in (3.5) and (3.6). From (3.6) and the inverse relation, in combination with the expressions

$$\mathbf{U}^-\cdot\mathbf{J} + \mathbf{J}^{\mathrm{T}}\cdot\mathbf{U}^- = -2\mathrm{i}\partial_x\mathbf{U}^- + \tfrac{1}{2}\mathrm{i}\mathbf{U}^-\cdot\mathbf{O}\cdot\mathbf{U}^+ + \tfrac{1}{2}\mathrm{i}\mathbf{U}^+\cdot\mathbf{O}\cdot\mathbf{U}^-\,, \tag{3.7}$$

$$\mathbf{U}^+\cdot\mathbf{J} + \mathbf{J}^{\mathrm{T}}\cdot\mathbf{U}^+ = -2p\mathbf{U}^- + \tfrac{1}{2}\mathrm{i}\mathbf{U}^+\cdot\mathbf{O}\cdot\mathbf{U}^+ - \tfrac{1}{2}\mathrm{i}\mathbf{U}^-\cdot\mathbf{O}\cdot\mathbf{U}^-\,, \tag{3.8}$$

which follow from (2.41), (2.42) and (2.48) after integration over the contour C, it can be shown that

$$\begin{aligned}2\mathrm{i}\partial_x(\tilde{\mathbf{U}}^- + \mathbf{U}^-) = {} & 2q(\tilde{\mathbf{U}}^- - \mathbf{U}^-) + \tfrac{1}{2}\mathrm{i}(\tilde{\mathbf{U}}^- - \mathbf{U}^-)\cdot\mathbf{O}\cdot(\tilde{\mathbf{U}}^+ - \mathbf{U}^+)\\ & + \tfrac{1}{2}\mathrm{i}(\tilde{\mathbf{U}}^+ - \mathbf{U}^+)\cdot\mathbf{O}\cdot(\tilde{\mathbf{U}}^- - \mathbf{U}^-)\,,\end{aligned} \tag{3.9}$$

$$\begin{aligned}2p(\tilde{\mathbf{U}}^- + \mathbf{U}^-) = {} & 2q(\tilde{\mathbf{U}}^+ - \mathbf{U}^+) - \tfrac{1}{2}\mathrm{i}(\tilde{\mathbf{U}}^- + \mathbf{U}^-)\cdot\mathbf{O}\cdot(\tilde{\mathbf{U}}^- + \mathbf{U}^-)\\ & + \tfrac{1}{2}\mathrm{i}(\tilde{\mathbf{U}}^+ - \mathbf{U}^+)\cdot\mathbf{O}\cdot(\tilde{\mathbf{U}}^+ - \mathbf{U}^+)\,.\end{aligned} \tag{3.10}$$

Taking the (0, 0) element of (3.10) one can solve

$$\tilde{u}^{+}_{0,0} - u^{+}_{0,0} = 2q\mathrm{i} \mp \mathrm{i}[4(q^2-p^2) - (\tilde{u}^{-}_{0,0} + u^{-}_{0,0} - 2p\mathrm{i})^2]^{1/2}\,. \tag{3.11}$$

Next from (3.10) and (3.11) one can evaluate the matrices $\mathbf{O}\cdot(\tilde{\mathbf{U}}^{+} - \mathbf{U}^{+})$ and $(\tilde{\mathbf{U}}^{+} - \mathbf{U}^{+})\cdot\mathbf{O}$ in terms of $\mathbf{U}^{-}$ and $\tilde{\mathbf{U}}^{-}$. Inserting the result in (3.9) one obtains

$$\begin{aligned} 2\mathrm{i}\partial_x(\tilde{\mathbf{U}}^{-} + \mathbf{U}^{-}) &= 2q(\tilde{\mathbf{U}}^{-} - \mathbf{U}^{-}) - 2(q \mp \tfrac{1}{2}[4(q^2-p^2) - (\tilde{u}^{-}_{0,0} + u^{-}_{0,0} - 2p\mathrm{i})^2]^{1/2}) \\ &\quad \times (\tilde{u}^{-}_{0,0} + u^{-}_{0,0})^{-1}[(\tilde{\mathbf{U}}^{-} - \mathbf{U}^{-})\cdot\mathbf{O}\cdot(\tilde{\mathbf{U}}^{-} + \mathbf{U}^{-})]^{\mathrm{s}}\,. \end{aligned} \tag{3.12}$$

Eq. (3.12) is the spatial part of the matrix BT for $\mathbf{U}^{-}$, associated with the integral equation (2.40). Eq. (3.12) again is common to all modified matrix PDE's which can be derived for various choices of the dispersion $\omega(k)$. (For the special case $\omega(k) = k^3$ the modified matrix PDE is given by (2.31)). Taking the (0, 0) element of (3.12) we have the BT for (2.31),

$$2\mathrm{i}\partial_x(\tilde{u}^{-}_{0,0} + u^{-}_{0,0}) = \pm\,(\tilde{u}^{-}_{0,0} - u^{-}_{0,0})[4(q^2-p^2) - (\tilde{u}^{-}_{0,0} + u^{-}_{0,0} - 2p\mathrm{i})^2]^{1/2}\,, \tag{3.13}$$

which for $p = 0$ reduces to the well-known BT for the MKdV[5]).

3.2. *Second modified PDE*

Introducing the matrices

$$\mathbf{U}^{-+} \equiv \tilde{\mathbf{U}}^{-} + \mathbf{U}^{-}, \qquad \mathbf{U}^{--} = \tilde{\mathbf{U}}^{-} - \mathbf{U}^{-}\,, \tag{3.14}$$

we have from (3.12)

$$2\mathrm{i}\partial_x\mathbf{U}^{-+} = 2q\mathbf{U}^{--} - \frac{2(q \mp \tfrac{1}{2}[4(q^2-p^2) - (u^{-+}_{0,0} - 2p\mathrm{i})^2]^{1/2})}{u^{-+}_{0,0}}(\mathbf{U}^{--}\cdot\mathbf{O}\cdot\mathbf{U}^{-+})^{\mathrm{s}}\,. \tag{3.15}$$

The (0, 0) element of $\mathbf{U}^{--}$ follows from (3.15), i.e.

$$u^{--}_{0,0} = \pm\,2\mathrm{i}[4(q^2-p^2) - (u^{-+}_{0,0} - 2p\mathrm{i})^2]^{-1/2}\,\partial_x u^{-+}_{0,0}\,. \tag{3.16}$$

From (3.15) and (3.16) one can solve the matrices $\mathbf{U}^{--}\cdot\mathbf{O}$ and $\mathbf{O}\cdot\mathbf{U}^{--}$ in terms of $\mathbf{U}^{-+}$. Inserting the result in (3.15) it follows that

$$\begin{aligned} 2q\mathbf{U}^{--} &= -4q\mathbf{U}^{-} + 2q\mathbf{U}^{-+} \\ &= 2\mathrm{i}\partial_x\mathbf{U}^{-+} + \frac{q \mp \tfrac{1}{2}[4(q^2-p^2) - (u^{-+}_{0,0} - 2p\mathrm{i})^2]^{1/2}}{q \pm \tfrac{1}{2}[4(q^2-p^2) - (u^{-+}_{0,0} - 2p\mathrm{i})^2]^{1/2}} \\ &\quad \times \Bigg[\frac{2\mathrm{i}\partial_x(\mathbf{U}^{-+}\cdot\mathbf{O}\cdot\mathbf{U}^{-+})}{u^{-+}_{0,0}} \pm \frac{4\mathrm{i}(\partial_x u^{-+}_{0,0})}{u^{-+2}_{0,0}} \\ &\quad \times \frac{(q \mp \tfrac{1}{2}[4(q^2-p^2) - (u^{-+}_{0,0} - 2p\mathrm{i})^2]^{1/2})}{[4(q^2-p^2) - (u^{-+}_{0,0} - 2p\mathrm{i})^2]^{1/2}}\mathbf{U}^{-+}\cdot\mathbf{O}\cdot\mathbf{U}^{-+}\Bigg]\,. \end{aligned} \tag{3.17}$$

Adding the matrix PDE (2.30) for $\mathbf{U}^-$ and its counterpart with $\mathbf{U}^- \to \tilde{\mathbf{U}}^-$ and using (3.17) one can derive a matrix PDE for $\mathbf{U}^{-+}$. Eq. (3.17) which is independent of $\omega(k)$, is for $q \neq 0$ a Miura transformation mapping a solution of the PDE for $\mathbf{U}^{-+}$ on a solution of the PDE for $\mathbf{U}^-$. Taking as an example $\omega(k) = k^3$, we obtain from (2.30)

$$\begin{aligned}(\partial_t - \partial_x^3)\mathbf{U}^{-+} = {}& \tfrac{3}{2}\mathrm{i}p[(\mathbf{U}^{-+}\cdot\mathbf{O}\cdot\partial_x\mathbf{U}^{-+})^{\mathrm{s}} + \mathbf{U}^{--}\cdot\mathbf{O}\cdot\partial_x\mathbf{U}^{--})^{\mathrm{s}}] \\ & - \tfrac{3}{8}[(\mathbf{U}^{-+}\cdot\mathbf{O}\cdot\mathbf{U}^{--} + \mathbf{U}^{--}\cdot\mathbf{O}\cdot\mathbf{U}^{-+})\cdot\mathbf{O}\cdot\partial_x\mathbf{U}^{--}]^{\mathrm{s}} \\ & - \tfrac{3}{8}[(\mathbf{U}^{-+}\cdot\mathbf{O}\cdot\mathbf{U}^{-+} + \mathbf{U}^{--}\cdot\mathbf{O}\cdot\mathbf{U}^{--})\cdot\mathbf{O}\cdot\partial_x\mathbf{U}^{-+}]^{\mathrm{s}}, \quad (3.18)\end{aligned}$$

in which the explicit expression for $\mathbf{U}^{--}\cdot\mathbf{O}$ must be inserted. The resulting matrix PDE, in terms of $\mathbf{U}^{-+}$ only, can be regarded as the second modified matrix PDE of (2.12).

For the (0, 0) element of (3.18) we obtain, using (3.16),

$$\begin{aligned}(\partial_t - \partial_x^3)u_{0,0}^{-+} = \frac{3}{2}\partial_x\Bigg[&\frac{(u_{0,0}^{-+} - 2\mathrm{i}p)(\partial_x u_{0,0}^{-+})^2}{4(q^2 - p^2) - (u_{0,0}^{-+} - 2\mathrm{i}p)^2} \\ & - \frac{1}{12}(u_{0,0}^{-+} - 2\mathrm{i}p)^3 - p^2 u_{0,0}^{-+}\Bigg], \qquad (3.19)\end{aligned}$$

which in terms of the new variable z, defined by

$$u_{0,0}^{-+} - 2\mathrm{i}p \equiv \mathrm{i}(q+p)\,\mathrm{e}^{z} - \mathrm{i}(q-p)\,\mathrm{e}^{-z}, \qquad (3.20)$$

can be written as

$$\begin{aligned}(\partial_t - \partial_x^3)z = {}& -\tfrac{1}{2}(\partial_x z)^3 + \tfrac{3}{8}[(p+q)^2\,\mathrm{e}^{2z} + (p-q)^2\,\mathrm{e}^{-2z} \\ & - 2(p^2 + q^2)]\,\partial_x z\,. \qquad (3.21)\end{aligned}$$

Eq. (3.21) is a completely integrable PDE, which can be called the second modified KdV and which has been treated in the literature[22,25,26], see also ref. 27, and for $q^2 \neq p^2$, eq. (3.21) may be reduced to eq. (2.32). The Miura transformation[22,25,26] mapping a solution of (3.21) on a solution of (2.31) is given by

$$u_{0,0}^{-} = \partial_x z + \tfrac{1}{2}\mathrm{i}\{(q+p)\,\mathrm{e}^{z} - (q-p)\,\mathrm{e}^{-z} + 2p\}\,, \qquad (3.22)$$

as follows from (3.16) and (3.20).

3.3. *Integral equation and linear problem for second modified PDE*

Defining

$$\boldsymbol{u}_k^{\alpha\pm} \equiv (q-k)\tilde{\boldsymbol{u}}_k^{\alpha} \pm (q+k)\boldsymbol{u}_k^{\alpha}, \quad \alpha = \pm\,, \qquad (3.23)$$

with inverse

$$\tilde{\boldsymbol{u}}_k^{\alpha} = \frac{\boldsymbol{u}_k^{\alpha+} + \boldsymbol{u}_k^{\alpha-}}{2(q-k)}, \quad \boldsymbol{u}_k^{\alpha} = \frac{\boldsymbol{u}_k^{\alpha+} - \boldsymbol{u}_k^{\alpha-}}{2(q+k)}, \qquad (3.24)$$

and introducing a new measure $\mathrm{d}\lambda_2(k)$ by

$$\mathrm{d}\lambda_2(k) = \frac{\mathrm{d}\tilde{\lambda}_1(k)}{q-k} = \frac{\mathrm{d}\lambda_1(k)}{q+k}, \tag{3.25}$$

we have

$$\mathbf{U}^{\alpha\pm} \equiv \int\limits_C \mathrm{d}\lambda_2(k)\boldsymbol{u}_k^{\alpha\pm}\boldsymbol{c}_k = \tilde{\mathbf{U}}^{\alpha} \pm \mathbf{U}^{\alpha}. \tag{3.26}$$

It is now straightforward to derive two coupled linear integral equations for $\boldsymbol{u}_k^{-+}$ and $\boldsymbol{u}_k^{--}$. Inserting (3.24) into (2.40) and its counterpart with $\boldsymbol{u}_k^- \to \tilde{\boldsymbol{u}}_k^-$, $\mathrm{d}\lambda_1(k) \to \mathrm{d}\tilde{\lambda}_1(k)$, and changing to the new measure (3.25) it is straightforward to show that

$$\begin{aligned}
&\boldsymbol{u}_k^{-\pm} + \int\limits_C \mathrm{d}\lambda_2(l) \int\limits_C \mathrm{d}\lambda_2(l')\frac{(p^2+kl')(q^2+kl')}{(k+l')(l'+l)}\rho_k\rho_{l'}\boldsymbol{u}_l^{-\pm} \\
&\quad + \rho_k\Bigg[\mathrm{i}p\int\limits_C \mathrm{d}\lambda_2(l)l^{-1}(q\boldsymbol{u}_l^{-\pm} - k\boldsymbol{u}_l^{-\mp}) - q\int\limits_C \mathrm{d}\lambda_2(l)\int\limits_C \mathrm{d}\lambda_2(l')\frac{(p^2+kl')\rho_{l'}}{(l'+l)}\boldsymbol{u}_l^{-\mp} \\
&\quad - p^2\int\limits_C \mathrm{d}\lambda_2(l)\int\limits_C \mathrm{d}\lambda_2(l')\frac{\rho_{l'}\{(q^2+kl')\boldsymbol{u}_l^{-\pm} - q(k+l')\boldsymbol{u}_l^{-\mp}\}}{l'(l'+l)}\Bigg] \\
&= -2\rho_k\boldsymbol{c}_k k[(q-k) \pm (q+k)],
\end{aligned} \tag{3.27}$$

which are two coupled linear integral equations for $\boldsymbol{u}_k^{-+}$ and $\boldsymbol{u}_k^{--}$. The solutions of the PDE for $\boldsymbol{U}^{-+}$ can be obtained from $\boldsymbol{u}_k^{-+}$ using the integration (3.26) and for any choice of contours and measure the $(0,0)$ element $u_{0,0}^{-+}$ is a solution of (3.19).

It is now straightforward to derive the linear problem for $u_{0,0}^{-+}$, or equivalently for the function z given by (3.20). From (3.5), using also (2.41), (2.42), (2.48) and the definitions (3.23) and (3.26) we have

$$\boldsymbol{u}_k^{--} = 2\mathrm{i}\partial_x\boldsymbol{u}_k^- + \tfrac{1}{2}\mathrm{i}\mathbf{U}^{+-}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^- + \tfrac{1}{2}\mathrm{i}\mathbf{U}^{--}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^+, \tag{3.28}$$

$$\boldsymbol{u}_k^{+-} = 2p\boldsymbol{u}_k^- + \tfrac{1}{2}\mathrm{i}\mathbf{U}^{+-}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^+ + \tfrac{1}{2}\mathrm{i}\mathbf{U}^{-+}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^-. \tag{3.29}$$

Multiplying (3.28) and (3.29) by $(q+k)$, and the inverse relations with $\boldsymbol{u}_k^{\pm} \leftrightarrow \tilde{\boldsymbol{u}}_k^{\pm}$, $\mathbf{U}^{\pm} \leftrightarrow \tilde{\mathbf{U}}^{\pm}$ by $(q-k)$, and using (3.23) and (3.26), we have the following relations between $\boldsymbol{u}_k^{--}$, $\boldsymbol{u}_k^{+-}$ and $\boldsymbol{u}_k^{++}$

$$((q-k)+\alpha(q+k))\boldsymbol{u}_k^{--} = 2\mathrm{i}\partial_x\boldsymbol{u}_k^{-\alpha} - \tfrac{1}{2}\mathrm{i}\mathbf{U}^{+-}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{-\bar{\alpha}} - \tfrac{1}{2}\mathrm{i}\mathbf{U}^{--}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{+\bar{\alpha}}, \tag{3.30}$$

$$((q-k)+\alpha(q+k))\boldsymbol{u}_k^{+-} = 2p\boldsymbol{u}_k^{-\alpha} - \tfrac{1}{2}\mathrm{i}\mathbf{U}^{+-}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{+\bar{\alpha}} + \tfrac{1}{2}\mathrm{i}\mathbf{U}^{-+}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{-\alpha}, \tag{3.31}$$

where $\alpha = \pm$, and $\bar{\alpha} = +$ if $\alpha = -$, and $\bar{\alpha} = -$ if $\alpha = +$.

From (2.47) and (2.48) and their Bäcklund transforms it can be shown that

$$(-k-\mathrm{i}\partial_x)\boldsymbol{u}_k^{-\alpha} = -\tfrac{1}{4}\mathrm{i}\mathbf{U}^{-+}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{+\alpha} - \tfrac{1}{4}\mathrm{i}\mathbf{U}^{--}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{+\bar{\alpha}}, \tag{3.32}$$

$$\mathrm{i}\partial_x\boldsymbol{u}_k^{+\alpha} = p\boldsymbol{u}_k^{-\alpha} + \tfrac{1}{4}\mathrm{i}\mathbf{U}^{-+}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{-\alpha} + \tfrac{1}{4}\mathrm{i}\mathbf{U}^{--}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{-\bar{\alpha}}. \tag{3.33}$$

Eqs. (3.30)–(3.33) can be simplified using (3.11) (cf. (3.26)) for $u_{0,0}^{+-}$. From (3.31) one can then express $\mathbf{O}\cdot\boldsymbol{u}_k^{+-}$ and $\mathbf{O}\cdot\boldsymbol{u}_k^{++}$ as linear combinations of $\mathbf{O}\cdot\boldsymbol{u}_k^{-+}$ and $\mathbf{O}\cdot\boldsymbol{u}_k^{--}$. Inserting these linear combinations in (3.32) one obtains

$$\begin{aligned}(-k-\mathrm{i}\partial_x)\boldsymbol{u}_k^{-\alpha} = &-\frac{(\tfrac{1}{4}\mathrm{i}u_{0,0}^{-+}+p)}{u_{0,0}^{+-}}\mathbf{U}^{-\alpha}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{--}\\ &-\frac{(\tfrac{1}{4}\mathrm{i}u_{0,0}^{+-}\mathbf{U}^{-\bar{\alpha}}+k\mathbf{U}^{-\alpha})\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{-+}}{u_{0,0}^{-+}},\end{aligned} \tag{3.34}$$

with

$$u_{0,0}^{+-} = 2q\mathrm{i} \mp \mathrm{i}[4(q^2-p^2)-(u_{0,0}^{-+}-2p\mathrm{i})^2]^{1/2}, \tag{3.35}$$

and $\mathbf{U}^{--}\cdot\mathbf{O}$ given by (3.15) and (3.16). Eqs. (3.34) are constitutive relations belonging to the linear problem associated with the matrix PDE (3.18) for $\mathbf{U}^{-+}$. The relations (3.34) are independent of the dispersion $\omega(k)$ and are valid for any matrix PDE for $\mathbf{U}^{-+}$ which can be derived for various $\omega(k)$. In addition to (3.34) one may also derive algebraic relations invovling the matrices $\mathbf{J}$ and $\mathbf{J}^{\mathrm{T}}$, but we do not go in further details.

For the special case $\omega(k)=k^3$, eq. (2.50) leads to (cf. (2.47))

$$\begin{aligned}(\partial_t-\partial_x^3)\boldsymbol{u}_k^{-\alpha} = \tfrac{3}{4}\mathrm{i}p[&\mathbf{U}^{-+}\cdot\mathbf{O}\cdot\partial_x\boldsymbol{u}_k^{-\alpha} + \mathbf{U}^{--}\cdot\mathbf{O}\cdot\partial_x\boldsymbol{u}_k^{-\bar{\alpha}} + (\partial_x\mathbf{U}^{-+})\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{-\alpha}\\ &+(\partial_x\mathbf{U}^{--})\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{-\bar{\alpha}}] - \frac{3}{16}(\mathbf{U}^{-+}\cdot\mathbf{O}\cdot\mathbf{U}^{-+}+\mathbf{U}^{--}\cdot\mathbf{O}\cdot\mathbf{U}^{--})\\ &\cdot\mathbf{O}\cdot\partial_x\boldsymbol{u}_k^{-\alpha} - \frac{3}{16}(\partial_x\mathbf{U}^{-+})\cdot\mathbf{O}\cdot(\mathbf{U}^{-+}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{-\alpha}\\ &+\mathbf{U}^{--}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{-\bar{\alpha}}) - \frac{3}{16}(\mathbf{U}^{-+}\cdot\mathbf{O}\cdot\mathbf{U}^{--}+\mathbf{U}^{--}\cdot\mathbf{O}\cdot\mathbf{U}^{-+})\\ &\cdot\mathbf{O}\cdot\partial_x\boldsymbol{u}_k^{-\bar{\alpha}} - \frac{3}{16}(\partial_x\mathbf{U}^{--})\cdot\mathbf{O}\cdot(\mathbf{U}^{--}\cdot\boldsymbol{u}_k^{-\alpha}+\mathbf{U}^{-+}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{-\bar{\alpha}}).\end{aligned} \tag{3.36}$$

The $n = 0$ components of (3.34) and (3.36) form the linear problem associated with the PDE (3.19) for $u_{0,0}^{-+}$, when we insert (3.35) and (3.16).

3.4. *Third modified PDE*

Using (3.23)–(3.25) in (2.37) we have the integral equations

$$\begin{aligned}&(\boldsymbol{u}_k^{++} + \epsilon_1 \boldsymbol{u}_k^{-+}) + \epsilon_2(\boldsymbol{u}_k^{+-} + \epsilon_1 \boldsymbol{u}_k^{--})\\ &\qquad + \mathrm{i}(p - \epsilon_1 k)(q - \epsilon_2 k)\rho_k \int_C \mathrm{d}\lambda_2(l)\, \frac{\{(\boldsymbol{u}_l^{++} + \epsilon_1 \boldsymbol{u}_l^{-+}) + \epsilon_2(\boldsymbol{u}_l^{+-} + \epsilon_1 \boldsymbol{u}_l^{--})\}}{k + l}\\ &\qquad = 4(p - \epsilon_1 k)(q - \epsilon_2 k)\rho_k \boldsymbol{c}_k \qquad (\epsilon_1, \epsilon_2 = \pm 1)\,. \end{aligned} \tag{3.37}$$

Eqs. (3.37) for ϵ_1, $\epsilon_2 = \pm 1$ have the same form as eq. (2.1) with

$$\boldsymbol{u}_k \to \tfrac{1}{4}[(\boldsymbol{u}_k^{++} + \epsilon_1 \boldsymbol{u}_k^{-+}) + \epsilon_2(\boldsymbol{u}_k^{+-} + \epsilon_1 \boldsymbol{u}_k^{--})], \qquad \rho_k \to (p - \epsilon_1 k)(q - \epsilon_2 k)\rho_k\,.$$

Applying the singular transformation of the measure

$$\mathrm{d}\tilde{\lambda}_2(k) = \frac{r - k}{r + k}\, \mathrm{d}\lambda_2(k)\,, \tag{3.38}$$

and introducing the solutions $\tilde{\boldsymbol{u}}_k^{++} + \epsilon_1 \tilde{\boldsymbol{u}}_k^{-+} + \epsilon_2(\tilde{\boldsymbol{u}}_k^{+-} + \epsilon_1 \tilde{\boldsymbol{u}}_k^{--})$ of (3.37) with $\mathrm{d}\lambda_2(k)$ replaced by $\mathrm{d}\tilde{\lambda}_2(k)$, we have the relations

$$\begin{aligned}&(r - k)\left[\frac{\tilde{\boldsymbol{u}}_k^{++} + \epsilon_1 \tilde{\boldsymbol{u}}_k^{-+} + \epsilon_2(\tilde{\boldsymbol{u}}_k^{+-} + \epsilon_1 \tilde{\boldsymbol{u}}_k^{--})}{4}\right]\\ &\quad = r\left[\frac{\boldsymbol{u}_k^{++} + \epsilon_1 \boldsymbol{u}_k^{-+} + \epsilon_2(\boldsymbol{u}_k^{+-} + \epsilon_1 \boldsymbol{u}_k^{--})}{4}\right]\\ &\qquad - \mathbf{J}^{\mathrm{T}} \cdot \left[\frac{\boldsymbol{u}_k^{++} + \epsilon_1 \boldsymbol{u}_k^{-+} + \epsilon_2(\boldsymbol{u}_k^{+-} + \epsilon_1 \boldsymbol{u}_k^{--})}{4}\right]\\ &\qquad + \mathrm{i}\left[\frac{\tilde{\mathbf{U}}^{++} + \epsilon_1 \tilde{\mathbf{U}}^{-+} + \epsilon_2(\tilde{\mathbf{U}}^{+-} + \epsilon_1 \tilde{\mathbf{U}}^{--})}{4}\right] \cdot \mathbf{O}\\ &\qquad \cdot \left[\frac{\boldsymbol{u}_k^{++} + \epsilon_1 \boldsymbol{u}_k^{-+} + \epsilon_2(\boldsymbol{u}_k^{+-} + \epsilon_1 \boldsymbol{u}_k^{--})}{4}\right], \end{aligned} \tag{3.39}$$

in which the four matrices $\tilde{\mathbf{U}}^{\alpha\pm}$ with $\alpha = \pm$ are given by

$$\tilde{\mathbf{U}}^{\alpha\pm} = \int_C \mathrm{d}\tilde{\lambda}_2(k) \tilde{\boldsymbol{u}}_k^{\alpha\pm} \boldsymbol{c}_k\,. \tag{3.40}$$

From (3.39) it is straightforward to show that (cf. appendix C for some details of the derivation)

$$
\begin{aligned}
&-[(\tilde{u}_{0,0}^{--}+u_{0,0}^{--})(\tilde{u}_{0,0}^{-+}-u_{0,0}^{-+})^{-1}([4(q^2-p^2)-(\tilde{u}_{0,0}^{-+}-2p\mathrm{i})^2]^{1/2}\\
&\quad+[4(q^2-p^2)-(u_{0,0}^{-+}-2p\mathrm{i})^2]^{1/2})]^2\\
&=[[4(q^2-p^2)-(\tilde{u}_{0,0}^{-+}-2p\mathrm{i})^2]^{1/2}-[4(q^2-p^2)-(u_{0,0}^{-+}-2p\mathrm{i})^2]^{1/2}]^2\\
&\quad+(\tilde{u}_{0,0}^{--}+u_{0,0}^{--})^2+(\tilde{u}_{0,0}^{-+}+u_{0,0}^{-+}-4\mathrm{i}p)^2+16(p^2-r^2)\,, \qquad (3.41)
\end{aligned}
$$

which in combination with (3.16) and its inverse with $u_{0,0}^{--}\to\tilde{u}_{0,0}^{--}$, $u_{0,0}^{-+}\to\tilde{u}_{0,0}^{-+}$ gives the BT for the PDE (3.19). The result can be further simplified introducing the variable z defined by (3.20). From (3.41), taking into account the relations

$$
\begin{aligned}
&u_{0,0}^{--}=-2\partial_x z,\\
&[4(q^2-p^2)-(u_{0,0}^{-+}-2p\mathrm{i})^2]^{1/2}=(q+p)\,\mathrm{e}^{z}+(q-p)\,\mathrm{e}^{-z}\,,
\end{aligned}
\qquad (3.42)
$$

(cf. (3.16)), one obtains after some straightforward algebra

$$
\begin{aligned}
(\partial_x(\tilde{z}+z))^2=(\sinh^2\tfrac{1}{2}(\tilde{z}-z))[16(p^2-r^2)\\
-\{(q+p)\,\mathrm{e}^{(\tilde{z}+z)/2}-(q-p)\,\mathrm{e}^{-(\tilde{z}+z)/2}\}^2]\,, \qquad (3.43)
\end{aligned}
$$

which is the BT for the second modified KdV (3.21). The BT for the special case (2.32) has been treated by Hirota's method in ref. 28 and in the context of prolongation structures in ref. 29.

The third modified KdV can be derived solving $(\tilde{z}-z)$ from (3.43) and inserting the result into the right-hand side of $(\partial_t-\partial_x^3)(z+\tilde{z})$, as given by (3.21) and its counterpart with $z\to\tilde{z}$.

The final result is in terms of the variable $y=\frac{1}{2}(\tilde{z}+z)$

$$
\begin{aligned}
(\partial_t-\partial_x^3)y&=-\tfrac{1}{2}(\partial_x y)^3+\tfrac{3}{8}\{(q+p)^2\,\mathrm{e}^{2y}+(q-p)^2\,\mathrm{e}^{-2y}-2(q^2+p^2)\}\,\partial_x y\\
&\quad+\frac{3}{2}D\partial_x\left[\frac{[(q+p)^2\,\mathrm{e}^{2y}-(q-p)^2\,\mathrm{e}^{-2y}](\partial_x y)^2}{1-D[(q+p)\,\mathrm{e}^{y}-(q-p)\,\mathrm{e}^{-y}]^2}\right]-\frac{3}{2}D(\partial_x y)\\
&\quad\times\left[\partial_x\left[\frac{1}{\sqrt{D}}\operatorname{arsinh}\left(\frac{2\sqrt{D}\,\partial_x y}{[1-D\{(q+p)\,\mathrm{e}^{y}-(q-p)\,\mathrm{e}^{-y}\}^2]^{1/2}}\right)\right]\right]^2,\\
&\qquad D\equiv\frac{1}{16(p^2-r^2)}\,. \qquad (3.44)
\end{aligned}
$$

Eq. (3.44) is an integrable equation, since its solutions can be obtained from a linear integral equation, such as e.g. (3.27), with measures $\mathrm{d}\lambda_2(k)$ and $\mathrm{d}\tilde{\lambda}_2(k)$, taking into account (3.26) and (3.20). In the limit $r\to\infty$, $(D\to 0)$, (3.44) reduces to the second modified KdV (3.21). A different version of the third modified KdV, in terms of the variable $\frac{1}{2}(\tilde{z} - z)$ has been given in ref. 28.

From this point one might continue to derive the integral equation for the third modified PDE, apply a singular transformation of measures and obtain the BT for the third modified PDE, as well as the PDE associated with the next level of modification. The procedure is systematic in terms of (matrix) functions $\mathbf{U}$ with a sequence of superscripts $+$ and $-$ that can be defined in an analogous way as in the preceding steps, cf. e.g. (2.36) and (3.14). It is not obvious, however, that the coupled PDE's for the different matrices $\mathbf{U}$ will lead to interesting closed PDE's in terms of only one function and we shall not go in further details here.

4. The NLS class

In the preceding sections we have considered the integral equation for the KdV class and by singular transformations of the measure we have obtained Bäcklund transformations and the first-, second–, and third modified KdV equation. In the present section we start from the integral equation for the NLS class and in subsection 4.1 we review some results obtained in ref. 18, which will be used in the following. By a singular transformation of the measure we shall derive the (matrix) BT for the NLS, as well as the matrix modified PDE, together with the associated linear integral equation. The PDE for the (0, 0) element of the matrix modified PDE will turn out to be equivalent to the equation of motion for the classical Heisenberg spin chain with uniaxial anisotropy, and at the end of the section a BT for the AHSC will be derived.

4.1. *Integral equation and matrix PDE*

For the NLS class we have the linear integral equation

$$\boldsymbol{\phi}_k(x,t) + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k(x,t)\rho^*_{l'}(x,t)}{(k-l')(l'-l)} \boldsymbol{\phi}_l(x,t) = \rho_k(x,t)\boldsymbol{c}_k\,, \tag{4.1}$$

where $\boldsymbol{\phi}_k$ and $\boldsymbol{c}_k$ are vectors with components $(\boldsymbol{\phi}_k)_n = \phi_k^{(n)}$, $(\boldsymbol{c}_k)_n = 1/k^n$, n being an integer, C and C^* are an arbitrary contour and its complex conjugate in the complex k-plane, $\mathrm{d}\lambda(l)$ and $\mathrm{d}\lambda^*(l')$ are an arbitrary measure and its complex conjugate, $\rho_k(x,t)$ is a plane-wave factor satisfying the linear differential equations

$$-\mathrm{i}\partial_x \rho_k(x,t) = k\rho_k(x,t), \quad \mathrm{i}\partial_t \rho_k(x,t) = \omega(k)\rho_k(x,t)\,, \tag{4.2}$$

where $\omega(k) = \Sigma_r \lambda_r k^r$ (r integer, λ_r real) is the dispersion.

Eq. (4.1) can be rewritten as

$$\boldsymbol{\phi}_k + \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k}{(k-l')} \boldsymbol{\psi}_{l'}^* = \rho_k \boldsymbol{c}_k \,, \tag{4.3a}$$

in which the vector $\boldsymbol{\psi}_k$ is defined by

$$\boldsymbol{\psi}_k - \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k}{(k-l')} \boldsymbol{\phi}_{l'}^* = 0 \,, \tag{4.3b}$$

and $\boldsymbol{\psi}_k$ satisfies the linear integral equation

$$\boldsymbol{\psi}_k + \int_{C} \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho_{l'}^*}{(k-l')(l'-l)} \boldsymbol{\psi}_l = \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho_{l'}^*}{(k-l')} \boldsymbol{c}_{l'} \,. \tag{4.4}$$

From now on the choice of measures and contours will be restricted by the condition that the solution $\boldsymbol{\phi}_k$ of the integral equation (4.1) is unique.

From eqs. (4.1) and (4.2), or alternatively from (4.3), (4.4) and (4.2), taking into account the uniqueness condition one can derive the constitutive relations

$$k^p \boldsymbol{\phi}_k = \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\phi}_k - \boldsymbol{\Psi}^* \cdot \boldsymbol{\Omega}_p \cdot \boldsymbol{\phi}_k - \boldsymbol{\Phi} \cdot \boldsymbol{\Omega}_p \cdot \boldsymbol{\psi}_k \,, \tag{4.5}$$

$$k^p \boldsymbol{\psi}_k = \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\psi}_k + \boldsymbol{\Phi}^* \cdot \boldsymbol{\Omega}_p \cdot \boldsymbol{\phi}_k - \boldsymbol{\Psi} \cdot \boldsymbol{\Omega}_p \cdot \boldsymbol{\psi}_k \quad (p \text{ integer}) \,, \tag{4.6}$$

$$-\mathrm{i}\partial_x \boldsymbol{\phi}_k = k \boldsymbol{\phi}_k + \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\psi}_k \,, \tag{4.7}$$

$$-\mathrm{i}\partial_x \boldsymbol{\psi}_k = \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k \,, \tag{4.8}$$

$$\mathrm{i}\partial_t \boldsymbol{\phi}_k = \omega(\mathbf{J}^{\mathrm{T}}) \cdot \boldsymbol{\phi}_k - \boldsymbol{\Psi}^* \cdot \boldsymbol{\Omega} \cdot \boldsymbol{\phi}_k \,, \tag{4.9}$$

$$\mathrm{i}\partial_t \boldsymbol{\psi}_k = \boldsymbol{\Phi}^* \cdot \boldsymbol{\Omega} \cdot \boldsymbol{\phi}_k \,, \tag{4.10}$$

in which the matrices $\mathbf{J}^{\mathrm{T}}$ and $\mathbf{O}$ have been defined by (2.7),

$$\boldsymbol{\Omega}_p \equiv (\operatorname{sgn} p) \sum_{j=0}^{|p|-1} \mathbf{J}^{j+p/2-|p|/2} \cdot \mathbf{O} \cdot \mathbf{J}^{\mathrm{T}-j-1+p/2+|p|/2}, \quad \boldsymbol{\Omega} \equiv \sum_r \lambda_r \boldsymbol{\Omega}_r \,. \tag{4.11}$$

The symmetric matrix $\boldsymbol{\Phi} = \boldsymbol{\Phi}^{\mathrm{T}}$ and the antihermitean matrix $\boldsymbol{\Psi}$ can be obtained from the vectors $\boldsymbol{\phi}_k$ and $\boldsymbol{\psi}_k$ by an integration over the same contour C that appears in the integral equation,

$$\boldsymbol{\Phi} = \int_C \mathrm{d}\lambda(k) \boldsymbol{\phi}_k \boldsymbol{c}_k, \quad \boldsymbol{\Psi} = \int_C \mathrm{d}\lambda(k) \boldsymbol{\psi}_k \boldsymbol{c}_k \,. \tag{4.12}$$

For dispersion of the type $\omega(k) = \Sigma_r \lambda_r k^r$, $\lambda_r = 0$ for $r < 0$, one can derive a closed PDE for the matrix $\boldsymbol{\Phi}$, containing only $\boldsymbol{\Phi}$ and its derivatives and $\mathbf{O}$, but not the matrices $\mathbf{J}$ and $\mathbf{J}^{\mathrm{T}}$. Taking as an example

$$\omega(k) = \lambda_2 k^2 + \lambda_3 k^3 , \tag{4.13}$$

we have e.g.

$$\begin{aligned} \mathrm{i}\partial_t \boldsymbol{\phi}_k = &- \lambda_2 \partial_x^2 \boldsymbol{\phi}_k - 2\lambda_2 \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k + \mathrm{i}\lambda_3 \partial_x^3 \boldsymbol{\phi}_k \\ &+ 3\mathrm{i}\lambda_3 (\partial_x \boldsymbol{\Phi}) \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k + 3\mathrm{i}\lambda_3 \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \partial_x \boldsymbol{\phi}_k , \end{aligned} \tag{4.14}$$

$$\begin{aligned} \mathrm{i}\partial_t \boldsymbol{\psi}_k = &\, \mathrm{i}\lambda_2 (\partial_x \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k - \mathrm{i}\lambda_2 \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \partial_x \boldsymbol{\phi}_k - \lambda_3 (\partial_x^2 \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k \\ &- \lambda_3 \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \partial_x^2 \boldsymbol{\phi}_k + \lambda_3 (\partial_x \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot \partial_x \boldsymbol{\phi}_k \\ &+ 3\lambda_3 \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k . \end{aligned} \tag{4.15}$$

For the (0, 0) element of $\boldsymbol{\Phi}$ we obtain

$$\mathrm{i}\partial_t \phi_{0,0} + \lambda_2 \partial_x^2 \phi_{0,0} - \mathrm{i}\lambda_3 \partial_x^3 \phi_{0,0} = - 2\lambda_2 |\phi_{0,0}|^2 \phi_{0,0} + 6\mathrm{i}\lambda_3 |\phi_{0,0}|^2 \partial_x \phi_{0,0} , \tag{4.16}$$

which is Hirota's equation.

4.2. *Singular transformation of the measure*

We introduce the singular transformation of the measure, cf. (A.10)–(A.11),

$$\mathrm{d}\lambda(k) \to \mathrm{d}\tilde{\lambda}(k) = \frac{p - k}{p^* - k} \, \mathrm{d}\lambda(k) , \tag{4.17}$$

and consider the solutions of (4.3a) and (4.3b) with the new measure, i.e.

$$\tilde{\boldsymbol{\phi}}_k + \int_{C^*} \mathrm{d}\tilde{\lambda}^*(l') \frac{\rho_k}{k - l'} \tilde{\boldsymbol{\psi}}_{l'}^* = \rho_k \boldsymbol{c}_k , \tag{4.18}$$

$$\tilde{\boldsymbol{\psi}}_k - \int_{C^*} \mathrm{d}\tilde{\lambda}^*(l') \frac{\rho_k}{k - l'} \tilde{\boldsymbol{\phi}}_{l'}^* = 0 . \tag{4.19}$$

Using (4.18) and (4.19) and a relation similar to (2.21) we have

$$(p - k)\tilde{\boldsymbol{\phi}}_k + \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k}{k - l'} (p^* - l') \, \tilde{\boldsymbol{\psi}}_{l'}^* = (p - k)\rho_k \boldsymbol{c}_k + \rho_k \tilde{\boldsymbol{\Psi}}^* \cdot \mathbf{O} \cdot \boldsymbol{c}_k , \tag{4.20}$$

$$(p - k)\tilde{\boldsymbol{\psi}}_k - \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k}{k - l'} (p^* - l') \tilde{\boldsymbol{\phi}}_{l'}^* = - \rho_k \tilde{\boldsymbol{\Phi}}^* \cdot \mathbf{O} \cdot \boldsymbol{c}_k , \tag{4.21}$$

in which the matrices $\tilde{\boldsymbol{\Phi}}$ and $\tilde{\boldsymbol{\Psi}}$ can be obtained from $\tilde{\boldsymbol{\phi}}_k$, $\tilde{\boldsymbol{\psi}}_k$ by an integration with the new measure, i.e.

$$\tilde{\boldsymbol{\Phi}} = \int_C \mathrm{d}\tilde{\lambda}(k)\tilde{\boldsymbol{\phi}}_k \boldsymbol{c}_k, \quad \tilde{\boldsymbol{\Psi}} = \int_C \mathrm{d}\tilde{\lambda}(k)\tilde{\boldsymbol{\psi}}_k \boldsymbol{c}_k . \tag{4.22}$$

From (4.20) and (4.21) we derive

$$\begin{aligned}(p-k)\tilde{\boldsymbol{\phi}}_k + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho_{l'}^*}{(k-l')(l'-l)}(p-l)\tilde{\boldsymbol{\phi}}_l \\ = \tilde{\boldsymbol{\Phi}} \cdot \mathbf{O} \cdot \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho_{l'}^*}{k-l'} \boldsymbol{c}_{l'} + (p-k)\rho_k \boldsymbol{c}_k + \rho_k \tilde{\boldsymbol{\Psi}}^* \cdot \mathbf{O} \cdot \boldsymbol{c}_k ,\end{aligned} \tag{4.23}$$

$$\begin{aligned}(p-k)\tilde{\boldsymbol{\psi}}_k + \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho_{l'}^*}{(k-l')(l'-l)}(p-l)\tilde{\boldsymbol{\psi}}_l \\ = \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho_{l'}^*}{k-l'}(p^*-l')\boldsymbol{c}_{l'} + \tilde{\boldsymbol{\Psi}} \cdot \mathbf{O} \cdot \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho_{l'}^*}{k-l'} \boldsymbol{c}_{l'} - \rho_k \tilde{\boldsymbol{\Phi}}^* \cdot \mathbf{O} \cdot \boldsymbol{c}_k .\end{aligned} \tag{4.24}$$

Taking into account that the homogeneous integral equation corresponding to (4.1) has only the zero solution, we obtain immediately

$$(p-k)\tilde{\boldsymbol{\phi}}_k = p\boldsymbol{\phi}_k - \mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{\phi}_k + \tilde{\boldsymbol{\Psi}}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k + \tilde{\boldsymbol{\Phi}} \cdot \mathbf{O} \cdot \boldsymbol{\psi}_k , \tag{4.25}$$

$$(p-k)\tilde{\boldsymbol{\psi}}_k = p^*\boldsymbol{\psi}_k - \mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{\psi}_k + \tilde{\boldsymbol{\Psi}} \cdot \mathbf{O} \cdot \boldsymbol{\psi}_k - \tilde{\boldsymbol{\Phi}}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k , \tag{4.26}$$

which are the basic relations for the BT (for the NLS class).

4.3. *Matrix Bäcklund transformation*

The inverse transformation of (4.17) can be obtained interchanging $p \leftrightarrow p^*$, $\mathrm{d}\lambda(k) \leftrightarrow \mathrm{d}\tilde{\lambda}(k)$. From (4.25) and (4.26) we thus obtain the inverse relations

$$(p-k)\tilde{\boldsymbol{\phi}}_k = p^*\tilde{\boldsymbol{\phi}}_k - \mathbf{J}^{\mathrm{T}} \cdot \tilde{\boldsymbol{\phi}}_k + \boldsymbol{\Psi}^* \cdot \mathbf{O} \cdot \tilde{\boldsymbol{\phi}}_k + \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \tilde{\boldsymbol{\psi}}_k , \tag{4.27}$$

$$(p^*-k)\boldsymbol{\psi}_k = p\tilde{\boldsymbol{\psi}}_k - \mathbf{J}^{\mathrm{T}} \cdot \tilde{\boldsymbol{\psi}}_k + \boldsymbol{\Psi} \cdot \mathbf{O} \cdot \tilde{\boldsymbol{\psi}}_k - \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \tilde{\boldsymbol{\phi}}_k . \tag{4.28}$$

Integrating (4.25) and (4.26) over the contour C with the measure $\mathrm{d}\lambda(k)$ and using the relation $(p^*-k)\,\mathrm{d}\tilde{\lambda}(k) = (p-k)\,\mathrm{d}\lambda(k)$ to evaluate the integrals on the left-hand side we have

$$p^*\tilde{\boldsymbol{\Phi}} - p\boldsymbol{\Phi} = \tilde{\boldsymbol{\Phi}} \cdot \mathbf{J} - \mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{\Phi} + \tilde{\boldsymbol{\Phi}} \cdot \mathbf{O} \cdot \boldsymbol{\Psi} + \tilde{\boldsymbol{\Psi}}^* \cdot \mathbf{O} \cdot \boldsymbol{\Phi} , \tag{4.29}$$

$$p^*(\tilde{\mathbf{\Psi}} - \mathbf{\Psi}) = \tilde{\mathbf{\Psi}} \cdot \mathbf{J} - \mathbf{J}^{\mathrm{T}} \cdot \mathbf{\Psi} + \tilde{\mathbf{\Psi}} \cdot \mathbf{O} \cdot \mathbf{\Psi} - \tilde{\mathbf{\Phi}}^* \cdot \mathbf{O} \cdot \mathbf{\Phi} \,. \tag{4.30}$$

The matrices $\mathbf{J}$ and $\mathbf{J}^{\mathrm{T}}$ in (4.29) and (4.30) can be eliminated with the relations

$$-2\mathrm{i}\partial_x \mathbf{\Phi} = \mathbf{J}^{\mathrm{T}} \cdot \mathbf{\Phi} + \mathbf{\Phi} \cdot \mathbf{J} + \mathbf{\Phi} \cdot \mathbf{O} \cdot \mathbf{\Psi} - \mathbf{\Psi}^* \cdot \mathbf{O} \cdot \mathbf{\Phi} \,, \tag{4.31}$$

and

$$\mathbf{\Psi} \cdot \mathbf{J} - \mathbf{J}^{\mathrm{T}} \cdot \mathbf{\Psi} + \mathbf{\Psi} \cdot \mathbf{O} \cdot \mathbf{\Psi} - \mathbf{\Phi}^* \cdot \mathbf{O} \cdot \mathbf{\Phi} = 0 \,, \tag{4.32}$$

which follow from (4.5), (4.7) and (4.6) resp. after integration over C. Using eqs. (4.29), (4.30) and the inverse equations with $p \leftrightarrow p^*$, $\mathbf{\Phi} \leftrightarrow \tilde{\mathbf{\Phi}}$, $\mathbf{\Psi} \leftrightarrow \tilde{\mathbf{\Psi}}$, in combination with (4.31) and (4.32), we obtain

$$\begin{aligned} 2(p^*\tilde{\mathbf{\Phi}} - p\mathbf{\Phi}) = &- 2\mathrm{i}\partial_x(\tilde{\mathbf{\Phi}} - \mathbf{\Phi}) + (\tilde{\mathbf{\Psi}}^* - \mathbf{\Psi}^*) \cdot \mathbf{O} \cdot (\tilde{\mathbf{\Phi}} + \mathbf{\Phi}) \\ &- (\tilde{\mathbf{\Phi}} + \mathbf{\Phi}) \cdot \mathbf{O} \cdot (\tilde{\mathbf{\Psi}} - \mathbf{\Psi}) \,, \end{aligned} \tag{4.33}$$

$$(p^* - p)(\tilde{\mathbf{\Psi}} - \mathbf{\Psi}) = (\tilde{\mathbf{\Phi}}^* - \mathbf{\Phi}^*) \cdot \mathbf{O} \cdot (\tilde{\mathbf{\Phi}} - \mathbf{\Phi}) - (\tilde{\mathbf{\Psi}} - \mathbf{\Psi}) \cdot \mathbf{O} \cdot (\tilde{\mathbf{\Psi}} - \mathbf{\Psi}) \,. \tag{4.34}$$

From eqs. (4.33) and (4.34) it is straightforward to derive a relation containing only $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$. In fact taking the $(0, 0)$ element of (4.34) we obtain a quadratic equation in $\tilde{\psi}_{0,0} - \psi_{0,0}$ which can be solved to give

$$\tilde{\psi}_{0,0} - \psi_{0,0} = \tfrac{1}{2}(p - p^*) \pm \frac{i}{2}\sqrt{|p - p^*|^2 - 4|\tilde{\phi}_{0,0} - \phi_{0,0}|^2} \,, \tag{4.35}$$

in which the left-hand side is imaginary as the matrix $\mathbf{\Psi}$ is antihermitean. From (4.34) one can solve the matrix $\mathbf{O} \cdot (\tilde{\mathbf{\Psi}} - \mathbf{\Psi})$, and inserting the result in (4.33) we arrive at

$$\begin{aligned} 2\mathrm{i}\partial_x(\tilde{\mathbf{\Phi}} - \mathbf{\Phi}) + (p^* + p)(\tilde{\mathbf{\Phi}} - \mathbf{\Phi}) = &- (p^* - p)(\tilde{\mathbf{\Phi}} + \mathbf{\Phi}) \\ &- 2\left(\tfrac{1}{2}(p - p^*) \pm \tfrac{1}{2}\mathrm{i}\sqrt{|p - p^*|^2 - 4|\tilde{\phi}_{0,0} - \phi_{0,0}|^2}\right)(\tilde{\phi}_{0,0} - \phi_{0,0})^{-1} \\ &\times [(\tilde{\mathbf{\Phi}} - \mathbf{\Phi}) \cdot \mathbf{O} \cdot (\tilde{\mathbf{\Phi}} + \mathbf{\Phi})]^{\mathrm{s}} \,. \end{aligned} \tag{4.36}$$

Eq. (4.36) is the spatial part of the matrix Bäcklund transformation associated with the integral equation (4.1). It is independent of the dispersion $\omega(k)$ and is common to all matrix PDE's which can be derived for various $\omega(k)$. The time-dependent part of the BT can be inferred from the matrix PDE and (4.36).

4.4. *Modified matrix PDE*

Introducing

$$\mathbf{\Phi}^{\pm} = \tilde{\mathbf{\Phi}} \pm \mathbf{\Phi} \,, \tag{4.37}$$

we have from the integrated version of (4.14) and its counterpart with $\mathbf{\Phi} \rightarrow \tilde{\mathbf{\Phi}}$

$$
\begin{aligned}
&\mathrm{i}\partial_t\boldsymbol{\Phi}^- + \lambda_2\partial_x^2\boldsymbol{\Phi}^- - \mathrm{i}\lambda_3\partial_x^3\boldsymbol{\Phi}^- \\
&\quad = -\tfrac{1}{2}\lambda_2(\boldsymbol{\Phi}^-\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{-*} + \boldsymbol{\Phi}^+\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{+*})\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^- \\
&\qquad -\tfrac{1}{2}\lambda_2(\boldsymbol{\Phi}^-\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{+*} + \boldsymbol{\Phi}^+\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{-*})\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^+ \\
&\qquad +\tfrac{3}{2}\mathrm{i}\lambda_3[(\boldsymbol{\Phi}^-\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{-*} + \boldsymbol{\Phi}^+\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{+*})\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\Phi}^-]^{\mathrm{s}} \\
&\qquad +\tfrac{3}{2}\mathrm{i}\lambda_3[(\boldsymbol{\Phi}^-\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{+*} + \boldsymbol{\Phi}^+\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{-*})\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\Phi}^+]^{\mathrm{s}}\,.
\end{aligned}
\tag{4.38}
$$

From the matrix BT (4.36) it is straightforward to express $\boldsymbol{\Phi}^+$ in terms of $\boldsymbol{\Phi}^-$. From the (0, 0) element we have

$$
\phi^+_{0,0} = \mp 2(\partial_x\phi^-_{0,0} - \tfrac{1}{2}\mathrm{i}(p+p^*)\,\phi^-_{0,0})/\sqrt{|p-p^*|^2 - 4|\phi^-_{0,0}|^2} \quad (p \neq p^*)\,. \tag{4.39}
$$

Next one can solve the matrices $(\tilde{\boldsymbol{\Phi}} + \boldsymbol{\Phi})\cdot\mathbf{O}$ and $\mathbf{O}\cdot(\tilde{\boldsymbol{\Phi}} + \boldsymbol{\Phi})$ from (4.36). Inserting the result in (4.36) one obtains after some algebra

$$
\begin{aligned}
(p^*-p)\boldsymbol{\Phi}^+ = (p^*-p)\boldsymbol{\Phi}^- + 2(p^*-p)\boldsymbol{\Phi} = &-2\mathrm{i}\partial_x\boldsymbol{\Phi}^- - (p+p^*)\boldsymbol{\Phi}^- \\
&+ 2\{\phi^-_{0,0}|\phi^-_{0,0}|^2\}^{-1}\{\tfrac{1}{2}(p-p^*) \pm \tfrac{1}{2}\mathrm{i}\sqrt{|p-p^*|^2 - 4|\phi^-_{0,0}|^2}\}^2 \\
&\times\{\mathrm{i}\partial_x(\boldsymbol{\Phi}^-\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^-) - \mathrm{i}(\partial_x \ln\phi^-_{0,0} + \tfrac{1}{2}\mathrm{i}(p+p^*))\boldsymbol{\Phi}^-\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^- \\
&\mp (p-p^*)\{|p-p^*|^2 - 4|\phi^-_{0,0}|^2\}^{-1/2} \\
&\times(\partial_x \ln\phi^-_{0,0} - \tfrac{1}{2}\mathrm{i}(p+p^*))\boldsymbol{\Phi}^-\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^-\}\,.
\end{aligned}
\tag{4.40}
$$

Inserting (4.40) in (4.38) we obtain the modified matrix PDE for $\boldsymbol{\Phi}^-$, and if $p \neq p^*$ (4.40) is the Miura transformation mapping a solution of the PDE for $\boldsymbol{\Phi}^-$ on a solution of the matrix PDE for $\boldsymbol{\Phi}$. (In the special case (4.13) the PDE for $\boldsymbol{\Phi}$ is the integrated version of (4.14), but (4.40) is independent of $\omega(k)$.)

As an example we write down the (0, 0) element of the matrix PDE in the case (4.13). The result is, cf. (4.38) and (4.39),

$$
\begin{aligned}
&\mathrm{i}\partial_t\phi^-_{0,0} + \lambda_2\partial_x^2\phi^-_{0,0} - \mathrm{i}\lambda_3\partial_x^3\phi^-_{0,0} \\
&\quad = ((-\tfrac{1}{2}\lambda_2 + \tfrac{3}{2}\mathrm{i}\lambda_3\partial_x)\phi^-_{0,0})\left(|\phi^-_{0,0}|^2 + \frac{|2\partial_x\phi^-_{0,0} - \mathrm{i}(p+p^*)\phi^-_{0,0}|^2}{|p-p^*|^2 - 4|\phi^-_{0,0}|^2}\right) \\
&\qquad + \left\{(-\tfrac{1}{2}\lambda_2 + \tfrac{3}{2}\mathrm{i}\lambda_3\partial_x)\left(\frac{2\partial_x\phi^-_{0,0} - \mathrm{i}(p+p^*)\phi^-_{0,0}}{\sqrt{|p-p^*|^2 - 4|\phi^-_{0,0}|^2}}\right)\right\}\frac{2\partial_x|\phi^-_{0,0}|^2}{\sqrt{|p-p^*|^2 - 4|\phi^-_{0,0}|^2}}\,,
\end{aligned}
\tag{4.41}
$$

and (4.39), with $\phi^+_{0,0} = \phi^-_{0,0} + 2\phi_{0,0}$ gives the Miura transformation mapping a solution of (4.41) on a solution of (4.16).

Taking $p^* \neq p$ and introducing the new variable

$$a(x,t) = 2|p - p^*|^{-1}\phi^{-}_{0,0}(x + \bar{\lambda}_2(p+p^*)t - \tfrac{3}{4}\lambda_3(p+p^*)^2 t, t)$$

$$\times \exp\left[-\tfrac{1}{2}\mathrm{i}(p+p^*)x - \frac{\mathrm{i}}{4}(p+p^*)^3\lambda_3 t + \frac{\mathrm{i}}{4}(p+p^*)^2\bar{\lambda}_2 t \right], \tag{4.42}$$

$$\bar{\lambda}_2 \equiv \lambda_2 + \tfrac{3}{2}(p+p^*),$$

eq. (4.41) can be simplified to

$$(\mathrm{i}\partial_t + \bar{\lambda}_2\partial_x^2 - \mathrm{i}\lambda_3\partial_x^3)a = \{(-\tfrac{1}{2}\bar{\lambda}_2 + \tfrac{3}{2}\mathrm{i}\lambda_3\partial_x)a\}\left\{\tfrac{1}{4}|p-p^*|^2|a|^2 + \frac{|\partial_x a|^2}{1-|a|^2}\right\}$$

$$+ \left\{(-\tfrac{1}{2}\bar{\lambda}_2 + \tfrac{3}{2}\mathrm{i}\lambda_3\partial_x)\frac{\partial_x a}{\sqrt{1-|a|^2}}\right\}\frac{\partial_x|\alpha|^2}{\sqrt{1-|a|^2}}. \tag{4.43}$$

In the special case $\lambda_3 = 0$, $\bar{\lambda}_2 = 1$ eq. (4.43) is equivalent to the AHSC

$$\partial_t \boldsymbol{S} = \boldsymbol{S} \times \partial_x^2 \boldsymbol{S} + \boldsymbol{S} \times (\tfrac{1}{4}|p-p^*|^2 S^z + b)\boldsymbol{e}^z, \quad \boldsymbol{S}\cdot\boldsymbol{S} = 1, \tag{4.44}$$

in which the polar angles of the spin vector $\boldsymbol{S}$, i.e.

$$\boldsymbol{S} = (\sin\theta\cos\alpha, \sin\theta\sin\alpha, \cos\theta), \tag{4.45}$$

can be expressed as, cf. eqs. (3.19) and (3.20) of ref. 17,

$$\theta = \arcsin|a|, \quad \partial_x\alpha = \frac{\operatorname{Im}\partial_x \ln a}{[1-|a|^2]^{1/2}},$$

$$\partial_t\alpha = \frac{\operatorname{Im}\partial_t \ln a}{[1-|a|^2]^{1/2}} - \frac{1}{2}\frac{|\partial_x a|^2}{[1-|a|^2]^{3/2}} - \frac{1}{8}|p-p^*|^2\frac{(2-|a|^2)}{[1-|a|^2]^{1/2}} - b. \tag{4.46}$$

The Miura transformation mapping a solution of the AHSC on a solution $\phi_{0,0}$ of the NLS has been worked out in ref. 17 and can be regarded as a direct generalization of the Lakshmanan result30) for the isotropic case. In the special case $\bar{\lambda}_2 = 0$, $\lambda_3 = 1$, eq. (4.43) may be regarded as a complex version of eq. (3.19) with $p = 0$.

4.5. *Integral equation and linear problem*

Defining

$$\begin{aligned} \boldsymbol{\phi}_k^{\pm} &= (p-k)\tilde{\boldsymbol{\phi}}_k \pm (p^*-k)\boldsymbol{\phi}_k, \\ \boldsymbol{\psi}_k^{\pm} &= (p-k)\tilde{\boldsymbol{\psi}}_k \pm (p^*-k)\boldsymbol{\psi}_k, \end{aligned} \tag{4.47}$$

with inverse

$$\begin{aligned} \tilde{\boldsymbol{\phi}}_k &= \frac{\boldsymbol{\phi}_k^+ + \boldsymbol{\phi}_k^-}{2(p-k)}, \qquad \boldsymbol{\phi}_k = \frac{\boldsymbol{\phi}_k^+ - \boldsymbol{\phi}_k^-}{2(p^*-k)}, \\ \tilde{\boldsymbol{\psi}}_k &= \frac{\boldsymbol{\psi}_k^+ + \boldsymbol{\psi}_k^-}{2(p-k)}, \qquad \boldsymbol{\psi}_k = \frac{\boldsymbol{\psi}_k^+ - \boldsymbol{\psi}_k^-}{2(p^*-k)}, \end{aligned} \tag{4.48}$$

and introducing a new measure $\mathrm{d}\lambda_1(k)$ by

$$\mathrm{d}\lambda_1(k) = \frac{\mathrm{d}\tilde{\lambda}(k)}{p-k} = \frac{\mathrm{d}\lambda(k)}{p^*-k}, \tag{4.49}$$

it is easy to show that

$$\begin{aligned} \boldsymbol{\Phi}^{\pm} &= \int_C \mathrm{d}\lambda_1(k) \boldsymbol{\phi}_k^{\pm} \boldsymbol{c}_k = \tilde{\boldsymbol{\Phi}} \pm \boldsymbol{\Phi}, \\ \boldsymbol{\Psi}^{\pm} &= \int_C \mathrm{d}\lambda_1(k) \boldsymbol{\psi}_k^{\pm} c_k = \tilde{\boldsymbol{\Psi}} \pm \boldsymbol{\Psi}. \end{aligned} \tag{4.50}$$

From (4.1) and the associated equation with $\boldsymbol{\phi}_k \to \tilde{\boldsymbol{\phi}}_k$, $\mathrm{d}\lambda(l) \to \mathrm{d}\tilde{\lambda}(l)$, it is straightforward to derive two coupled integral equations for $\boldsymbol{\phi}_k^+$ and $\boldsymbol{\phi}_k^-$. In fact, inserting (4.48) in (4.1) and the associated equation, and changing to the new measure (4.49), we have

$$\begin{aligned} &(\boldsymbol{\phi}_k^+ + \boldsymbol{\phi}_k^-) + (p-k) \int_C \mathrm{d}\lambda_1(l) \int_{C^*} \mathrm{d}\lambda_1^*(l') \frac{(p^*-l')\rho_k \rho_{l'}^*}{(k-l')(l'-l)} (\boldsymbol{\phi}_l^+ + \boldsymbol{\phi}_l^-) \\ &\quad = 2(p-k)\rho_k \boldsymbol{c}_k, \\ &(\boldsymbol{\phi}_k^+ - \boldsymbol{\phi}_k^-) + (p^*-k) \int_C \mathrm{d}\lambda_1(l) \int_{C^*} \mathrm{d}\lambda_1^*(l') \frac{(p-l')\rho_k \rho_{l'}^*}{(k-l')(l'-l)} (\boldsymbol{\phi}_l^+ - \boldsymbol{\phi}_l^-) \\ &\quad = 2(p^*-k)\rho_k \boldsymbol{c}_k, \end{aligned} \tag{4.51}$$

leading to

$$\begin{aligned} &\boldsymbol{\phi}_k^{\pm} + \int_C \mathrm{d}\lambda_1(l) \int_{C^*} \mathrm{d}\lambda_1^*(l') \frac{(|p|^2 + (k-p-p^*)l')\rho_k \rho_{l'}^*}{(k-l')(l'-l)} \boldsymbol{\phi}_l^{\pm} \\ &\qquad - \int_C \mathrm{d}\lambda_1(l) \int_{C^*} \mathrm{d}\lambda_1^*(l') \frac{\rho_k \rho_{l'}^*}{l'-l} [\tfrac{1}{2}(p^*+p)\boldsymbol{\phi}_l^{\pm} + \tfrac{1}{2}(p-p^*)\boldsymbol{\phi}_l^{\mp}] \\ &\quad = ((p-k) \pm (p^*-k))\rho_k \boldsymbol{c}_k. \end{aligned} \tag{4.52}$$

Eqs. (4.52) are two coupled linear integral equations and $\boldsymbol{\Phi}^-$, i.e. the solution of the modified matrix PDE, can be obtained from $\boldsymbol{\phi}_k^-$ integrating over the contour C, as in (4.50). These equations which were introduced in a slightly different notation in ref. 20, give a complete linearization of the AHSC (eq. (4.41) with $\lambda_3 = 0$, $\lambda_2 = 1$) as well as of other modified PDE's which may be derived for other choices of $\omega(k)$. (In this context we note that the bilinearization of the Heisenberg spin chain with orthorhombic anisotropy was given in ref. 31, and very recently the Riemann–Hilbert problem was settled by Mikhailov[32]).)

It is straightforward to derive the linear problem for $\phi^{-}_{0,0}$. From (4.5)–(4.7), (4.25), (4.26), (4.47) and (4.50) we have

$$\boldsymbol{\phi}_k^{+} = 2\mathrm{i}\partial_x \boldsymbol{\phi}_k + (p + p^*)\boldsymbol{\phi}_k + \boldsymbol{\Psi}^{-*}\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k + \boldsymbol{\Phi}^{+}\cdot\mathbf{O}\cdot\boldsymbol{\psi}_k\,, \tag{4.53}$$

$$\boldsymbol{\psi}_k^{-} = \boldsymbol{\Psi}^{-}\cdot\mathbf{O}\cdot\boldsymbol{\psi}_k - \boldsymbol{\Phi}^{-*}\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k\,. \tag{4.54}$$

Multiplying (4.53) and (4.54) by $(p^* - k)$ and using also the inverse relations with $p \leftrightarrow p^*$, $\boldsymbol{\phi}_k \leftrightarrow \tilde{\boldsymbol{\phi}}_k$, $\boldsymbol{\psi}_k \leftrightarrow \tilde{\boldsymbol{\psi}}_k$, $\boldsymbol{\Phi} \leftrightarrow \tilde{\boldsymbol{\Phi}}$. $\boldsymbol{\Psi} \leftrightarrow \tilde{\boldsymbol{\Psi}}$, it can easily be shown that

$$\begin{aligned} (p - p^*)\boldsymbol{\phi}_k^{+} &= 2\mathrm{i}\partial_x \boldsymbol{\phi}_k^{-} + (p + p^*)\boldsymbol{\phi}_k^{-} - \boldsymbol{\Psi}^{-*}\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k^{+} + \boldsymbol{\Phi}^{+}\cdot\mathbf{O}\cdot\boldsymbol{\psi}_k^{-}\,,\\ -2k\boldsymbol{\phi}_k^{+} &= 2\mathrm{i}\partial_x \boldsymbol{\phi}_k^{+} - \boldsymbol{\Psi}^{-*}\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k^{-} + \boldsymbol{\Phi}^{+}\cdot\mathbf{O}\cdot\boldsymbol{\psi}_k^{+}\,, \end{aligned} \tag{4.55}$$

and

$$\begin{aligned} (p - p^*)\boldsymbol{\psi}_k^{-} &= \boldsymbol{\Psi}^{-}\cdot\mathbf{O}\cdot\boldsymbol{\psi}_k^{-} - \boldsymbol{\Phi}^{-*}\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k^{-}\,,\\ -2k\boldsymbol{\psi}_k^{-} &= \boldsymbol{\Psi}^{-}\cdot\mathbf{O}\cdot\boldsymbol{\psi}_k^{+} - \boldsymbol{\Phi}^{-*}\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k^{+} - (p + p^*)\boldsymbol{\psi}_k^{-}\,. \end{aligned} \tag{4.56}$$

From (4.7) and (4.8) and their counterparts with $\boldsymbol{\phi}_k \to \tilde{\boldsymbol{\phi}}_k$ etc., it can be shown that

$$(-k - \mathrm{i}\partial_x)\boldsymbol{\phi}_k^{\mp} = \tfrac{1}{2}\boldsymbol{\Phi}^{+}\cdot\mathbf{O}\cdot\boldsymbol{\psi}_k^{\mp} + \tfrac{1}{2}\boldsymbol{\Phi}^{-}\cdot\mathbf{O}\cdot\boldsymbol{\psi}_k^{\pm}\,, \tag{4.57}$$

$$-\mathrm{i}\partial_x \boldsymbol{\psi}_k^{\mp} = \tfrac{1}{2}\boldsymbol{\Phi}^{+*}\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k^{\mp} + \tfrac{1}{2}\boldsymbol{\Phi}^{-*}\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k^{\pm}\,. \tag{4.58}$$

Using (4.56) one can solve the vectors $\mathbf{O}\cdot\boldsymbol{\psi}_k^{\mp}$ as linear combinations of $\mathbf{O}\cdot\boldsymbol{\phi}_k^{-}$ and $\mathbf{O}\cdot\boldsymbol{\phi}_k^{+}$. Inserting the result into (4.57) one obtains after some straightforward algebra

$$\begin{aligned} (-k - \mathrm{i}\partial_x)\boldsymbol{\phi}_k^{\mp} &= \tfrac{1}{2}\left[\frac{\psi^{-}_{0,0} - (p - p^*)}{\phi^{-}_{0,0}}\right]\boldsymbol{\Phi}^{\mp}\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k^{+}\\ &\quad + \tfrac{1}{2}\left[\frac{\psi^{-}_{0,0}}{\phi^{-}_{0,0}}\boldsymbol{\Phi}^{\pm} + \left(\frac{p + p^* - 2k}{\phi^{-}_{0,0}}\right)\boldsymbol{\Phi}^{\mp}\right]\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k^{-}\,, \end{aligned} \tag{4.59}$$

with $\psi^{-}_{0,0} = \tilde{\psi}_{0,0} - \psi_{0,0}$ given by (4.35) with $\tilde{\phi}_{0,0} - \phi_{0,0} \equiv \phi^{-}_{0,0}$.

Eq. (4.59) is independent of $\omega(k)$. For the special case (4.13) the time-dependent part of the linear problem of the AHSC can be inferred from (4.14) and its counterpart with $\boldsymbol{\Phi} \to \tilde{\boldsymbol{\Phi}}$, $\boldsymbol{\phi}_k \to \tilde{\boldsymbol{\phi}}_k$.

We have

$$
\begin{aligned}
&(\mathrm{i}\partial_t + \lambda_2\partial_x^2 - \mathrm{i}\lambda_3\partial_x^3)\boldsymbol{\phi}_k^{\mp} \\
&\quad = -\tfrac{1}{2}\lambda_2[(\boldsymbol{\Phi}^+\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{+*} + \boldsymbol{\Phi}^-\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{-*})\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k^{\mp} \\
&\qquad + (\boldsymbol{\Phi}^+\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{-*} + \boldsymbol{\Phi}^-\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{+*})\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k^{\pm}] \\
&\qquad + \tfrac{3}{4}\mathrm{i}\lambda_3[(\partial_x\boldsymbol{\Phi}^+)\cdot\mathbf{O}\cdot(\boldsymbol{\Phi}^{+*}\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k^{\mp} + \boldsymbol{\Phi}^{-*}\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k^{\pm}) \\
&\qquad + (\partial_x\boldsymbol{\Phi}^-)\cdot\mathbf{O}\cdot(\boldsymbol{\Phi}^{+*}\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k^{\pm} + \boldsymbol{\Phi}^{-*}\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k^{\mp}) \\
&\qquad + (\boldsymbol{\Phi}^+\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{+*} + \boldsymbol{\Phi}^-\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{-*})\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k^{\mp} \\
&\qquad + (\boldsymbol{\Phi}^+\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{-*} + \boldsymbol{\Phi}^-\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^{+*})\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k^{\pm}]\,.
\end{aligned}
\tag{4.60}
$$

It is also straightforward to derive the remaining constitutive relations for the matrix $\boldsymbol{\Phi}^-$, but the results will not be presented here.

4.6. *Bäcklund transformation for the AHSC*

From (4.51) it is clear that $\frac{1}{2}(\boldsymbol{\phi}_k^+ + \boldsymbol{\phi}_k^-)$ and $\frac{1}{2}(\boldsymbol{\phi}_k^+ - \boldsymbol{\phi}_k^-)$ satisfy an integral equation of the form (4.1) with $\mathrm{d}\lambda(l)$ replaced by $\mathrm{d}\lambda_1(l)$ and ρ_k replaced by $(p-k)\rho_k$ and $(p^*-k)\rho_k$, respectively. Similarly, it can be shown that $\frac{1}{2}(\boldsymbol{\psi}_k^+ + \boldsymbol{\psi}_k^-)$ and $\frac{1}{2}(\boldsymbol{\psi}_k^+ - \boldsymbol{\psi}_k^-)$ satisfy (4.4) with $\mathrm{d}\lambda(l)$ replaced by $\mathrm{d}\lambda_1(l)$ and ρ_k by $(p-k)\rho_k$ and $(p^*-k)\rho_k$.

We now apply the singular transformation of measures

$$
\mathrm{d}\lambda_1(k)\to\mathrm{d}\tilde{\lambda}_1(k) = \frac{q-k}{q^*-k}\,\mathrm{d}\lambda_1(k)\,, \tag{4.61}
$$

and introduce the functions $\frac{1}{2}(\tilde{\boldsymbol{\phi}}_k^+ \pm \tilde{\boldsymbol{\phi}}_k^-)$, $\frac{1}{2}(\tilde{\boldsymbol{\psi}}_k^+ \pm \tilde{\boldsymbol{\psi}}_k^-)$ that are the solutions of the integral equations mentioned above with $\mathrm{d}\lambda_1(l)$ replaced by $\mathrm{d}\tilde{\lambda}_1(l)$.

From (4.25) and (4.26) we immediately obtain the basic relations of the BT, viz.

$$
\begin{aligned}
&\tfrac{1}{2}(q-k)(\tilde{\boldsymbol{\phi}}_k^+ + \epsilon\tilde{\boldsymbol{\phi}}_k^-) = \tfrac{1}{2}q(\boldsymbol{\phi}_k^+ + \epsilon\boldsymbol{\phi}_k^-) - \tfrac{1}{2}\mathbf{J}^{\mathrm{T}}\cdot(\boldsymbol{\phi}_k^+ + \epsilon\boldsymbol{\phi}_k^-) \\
&\quad + \tfrac{1}{4}(\tilde{\boldsymbol{\Psi}}^{+*} + \epsilon\tilde{\boldsymbol{\Psi}}^{-*})\cdot\mathbf{O}\cdot(\boldsymbol{\phi}_k^+ + \epsilon\boldsymbol{\phi}_k^-) \\
&\quad + \tfrac{1}{4}(\tilde{\boldsymbol{\Phi}}^+ + \epsilon\tilde{\boldsymbol{\Phi}}^-)\cdot\mathbf{O}\cdot(\boldsymbol{\psi}_k^+ + \epsilon\boldsymbol{\psi}_k^-)\,,
\end{aligned}
\tag{4.62}
$$

$$
\begin{aligned}
&\tfrac{1}{2}(q-k)(\tilde{\boldsymbol{\psi}}_k^+ + \epsilon\tilde{\boldsymbol{\psi}}_k^-) = \tfrac{1}{2}q^*(\boldsymbol{\psi}_k^+ + \epsilon\boldsymbol{\psi}_k^-) \\
&\quad - \tfrac{1}{2}\mathbf{J}^{\mathrm{T}}\cdot(\boldsymbol{\psi}_k^+ + \epsilon\boldsymbol{\psi}_k^-) + \tfrac{1}{4}(\tilde{\boldsymbol{\Psi}}^+ + \epsilon\tilde{\boldsymbol{\Psi}}^-)\cdot\mathbf{O}\cdot(\boldsymbol{\psi}_k^+ + \epsilon\boldsymbol{\psi}_k^-) \\
&\quad - \tfrac{1}{4}(\tilde{\boldsymbol{\Phi}}^{+*} + \epsilon\tilde{\boldsymbol{\Phi}}^{-*})\cdot\mathbf{O}\cdot(\boldsymbol{\phi}_k^+ + \epsilon\boldsymbol{\phi}_k^-)\,, \quad \epsilon = \pm 1\,,
\end{aligned}
\tag{4.63}
$$

in which the matrices $\tilde{\boldsymbol{\Phi}}^{\pm}$ and $\tilde{\boldsymbol{\Psi}}^{\pm}$ are defined by

$$
\tilde{\boldsymbol{\Phi}}^{\pm} = \int\limits_C \mathrm{d}\tilde{\lambda}_1(k)\tilde{\boldsymbol{\phi}}_k^{\pm}\boldsymbol{c}_k, \quad \tilde{\boldsymbol{\Psi}}^{\pm} = \int\limits_C \mathrm{d}\tilde{\lambda}_1(k)\tilde{\boldsymbol{\psi}}_k^{\pm}\boldsymbol{c}_k\,. \tag{4.64}
$$

From (4.62) and (4.63) one can derive the spatial part of the BT for the PDE (4.41).

The result is given by

$$\begin{aligned}
&-\left\{\left(\frac{\partial_x\tilde{\phi}_{0,\bar{0}}-\frac{1}{2}\mathrm{i}(p+p^*)\tilde{\phi}_{0,\bar{0}}}{[|p-p^*|^2-4|\tilde{\phi}_{0,\bar{0}}|^2]^{1/2}}-\frac{\partial_x\phi_{0,\bar{0}}-\frac{1}{2}\mathrm{i}(p+p^*)\phi_{0,\bar{0}}}{[|p-p^*|^2-4|\phi_{0,\bar{0}}|^2]^{1/2}}\right)\left(\frac{1}{\tilde{\phi}_{0,\bar{0}}+\phi_{0,\bar{0}}}\right)\right.\\
&\qquad\times([|p-p^*|^2-4|\tilde{\phi}_{0,\bar{0}}|^2]^{1/2}+[|p-p^*|^2-4|\phi_{0,\bar{0}}|^2]^{1/2})\\
&\qquad\left.+\mathrm{i}(p+p^*-q-q^*)\left(\frac{\tilde{\phi}_{0,\bar{0}}-\phi_{0,\bar{0}}}{\tilde{\phi}_{0,\bar{0}}+\phi_{0,\bar{0}}}\right)\right\}^2\\
&\quad=(q^*-q)^2+|\tilde{\phi}_{0,\bar{0}}-\phi_{0,\bar{0}}|^2+\tfrac{1}{4}([|p-p^*|^2-4|\tilde{\phi}_{0,\bar{0}}|^2]^{1/2}\\
&\qquad-[|p-p^*|^2-4|\phi_{0,\bar{0}}|^2]^{1/2})^2\\
&\qquad+4\left|\frac{\partial_x\tilde{\phi}_{0,\bar{0}}-\frac{1}{2}\mathrm{i}(p+p^*)\tilde{\phi}_{0,\bar{0}}}{[|p-p^*|^2-4|\tilde{\phi}_{0,\bar{0}}|^2]^{1/2}}-\frac{\partial_x\phi_{0,\bar{0}}-\frac{1}{2}\mathrm{i}(p+p^*)\phi_{0,\bar{0}}}{[|p-p^*|^2-4|\phi_{0,\bar{0}}|^2]^{1/2}}\right|^2 .
\end{aligned}\tag{4.65}$$

Some details of the derivation are presented in appendix D. The BT for the AHSC is rather complicated and it does not seem to be easy to derive a second modified NLS equation in terms of only one function.

5. The sine-Gordon equation

In the preceding section we have described a general scheme, independent of the dispersion $\omega(k)$, of deriving Bäcklund transformations and modified PDE's, on the basis of a singular transformation of the measure in the integral equation associated with the original PDE's. As specific examples we have treated $\omega(k)=k^3$ in the case of the integral equation (2.1) for the KdV class and $\omega(k)=\lambda_2k^2+\lambda_3k^3$ in the case of the integral equation (4.1) for the NLS class. In this section we present some results on Bäcklund transformations and modified equations for $\omega(k)=k^{-1}$ starting from the integral equations (4.1) and (2.40) with $p=0$, for the NLS class and the MKdV class, respectively.

5.1. *The NLS class*

In the special case $\omega(k)=k^{-1}$ the function $\phi_{0,0}$, defined by (4.1) and by the (0, 0) element of (4.12) satisfies the PDE

$$\partial_x\partial_t\phi_{0,0}=\phi_{0,0}[1-4|\partial_t\phi_{0,0}|^2]^{1/2}\,,\tag{5.1}$$

cf. eqs. (6.3a) and (6.5) of ref. 18. In terms of the variable $\chi=2\partial_t\phi_{0,0}$, eq. (5.1) can be expressed as

$$\chi = \partial_t\left(\frac{\partial_x \chi}{[1-|\chi|^2]^{1/2}}\right), \tag{5.2}$$

which may be regarded as the complex sine-Gordon equation.

The spatial part of the BT of (5.1), cf. also refs. 33 and 34, is given by the (0, 0) element of (4.36), i.e.

$$\begin{aligned} 2\mathrm{i}\partial_x(\tilde{\phi}_{0,0} - \phi_{0,0}) = &-(p^* + p)(\tilde{\phi}_{0,0} - \phi_{0,0}) \\ &\mp \mathrm{i}(\tilde{\phi}_{0,0} + \phi_{0,0})[|p-p^*|^2 - 4|\tilde{\phi}_{0,0} - \phi_{0,0}|^2]^{1/2} . \end{aligned} \tag{5.3}$$

The time-dependent part of the BT can be inferred from (5.1), inserting (5.3) for $\partial_x(\tilde{\phi}_{0,0} - \phi_{0,0})$ in order to evaluate $\partial_x\partial_t(\tilde{\phi}_{0,0} - \phi_{0,0})$.

The modified PDE corresponding to (5.1) can be found inserting (4.39) into

$$\begin{aligned} \partial_x\partial_t\phi^{-}_{0,0} = &\tfrac{1}{2}(\phi^{+}_{0,0} + \phi^{-}_{0,0})[1 - |\partial_t(\phi^{+}_{0,0} + \phi^{-}_{0,0})|^2]^{1/2} \\ &- \tfrac{1}{2}(\phi^{+}_{0,0} - \phi^{-}_{0,0})[1 - |\partial_t(\phi^{+}_{0,0} - \phi^{-}_{0,0})|^2]^{1/2} . \end{aligned} \tag{5.4}$$

The result in terms of the variable

$$a(x,t) \equiv 2|p-p^*|^{-1}\phi^{-}_{0,0}\,\mathrm{e}^{-(\mathrm{i}/2)(p+p^*)x} \qquad (p^* \neq p), \tag{5.5}$$

can be expressed as

$$\begin{aligned} &\partial_x\partial_t a + \frac{\mathrm{i}}{2}(p+p^*)\partial_t a \\ &= \left[\tfrac{1}{2}a \mp \frac{|p-p^*|^{-1}\partial_x a}{[1-|a|^2]^{1/2}}\right]\left[1 - \left|\tfrac{1}{2}(p-p^*)\partial_t a \mp \partial_t\left(\frac{\partial_x a}{[1-|a|^2]^{1/2}}\right)\right|^2\right]^{1/2} \\ &\quad + \left[\tfrac{1}{2}a \pm \frac{|p-p^*|^{-1}\partial_x a}{[1-|a|^2]^{1/2}}\right]\left[1 - \left|\tfrac{1}{2}(p-p^*)\partial_t a \pm \partial_t\left(\frac{\partial_x a}{[1-|a|^2]^{1/2}}\right)\right|^2\right]^{1/2} . \end{aligned} \tag{5.6}$$

Eq. (4.65) is again the spatial part of the BT for the PDE for $\phi^{-}_{0,0}$.

5.2. *The MKdV class*

From (2.40), in the special case $p = 0$, we have the integral equation

$$\boldsymbol{v}_k + \int_C \mathrm{d}\lambda(l) \int_C \mathrm{d}\lambda(l') \frac{\rho_k \rho_{l'}}{(k+l')(l'+l)} \boldsymbol{v}_l = \rho_k \boldsymbol{c}_k, \quad \boldsymbol{v}_k \equiv -\frac{\boldsymbol{u}_k^{-}}{2k}, \tag{5.7}$$

which is the integral equation of type II of ref. 18. For the (0, 0) element of the matrix

$$\mathbf{V} = \int_C \mathrm{d}\lambda(l) \boldsymbol{v}_l \boldsymbol{c}_l , \tag{5.8}$$

we have, in the special case $\omega(k) = k^{-1}$, the PDE, cf. eq. (6.17) of ref. 18,

$$\partial_x \partial_t v_{0,0} = v_{0,0}[1 + 4(\partial_t v_{0,0})^2]^{1/2}\,, \tag{5.9}$$

Eq. (5.9) can be regarded as the potential sine-Gordon equation, since the substitution

$$\theta = \arcsin 2\mathrm{i}\partial_t v_{0,0} \tag{5.10}$$

gives the sine-Gordon equation

$$\partial_x \partial_t \theta = \sin\theta\,. \tag{5.11}$$

For the quantity $v_{0,0} = -\frac{1}{2} u_{0,0}^{-}$ we have the BT

$$\mathrm{i}\partial_x(\tilde{v}_{0,0} + v_{0,0}) = \mp(\tilde{v}_{0,0} - v_{0,0})[p^2 - (\tilde{v}_{0,0} + v_{0,0})^2]^{1/2}\,, \tag{5.12}$$

cf. (3.13) with $p = 0$ and the change of notation $q \rightarrow p$. For the quantity

$$w = \frac{1}{p}(\tilde{v}_{0,0} + v_{0,0}) \tag{5.13}$$

one can derive the modified PDE

$$\begin{aligned}\partial_x \partial_t w = {} & \left[\tfrac{1}{2} w \mp \frac{\frac{1}{2}\mathrm{i}p^{-1}\partial_x w}{[1 - w^2]^{1/2}}\right]\left[1 + \left[p\partial_t w \mp \mathrm{i}\partial_t\left(\frac{\partial_x w}{[1 - w^2]^{1/2}}\right)\right]^2\right]^{1/2} \\ & + \left[\tfrac{1}{2} w \pm \frac{\frac{1}{2}\mathrm{i}p^{-1}\partial_x w}{[1 - w^2]^{1/2}}\right]\left[1 + \left[p\partial_t w \pm \mathrm{i}\partial_t\left(\frac{\partial_x w}{[1 - w^2]^{1/2}}\right)\right]^2\right]^{1/2}, \end{aligned} \tag{5.14}$$

which, for imaginary p, may be regarded as a real counterpart of (5.6). Solutions of (5.14) can be found from the linear integral equation (3.27) with $p = 0$ and $q \rightarrow p$, noting that $w = -\frac{1}{2} p^{-1} u_{0,0}^{-+}$.

Instead of the BT for the potential sine-Gordon equation one can consider the BT for the SG equation (5.11) itself. From (5.9) and (5.12) one can show that

$$\begin{aligned} & \mathrm{i}v_{0,0}[1 + 4(\partial_t v_{0,0})^2]^{1/2} + \mathrm{i}\tilde{v}_{0,0}[1 + 4(\partial_t \tilde{v}_{0,0})^2]^{1/2} \\ & \quad = \mp\left[(\partial_t(\tilde{v}_{0,0} - v_{0,0}))[p^2 - (\tilde{v}_{0,0} + v_{0,0})^2]^{1/2}\right. \\ & \qquad \left. - \frac{\tilde{v}_{0,0}^2 - v_{0,0}^2}{[p^2 - (\tilde{v}_{0,0} + v_{0,0})^2]^{1/2}}\,\partial_t(\tilde{v}_{0,0} + v_{0,0})\right]. \end{aligned} \tag{5.15}$$

In terms of the variable θ, defined by (5.10), eq. (5.15) can be rewritten using (5.11) as

$$\begin{aligned} \cos\theta(\partial_x\theta) + \cos\tilde{\theta}(\partial_x\tilde{\theta}) = \pm\mathrm{i} & \left[(\sin\tilde{\theta} - \sin\theta)[p^2 + \tfrac{1}{4}(\partial_x(\tilde{\theta} + \theta))^2]^{1/2}\right. \\ & \left. + \frac{(\sin\tilde{\theta} + \sin\theta)[(\partial_x\tilde{\theta})^2 - (\partial_x\theta)^2]}{4[p^2 + \frac{1}{4}(\partial_x(\tilde{\theta} + \theta))^2]^{1/2}}\right]. \end{aligned} \tag{5.16}$$

Writing $2ip \sin\alpha \equiv \partial_x(\tilde{\theta}+\theta)$, eq. (5.16) can be worked out to give

$$[\tan\alpha \mp \tan\tfrac{1}{2}(\tilde{\theta}-\theta)][ip\cos\alpha \pm \tfrac{1}{2}\tan\tfrac{1}{2}(\tilde{\theta}+\theta)\partial_x(\tilde{\theta}-\theta)] \times \cos\tfrac{1}{2}(\tilde{\theta}+\theta)\cos\tfrac{1}{2}(\tilde{\theta}-\theta)=0\,, \tag{5.17}$$

whereas eq. (5.12) with (5.10) and (5.9) leads to

$$\partial_x^2(\tilde{\theta}+\theta) = \pm\, ip(\cos\alpha)\partial_x(\tilde{\theta}-\theta)\,. \tag{5.18}$$

The only consistent solution of (5.17) and (5.18) is $\alpha = \pm\frac{1}{2}(\tilde{\theta}-\theta)$, leading to

$$\partial_x(\tilde{\theta}+\theta) = \pm\, 2ip \sin\tfrac{1}{2}(\tilde{\theta}-\theta)\,. \tag{5.19}$$

From (5.19) together with (5.11) we have

$$\partial_t(\tilde{\theta}-\theta) = \mp\, 2ip^{-1}\sin\tfrac{1}{2}(\tilde{\theta}+\theta)\,. \tag{5.20}$$

Eqs. (5.19) and (5.20) form the well-known BT[1]) for the SG.

In terms of the variable $\alpha = \frac{1}{2}(\tilde{\theta}-\theta)$ eqs. (5.19) and (5.20) yield the PDE

$$\partial_x\partial_t\alpha = [1+p^2(\partial_t\alpha)^2]^{1/2}\sin\alpha\,, \tag{5.21}$$

which is the modified sine-Gordon equation (MSG). The solutions of (5.21) can be inferred from the solutions of the linear integral equation (3.27) with $p=0$ and $q\to p$. From (3.26) one obtains the function $u_{\bar{0},0}^{-+}$ leading with (3.16) and the 0,0 element of (3.14) to $u_{0,0}^{-}(=-2v_{0,0})$ and $\tilde{u}_{0,0}^{-}(=-2\tilde{v}_{0,0})$, and $\alpha = \pm\frac{1}{2}(\tilde{\theta}-\theta)$ is determined by (5.10). Multisoliton solutions were obtained by bilinearization in ref. 35.

Remark. From eq. (5.7) it can be shown that the integral equation (2.1) with dispersion $\omega(k)=k^{-1}$ yields solutions of the sine-Gordon equation as well as of a shallow water wave equation. Some details are presented in appendix E.

6. Bäcklund transformations for the wave functions

In the preceding sections we have given a general method to derive Bäcklund transformations for integrable PDE's using a singular transformation of measures in the corresponding linear integral equation. From this transformation one immediately obtains the basic Bäcklund relations containing both potentials and wave functions. Here the term wave functions is used to denote the solutions of the linear integral equations which depend on the (spectral) parameter k and which appear as eigenfunctions in the associated linear problems. The potentials denote the functions, obtained through integration of the wave functions over the contour C in the complex k-plane.

So far we have considered PDE's and BT's containing only the potentials, and which have been obtained after eliminating the wave functions by integration over the contour C. In the present section we will investigate PDE's and BT's in terms of only the wave functions, which in a number of cases can be derived by eliminating the potentials from the linear problem and the basic relations of the BT. Examples include e.g. the BT for the potential MKdV and the AHSC.

6.1. *The KdV class*

The linear problem for the KdV is given by the $n = 0$ components of eqs. (2.9) and (2.11). From eq (2.9), writing $a \equiv \ln u_k^{(0)}$, we have the relation

$$\partial_x u_{0,0} = \partial_x^2 a + (\partial_x a)^2 - \mathrm{i}k\partial_x a\,, \tag{6.1}$$

mapping the wave function $u_k^{(0)}$ on the potential $\partial_x u_{0,0}$ that satisfies the Korteweg–de Vries equation. Inserting (6.1) in (2.11) it is clear that $a = \ln u_k^{(0)}$ satisfies the PDE

$$(\partial_t - \partial_x^3)a = 3\mathrm{i}k(\partial_x a)^2 - 2(\partial_x a)^3\,, \tag{6.2}$$

which is equivalent to the potential MKdV, i.e. the PDE for $\partial_x a$ is equivalent to the MKdV.

From the basic relation of the BT (2.23), using also (2.3), it follows that

$$(p-k)\tilde{\boldsymbol{u}}_k = (p+k)\boldsymbol{u}_k + 2\mathrm{i}\partial_x \boldsymbol{u}_k + \mathrm{i}(\tilde{\mathbf{U}} - \mathbf{U})\cdot\mathbf{O}\cdot\boldsymbol{u}_k\,, \tag{6.3}$$

implying that

$$(p-k)\,\mathrm{e}^{(\tilde{a}-a)} = p + k + 2\mathrm{i}\partial_x a + \mathrm{i}(\tilde{u}_{0,0} - u_{0,0})\,. \tag{6.4}$$

From (6.4) and the inverse relation with $p \leftrightarrow -p$, $a \leftrightarrow \tilde{a}$, $u_{0,0} \leftrightarrow \tilde{u}_{0,0}$, it can be shown that

$$2\mathrm{i}\partial_x(a + \tilde{a}) = -2k + (p-k)\,\mathrm{e}^{(\tilde{a}-a)} - (p+k)\,\mathrm{e}^{-(\tilde{a}-a)}\,, \tag{6.5}$$

which is the spatial part of the BT for the potential MKdV (6.2).

Substituting $z = \tilde{a} - a$ in (6.5), we have

$$\partial_x a = -\tfrac{1}{2}\partial_x z + \tfrac{1}{2}\mathrm{i}k - \tfrac{1}{4}\mathrm{i}(p-k)\,\mathrm{e}^{z} + \tfrac{1}{4}\mathrm{i}(p+k)\,\mathrm{e}^{-z}\,. \tag{6.6}$$

From (6.2) and its counterpart with $a \to \tilde{a}$, and using (6.6), we obtain again the second modified KdV (3.21), i.e.

$$(\partial_t - \partial_x^3)z = -\tfrac{1}{2}(\partial_x z)^3 + \tfrac{3}{8}[(p-k)^2\,\mathrm{e}^{2z} + (p+k)^2\,\mathrm{e}^{-2z} - 2(p^2 + k^2)]\partial_x z\,. \tag{6.7}$$

6.2. *The NLS class*

For the function

$$u(x,t) \equiv -\phi_k^{(0)}(x,t)/\psi_k^{(0)}(x,t)\,, \tag{6.8}$$

defined in terms of the wave fucntions $\phi_k^{(0)}$ and $\psi_k^{(0)}$ of the linear problem of the NLS, cf. the $n=0$ components of (4.7), (4.8), (4.14) and (4.15), in the special case $\lambda_2=1$, $\lambda_3=0$, one derives the PDE

$$\begin{aligned}\partial_t u = {} & \mathrm{i}\partial_x\left(\frac{\partial_x u - u^2\partial_x u^*}{1-|u|^4}\right) - \mathrm{i}u^2\partial_x\left(\frac{\partial_x u^* - u^{*2}\partial_x u}{1-|u|^4}\right) \\ & + \frac{2\mathrm{i}u}{(1-|u|^4)^2}(\partial_x u - u^2\partial_x u^*)(\partial_x u^* - u^{*2}\partial_x u) \\ & + \frac{2(k+k^*)u^2\partial_x u^*}{(1-|u|^2)^2} + \frac{2\mathrm{i}(k+k^*)^2 u^3 u^{*2}}{(1-|u|^4)^2} + \frac{2\mathrm{i}kk^* u^2 u^*}{(1+|u|^2)^2}\,, \end{aligned} \tag{6.9}$$

cf. eq. (2.17) of ref. 17 with $\alpha=2$, apart from a misprint. In ref. 5 the invariance of (6.9) under $k \leftrightarrow k^*$ was used to obtain the BT for the NLS.

From the basic relations (4.25), (4.26) of the BT for the NLS, together with eqs. (4.5)–(4.8) and (6.8) we have, noting that $\psi_{0,0}$ is imaginary,

$$\mathrm{i}\partial_x u = (p-k)\tilde{u}\frac{\tilde{\psi}_k^{(0)}}{\psi_k^{(0)}} + (p-k)u\frac{\tilde{\psi}_k^{(0)}}{\psi_k^{(0)}} - (p^*-k)u - pu + \tilde{\phi}_{0,0} - \tilde{\phi}_{0,0}^* u^2\,. \tag{6.10}$$

Solving $\tilde{\psi}_k^{(0)}/\psi_k^{(0)}$ from (6.10) and inserting the result in the inverse relation with $p \leftrightarrow p^*$, $u \leftrightarrow \tilde{u}$, $\psi_k^{(0)} \leftrightarrow \tilde{\psi}_k^{(0)}$, $\phi_{0,0} \leftrightarrow \tilde{\phi}_{0,0}$, we find

$$\begin{aligned} & (\mathrm{i}\partial_x\tilde{u} + (p-k)\tilde{u} - \phi_{0,0} + \phi_{0,0}^*\tilde{u}^2 + p^*\tilde{u}) \\ & \quad \times (\mathrm{i}\partial_x u + (p^*-k)u - \tilde{\phi}_{0,0} + \tilde{\phi}_{0,0}^* u^2 + pu) \\ & = (p^*-k)(p-k)(\tilde{u}+u)^2\,. \end{aligned} \tag{6.11}$$

From eqs. (4.7) and (4.8) one also has the relation

$$\mathrm{i}\partial_x u = -ku + \phi_{0,0} - \phi_{0,0}^* u^2\,, \tag{6.12}$$

from which one can express $\phi_{0,0}$ in terms of u:

$$\phi_{0,0} = \frac{\mathrm{i}\partial_x u + ku - u^2(\mathrm{i}\partial_x u^* - k^*u^*)}{1-|u|^4}\,. \tag{6.13}$$

Inserting (6.13), its complex conjugate, and the inverse relations with $\phi_{0,0} \to \tilde{\phi}_{0,0}$, $u \to \tilde{u}$, in (6.11), we have

$$\begin{aligned} & \left[\mathrm{i}\partial_x\tilde{u} + (p-k)\tilde{u} + p^*\tilde{u} + \frac{(\mathrm{i}\partial_x u + ku)(-1+\tilde{u}^2u^{*2})}{1-|u|^4}\right. \\ & \quad \left. - \frac{(\mathrm{i}\partial_x u^* - k^*u^*)(\tilde{u}^2-u^2)}{1-|u|^4}\right] \times \left[\mathrm{i}\partial_x u + (p^*-k)u + pu\right. \\ & \quad \left. + \frac{(\mathrm{i}\partial_x\tilde{u} + k\tilde{u})(-1+u^2\tilde{u}^{*2})}{1-|\tilde{u}|^4} - \frac{(\mathrm{i}\partial_x\tilde{u}^* - k^*\tilde{u}^*)(u^2-\tilde{u}^2)}{1-|\tilde{u}|^4}\right] \\ & = (p^*-k)(p-k)(\tilde{u}+u)^2\,, \end{aligned} \tag{6.14}$$

which is the BT for the PDE (6.9).

In the special case $p^* = k$, eq. (6.11) leads to

$$\mathrm{i}\partial_x u = \tilde{\phi}_{0,0} - \tilde{\phi}^*_{0,0} u^2 - k^* u \,, \tag{6.15}$$

cf. eq. (2.19) of ref. 17. Eqs. (6.12) and (6.15) lead to

$$\phi_{\bar{0},0} = \frac{(p - p^*)u}{1 + |u|^2} \,, \tag{6.16}$$

showing that the PDE (6.9) with $k \to p^*$ is equivalent to eq. (4.41) for $\lambda_2 = 1$, $\lambda_3 = 0$, and thus to the AHSC (4.44), cf. ref. 17. Eq. (6.14) (cf. eq. (4.65)) is thus an alternative expression for the BT of the AHSC.

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Appendix A

In this appendix we argue that the transformation (2.18) can increase the number of solitons by 1, and we also discuss another way of obtaining the Bäcklund transformation.

N-soliton solutions can be obtained from the integral equation (2.1) together with (2.6), choosing as a measure a linear combination of N simple poles, i.e.

$$\mathrm{d}\lambda(k) = \frac{1}{2\pi\mathrm{i}} \sum_{\alpha=1}^{N} \frac{g_\alpha}{k - k_\alpha} \mathrm{d}k \,, \tag{A.1}$$

and a contour C enclosing $k_1, k_2, \ldots, k_N$. For an analytic function $f(k)$ we then have

$$\int_C f(k)\,\mathrm{d}\lambda(k) = \sum_{\alpha=1}^{N} g_\alpha f(k_\alpha) \,, \tag{A.2}$$

and (2.1) reduces to a set of N linear algebraic equations. (In view of (A.2) $\mathrm{d}\lambda(k)$ may also be regarded as a linear combination of delta functions $\mathrm{d}\lambda(k) = \Sigma_{\alpha=1}^{N} g_\alpha \delta(k - k_\alpha)\,\mathrm{d}k$, cf. ref. 15.)

Consider now the transformation (2.18) with $-p \neq k_\alpha$, then

$$\mathrm{d}\tilde{\lambda}(k) = \frac{1}{2\pi\mathrm{i}} \frac{\tilde{g}_p}{k + p} \mathrm{d}k + \frac{1}{2\pi\mathrm{i}} \sum_{\alpha=1}^{N} \frac{\tilde{g}_\alpha}{k - k_\alpha} \mathrm{d}k \,, \tag{A.3}$$

$$\tilde{g}_p = -2p \sum_{\alpha=1}^{N} \frac{g_\alpha}{p + k_\alpha}, \quad \tilde{g}_\alpha = \frac{g_\alpha(p - k_\alpha)}{p + k_\alpha}. \tag{A.4}$$

Eq. (A.3) is a linear combination of $N + 1$ simple poles, so that the new function $\tilde{\mathbf{U}}$, which can be found from (2.6) with the measure (A.3) is an $(N + 1)$-soliton solution.

In order to give a different, but equivalent, way of getting the BT, let us consider the singular transformation of the plane-wave factor

$$\rho_k \to \hat{\rho}_k = \frac{p - k}{p + k} \rho_k. \tag{A.5}$$

It is clear that $\hat{\rho}_k$ obeys the differential relations (2.2) as well and by defining a solution $\hat{\boldsymbol{u}}_k(x, t)$ of (2.1) with $\hat{\rho}_k$ instead of ρ_k, i.e.

$$\hat{\boldsymbol{u}}_k + \mathrm{i}\hat{\rho}_k \int_C \mathrm{d}\lambda(l) \frac{\hat{\boldsymbol{u}}_l}{k + l} = \hat{\rho}_k \boldsymbol{c}_k, \tag{A.6}$$

we obtain a solution

$$\hat{\mathbf{U}} = \int_C \mathrm{d}\lambda(k) \hat{\boldsymbol{u}}_k \boldsymbol{c}_k \tag{A.7}$$

of the matrix PDE (2.12). On the other hand, multiplying (A.6) by $(p + k)(p - k)^{-1}$, it is easy to show that $\hat{\boldsymbol{u}}_k(p + k)(p - k)^{-1}$ satisfies the integral equation (2.19). As a consequence we have

$$\hat{\boldsymbol{u}}_k = \frac{(p - k)}{(p + k)} \tilde{\boldsymbol{u}}_k. \tag{A.8}$$

Inserting (A.8) in (A.7) and using (2.18) and (2.20) we immediately obtain

$$\hat{\mathbf{U}} = \tilde{\mathbf{U}}, \tag{A.9}$$

implying that the Bäcklund transformation derived in subsection 2.4 can also be obtained as a result of the transformation (A.5) of the plane-wave factor ρ_k.

A similar result can be derived for the integral equation of the NLS type treated in section 4. One has the relations

$$\tilde{\mathbf{\Phi}} = \int_C \mathrm{d}\lambda(k) \hat{\boldsymbol{\phi}}_k \boldsymbol{c}_k, \quad \tilde{\mathbf{\Psi}} = \int_C \mathrm{d}\lambda(k) \hat{\boldsymbol{\psi}}_k \boldsymbol{c}_k, \tag{A.10}$$

in which $\hat{\boldsymbol{\phi}}_k$ and $\hat{\boldsymbol{\psi}}_k$ are the solutions of (4.2) and (4.3) with

$$\rho_k \to \hat{\rho}_k = \frac{p - k}{p^* - k} \rho_k. \tag{A.11}$$

Appendix B

Eq. (2.32) is most easily derived using two relations of the linear problem in subsection 2.6. From the integrated version of (2.48) we have

$$\mathrm{i}\partial_x u^-_{1,1} = -\,u^-_{1,0} + \tfrac{1}{2}\mathrm{i}u^-_{1,0}u^+_{0,1}\,, \tag{B.1}$$

$$\mathrm{i}\partial_x u^-_{0,1} = -\,u^-_{0,0} + \tfrac{1}{2}\mathrm{i}u^-_{0,0}u^+_{0,1}\,, \tag{B.2}$$

from which

$$u^-_{0,0} = \frac{\partial_x (u^-_{1,0})^2}{2\partial_x u^-_{1,1}}\,. \tag{B.3}$$

From the integrated version of (2.41), using also (B.1) it follows that

$$2pu^-_{1,1} = \tfrac{1}{2}\mathrm{i}(u^+_{1,0} + 2\mathrm{i})^2 - \tfrac{1}{2}\mathrm{i}(u^-_{1,0})^2 + 2\mathrm{i} = \frac{2\mathrm{i}(\partial_x u^-_{1,1})^2}{(u^-_{1,0})^2} - \tfrac{1}{2}\mathrm{i}(u^-_{1,0})^2 + 2\mathrm{i}\,, \tag{B.4}$$

from which one can solve

$$(u^-_{1,0})^2 = 2 + 2\mathrm{i}pu^-_{1,1} \pm 2[(1 + \mathrm{i}pu^-_{1,1})^2 + (\partial_x u^-_{1,1})^2]^{1/2}\,. \tag{B.5}$$

Substituting (B.3) and (B.5) in the (1, 1) element of (2.30), i.e.

$$(\partial_t - \partial_x^3)u^-_{1,1} = (3\mathrm{i}pu^-_{1,0} - \tfrac{3}{2}u^-_{1,0}u^-_{0,0})\partial_x u^-_{0,1}\,, \tag{B.6}$$

one obtains

$$(\partial_t - \partial_x^3)u^-_{1,1} = -\,\tfrac{3}{2}(\partial_x u^-_{1,1})^{-1}[\partial_x[(1 + \mathrm{i}pu^-_{1,1})^2 + (\partial_x u^-_{1,1})^2]^{1/2}]^2 - \tfrac{3}{2}p^2\partial_x u^-_{1,1}\,. \tag{B.7}$$

Note that eq. (B.7) can also be derived from the Bäcklund transformation for eq. (2.15)

$$(1 - 2\partial_x\tilde{u}_{1,1})(1 - 2\partial_x u_{1,1}) = [1 + \mathrm{i}p(\tilde{u}_{1,1} - u_{1,1})]^2\,, \tag{B.8}$$

which can be inferred from the (1,1) element of (2.29), using also (2.17).

It is now easy to show that the quantity

$$b \equiv 2\partial_x \ln\left(u^-_{1,1} - \frac{\mathrm{i}}{p}\right) \tag{B.9}$$

obeys the following PDE:

$$(\partial_t - \partial_x^3)b = \tfrac{3}{2}\partial_x\left[\frac{b(\partial_x b)^2}{4p^2 - b^2} - \frac{1}{12}b^3\right], \tag{B.10}$$

and eq. (2.32) is obtained substituting $b = 2\mathrm{i}p \sinh z$.

Appendix C

In this appendix we give the derivation of the Bäcklund transformation (3.41) for the PDE (3.19). As a first step, eq. (3.39) is rewritten

$$\begin{aligned}(r-k)\tilde{\boldsymbol{u}}_k^{+\alpha} &= r\boldsymbol{u}_k^{+\alpha} - \mathbf{J}^{\mathrm{T}}\cdot\boldsymbol{u}_k^{+\alpha} + \tfrac{1}{4}\mathrm{i}[\tilde{\mathbf{U}}^{++}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{+\alpha} \\ &\quad + \tilde{\mathbf{U}}^{+-}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{+\bar{\alpha}} + \tilde{\mathbf{U}}^{-+}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{-\alpha} + \tilde{\mathbf{U}}^{--}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{-\bar{\alpha}}]\,, \qquad \text{(C.1)}\end{aligned}$$

$$\begin{aligned}(r-k)\tilde{\boldsymbol{u}}_k^{-\alpha} &= r\boldsymbol{u}_k^{-\alpha} - \mathbf{J}^{\mathrm{T}}\cdot\boldsymbol{u}_k^{-\alpha} + \tfrac{1}{4}\mathrm{i}[\tilde{\mathbf{U}}^{++}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{-\alpha} + \tilde{\mathbf{U}}^{+-}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{-\bar{\alpha}} \\ &\quad + \tilde{\mathbf{U}}^{-+}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{+\alpha} + \tilde{\mathbf{U}}^{--}\cdot\mathbf{O}\cdot\boldsymbol{u}_k^{+\bar{\alpha}}]\,. \qquad \text{(C.2)}\end{aligned}$$

Integrating (C.1) over $\mathrm{d}\lambda_2(k)$, and using the relation $(r-k)\,\mathrm{d}\lambda_2(k) = (r+k)\,\mathrm{d}\tilde{\lambda}_2(k)$ to evaluate the left-hand side, we have

$$\begin{aligned}r(\tilde{\mathbf{U}}^{++} - \mathbf{U}^{++}) &= -\mathbf{J}^{\mathrm{T}}\mathbf{U}^{++} - \tilde{\mathbf{U}}^{++}\cdot\mathbf{J} + \tfrac{1}{4}\mathrm{i}[\tilde{\mathbf{U}}^{++}\cdot\mathbf{O}\cdot\mathbf{U}^{++} \\ &\quad + \tilde{\mathbf{U}}^{+-}\cdot\mathbf{O}\cdot\mathbf{U}^{+-} + \tilde{\mathbf{U}}^{-+}\cdot\mathbf{O}\cdot\mathbf{U}^{-+} + \tilde{\mathbf{U}}^{--}\cdot\mathbf{O}\cdot\mathbf{U}^{--}]\,. \qquad \text{(C.3)}\end{aligned}$$

Using (C.3), its inverse with $r \leftrightarrow -r$, $\mathbf{U}^{\lambda\mu} \leftrightarrow \tilde{\mathbf{U}}^{\lambda\mu}$, and the relation

$$\begin{aligned}\mathbf{J}^{\mathrm{T}}\cdot\mathbf{U}^{++} + \mathbf{U}^{++}\cdot\mathbf{J} &= -2p\mathbf{U}^{-+} + \tfrac{1}{4}\mathrm{i}[\mathbf{U}^{++}\cdot\mathbf{O}\cdot\mathbf{U}^{++} + \mathbf{U}^{+-}\cdot\mathbf{O}\cdot\mathbf{U}^{+-} \\ &\quad - \mathbf{U}^{-+}\cdot\mathbf{O}\cdot\mathbf{U}^{-+} - \mathbf{U}^{--}\cdot\mathbf{O}\cdot\mathbf{U}^{--}]\,, \qquad \text{(C.4)}\end{aligned}$$

cf. eqs. (3.8) and (3.14), we find

$$\begin{aligned}2r(\tilde{\mathbf{U}}^{++} - \mathbf{U}^{++}) &= 2p(\tilde{\mathbf{U}}^{-+} + \mathbf{U}^{-+}) - \tfrac{1}{4}\mathrm{i}(\tilde{\mathbf{U}}^{++} - \mathbf{U}^{++})\cdot\mathbf{O}\cdot(\tilde{\mathbf{U}}^{++} - \mathbf{U}^{++}) \\ &\quad - \tfrac{1}{4}\mathrm{i}(\tilde{\mathbf{U}}^{+-} - \mathbf{U}^{+-})\cdot\mathbf{O}\cdot(\tilde{\mathbf{U}}^{+-} - \mathbf{U}^{+-}) \\ &\quad + \tfrac{1}{4}\mathrm{i}(\tilde{\mathbf{U}}^{-+} + \mathbf{U}^{-+})\cdot\mathbf{O}\cdot(\tilde{\mathbf{U}}^{-+} + \mathbf{U}^{-+}) \\ &\quad + \tfrac{1}{4}\mathrm{i}(\tilde{\mathbf{U}}^{--} + \mathbf{U}^{--})\cdot\mathbf{O}\cdot(\tilde{\mathbf{U}}^{--} + \mathbf{U}^{--})\,. \qquad \text{(C.5)}\end{aligned}$$

From (C.2) we obtain after integration over $\mathrm{d}\lambda_2(k)$

$$\begin{aligned}r(\tilde{\mathbf{U}}^{-+} - \mathbf{U}^{-+}) &= -\mathbf{J}^{\mathrm{T}}\cdot\mathbf{U}^{-+} - \tilde{\mathbf{U}}^{-+}\cdot\mathbf{J} + \tfrac{1}{4}\mathrm{i}[\tilde{\mathbf{U}}^{++}\cdot\mathbf{O}\cdot\mathbf{U}^{-+} \\ &\quad + \tilde{\mathbf{U}}^{+-}\cdot\mathbf{O}\cdot\mathbf{U}^{--} + \tilde{\mathbf{U}}^{-+}\cdot\mathbf{O}\cdot\mathbf{U}^{++} + \tilde{\mathbf{U}}^{--}\cdot\mathbf{O}\cdot\mathbf{U}^{+-}]\,, \qquad \text{(C.6)}\end{aligned}$$

and from (C.6), the inverse relation with $r \leftrightarrow -r$, $\mathbf{U}^{\lambda\mu} \leftrightarrow \tilde{\mathbf{U}}^{\lambda\mu}$, and the relation

$$\begin{aligned}\mathbf{J}^{\mathrm{T}}\cdot\mathbf{U}^{-+} + \mathbf{U}^{-+}\cdot\mathbf{J} &= -2\mathrm{i}\partial_x\mathbf{U}^{-+} + \tfrac{1}{4}\mathrm{i}[\mathbf{U}^{-+}\cdot\mathbf{O}\cdot\mathbf{U}^{++} + \mathbf{U}^{--}\cdot\mathbf{O}\cdot\mathbf{U}^{+-} \\ &\quad + \mathbf{U}^{++}\cdot\mathbf{O}\cdot\mathbf{U}^{-+} + \mathbf{U}^{+-}\cdot\mathbf{O}\cdot\mathbf{U}^{--}]\,, \qquad \text{(C.7)}\end{aligned}$$

cf. (3.7) and (3.14), it can be shown that

$$\begin{aligned}2\mathrm{i}\partial_x(\tilde{\mathbf{U}}^{-+}+\mathbf{U}^{-+}) &= 2r(\tilde{\mathbf{U}}^{-+}-\mathbf{U}^{-+})+\tfrac{1}{2}\mathrm{i}[(\tilde{\mathbf{U}}^{-+}-\mathbf{U}^{-+})\cdot\mathbf{O}\cdot(\tilde{\mathbf{U}}^{++}\\ &\quad -\mathbf{U}^{++})]^{\mathrm{s}}+\tfrac{1}{2}\mathrm{i}[(\tilde{\mathbf{U}}^{--}-\mathbf{U}^{--})\cdot\mathbf{O}\cdot(\tilde{\mathbf{U}}^{+-}-\mathbf{U}^{+-})]^{\mathrm{s}}\,.\end{aligned} \tag{C.8}$$

In a similar way one can derive two relations for $\tilde{\mathbf{U}}^{+-}-\mathbf{U}^{+-}$ and $\tilde{\mathbf{U}}^{--}-\mathbf{U}^{--}$, but we shall not give them here.

From the (0, 0) element of (C.8), using (3.16) and (3.35) for $\mathrm{i}\partial_x u_{0,0}^{-+}$ and $u_{0,0}^{+-}$, respectively, one has

$$\begin{aligned}\tfrac{1}{2}\mathrm{i}(\tilde{u}_{0,0}^{++}-u_{0,0}^{++})+2r &= \pm\frac{(\tilde{u}_{0,0}^{--}+u_{0,0}^{--})}{2(\tilde{u}_{0,0}^{-+}-u_{0,0}^{-+})}\{[4(q^2-p^2)-(\tilde{u}_{0,0}^{-+}-2p\mathrm{i})^2]^{1/2}\\ &\quad +[4(q^2-p^2)-(u_{0,0}^{-+}-2p\mathrm{i})^2]^{1/2}\}\,,\end{aligned} \tag{C.9}$$

and eq. (3.41) can be derived inserting (C.9) and (3.35) in (C.5).

Appendix D

In this appendix we derive eq. (4.65). From (4.62) and (4.63) we obtain the relations

$$\begin{aligned}(q-k)\tilde{\phi}_k^{\pm} &= q\phi_k^{\pm}-\mathbf{J}^{\mathrm{T}}\cdot\phi_k^{\pm}+\tfrac{1}{2}(\tilde{\mathbf{\Psi}}^{+*}\cdot\mathbf{O}\cdot\phi_k^{\pm}+\tilde{\mathbf{\Psi}}^{-*}\cdot\mathbf{O}\cdot\phi_k^{\mp}\\ &\quad +\tilde{\mathbf{\Phi}}^{+}\cdot\mathbf{O}\cdot\psi_k^{\pm}+\tilde{\mathbf{\Phi}}^{-}\cdot\mathbf{O}\cdot\psi_k^{\mp})\,,\end{aligned} \tag{D.1}$$

$$\begin{aligned}(q-k)\tilde{\psi}_k^{\pm} &= q^*\psi_k^{\pm}-\mathbf{J}^{\mathrm{T}}\cdot\psi_k^{\pm}+\tfrac{1}{2}(\tilde{\mathbf{\Psi}}^{+}\cdot\mathbf{O}\cdot\psi_k^{\pm}+\tilde{\mathbf{\Psi}}^{-}\cdot\mathbf{O}\cdot\psi_k^{\mp}\\ &\quad -\tilde{\mathbf{\Phi}}^{+*}\cdot\mathbf{O}\cdot\phi_k^{\pm}-\tilde{\mathbf{\Phi}}^{-*}\cdot\mathbf{O}\cdot\phi_k^{\mp})\,.\end{aligned} \tag{D.2}$$

Integrating (D.1) over $\mathrm{d}\lambda_1(k)$ and using $(q-k)\,\mathrm{d}\lambda_1(k)=(q^*-k)\,\mathrm{d}\tilde{\lambda}_1(k)$ to evaluate the left-hand side, we find

$$\begin{aligned}q^*\tilde{\mathbf{\Phi}}^{-}-q\mathbf{\Phi}^{-} &= -(\mathbf{J}^{\mathrm{T}}\cdot\mathbf{\Phi}^{-}-\tilde{\mathbf{\Phi}}^{-}\cdot\mathbf{J})+\tfrac{1}{2}(\tilde{\mathbf{\Psi}}^{-*}\cdot\mathbf{O}\cdot\mathbf{\Phi}^{+}\\ &\quad +\tilde{\mathbf{\Psi}}^{+*}\cdot\mathbf{O}\cdot\mathbf{\Phi}^{-}+\tilde{\mathbf{\Phi}}^{-}\cdot\mathbf{O}\cdot\mathbf{\Psi}^{+}+\tilde{\mathbf{\Phi}}^{+}\cdot\mathbf{O}\cdot\mathbf{\Psi}^{-})\,.\end{aligned} \tag{D.3}$$

From (D.3) and the relation, cf. (4.31) and (4.50),

$$\begin{aligned}\mathbf{J}^{\mathrm{T}}\cdot\mathbf{\Phi}^{-}+\mathbf{\Phi}^{-}\cdot\mathbf{J} &= -2\mathrm{i}\partial_x\mathbf{\Phi}^{-}-\tfrac{1}{2}(\mathbf{\Phi}^{-}\cdot\mathbf{O}\cdot\mathbf{\Psi}^{+}+\mathbf{\Phi}^{+}\cdot\mathbf{O}\cdot\mathbf{\Psi}^{-})\\ &\quad +\tfrac{1}{2}(\mathbf{\Psi}^{-*}\cdot\mathbf{O}\cdot\mathbf{\Phi}^{+}+\mathbf{\Psi}^{+*}\cdot\mathbf{O}\cdot\mathbf{\Phi}^{-})\,.\end{aligned} \tag{D.4}$$

and the inverse relations with $q\leftrightarrow q^*$, $\mathbf{\Phi}^{\pm}\leftrightarrow\tilde{\mathbf{\Phi}}^{\pm}$, $\mathbf{\Psi}^{\pm}\leftrightarrow\tilde{\mathbf{\Psi}}^{\pm}$ it can be shown that

$$\begin{aligned}2(q^*\tilde{\mathbf{\Phi}}^- - q\mathbf{\Phi}^-) = &- 2\mathrm{i}\partial_x(\tilde{\mathbf{\Phi}}^- - \mathbf{\Phi}^-) - \tfrac{1}{2}(\tilde{\mathbf{\Phi}}^- + \mathbf{\Phi}^-)\cdot\mathbf{O}\cdot(\tilde{\mathbf{\Psi}}^+ - \mathbf{\Psi}^+)\\ &- \tfrac{1}{2}(\tilde{\mathbf{\Phi}}^+ + \mathbf{\Phi}^+)\cdot\mathbf{O}\cdot(\tilde{\mathbf{\Psi}}^- - \mathbf{\Psi}^-)\\ &+ \tfrac{1}{2}(\tilde{\mathbf{\Psi}}^{-*} - \mathbf{\Psi}^{-*})\cdot\mathbf{O}\cdot(\tilde{\mathbf{\Phi}}^+ + \mathbf{\Phi}^+)\\ &+ \tfrac{1}{2}(\tilde{\mathbf{\Psi}}^{+*} - \mathbf{\Psi}^{+*})\cdot\mathbf{O}\cdot(\tilde{\mathbf{\Phi}}^- + \mathbf{\Phi}^-)\,. \qquad \text{(D.5)}\end{aligned}$$

Integrating (D.2) over $\mathrm{d}\lambda_1(k)$ we have

$$\begin{aligned}q^*(\tilde{\mathbf{\Psi}}^+ - \mathbf{\Psi}^+) = &- (\mathbf{J}^{\mathrm{T}}\cdot\mathbf{\Psi}^+ - \tilde{\mathbf{\Psi}}^+\cdot\mathbf{J}) + \tfrac{1}{2}(\tilde{\mathbf{\Psi}}^-\cdot\mathbf{O}\cdot\mathbf{\Psi}^- + \tilde{\mathbf{\Psi}}^+\cdot\mathbf{O}\cdot\mathbf{\Psi}^+)\\ &- \tfrac{1}{2}(\tilde{\mathbf{\Phi}}^{-*}\cdot\mathbf{O}\cdot\mathbf{\Phi}^- + \tilde{\mathbf{\Phi}}^{+*}\cdot\mathbf{O}\cdot\mathbf{\Phi}^+)\,. \qquad \text{(D.6)}\end{aligned}$$

From (D.5) and the relation, cf. (4.32) and (4.50),

$$\begin{aligned}\mathbf{\Psi}^+\cdot\mathbf{J} - \mathbf{J}^{\mathrm{T}}\cdot\mathbf{\Psi}^+ = &- \tfrac{1}{2}(\mathbf{\Psi}^-\cdot\mathbf{O}\cdot\mathbf{\Psi}^- + \mathbf{\Psi}^+\cdot\mathbf{O}\cdot\mathbf{\Psi}^+)\\ &+ \tfrac{1}{2}(\mathbf{\Phi}^{-*}\cdot\mathbf{O}\cdot\mathbf{\Phi}^- + \mathbf{\Phi}^{+*}\cdot\mathbf{O}\cdot\mathbf{\Phi}^+)\,, \qquad \text{(D.7)}\end{aligned}$$

in combination with the inverse relations, it can be shown that

$$\begin{aligned}(q^* - q)(\tilde{\mathbf{\Psi}}^+ - \mathbf{\Psi}^+) = &- \tfrac{1}{2}(\tilde{\mathbf{\Psi}}^- - \mathbf{\Psi}^-)\cdot\mathbf{O}\cdot(\tilde{\mathbf{\Psi}}^- - \mathbf{\Psi}^-)\\ &- \tfrac{1}{2}(\tilde{\mathbf{\Psi}}^+ - \mathbf{\Psi}^+)\cdot\mathbf{O}\cdot(\tilde{\mathbf{\Psi}}^+ - \mathbf{\Psi}^+)\\ &+ \tfrac{1}{2}(\tilde{\mathbf{\Phi}}^{-*} - \mathbf{\Phi}^{-*})\cdot\mathbf{O}\cdot(\tilde{\mathbf{\Phi}}^- - \mathbf{\Phi}^-)\\ &+ \tfrac{1}{2}(\tilde{\mathbf{\Phi}}^{+*} - \mathbf{\Phi}^{+*})\cdot\mathbf{O}\cdot(\tilde{\mathbf{\Phi}}^+ - \mathbf{\Phi}^+)\,. \qquad \text{(D.8)}\end{aligned}$$

Eq. (4.65) can now be derived solving the imaginary quantity $\tilde{\psi}^+_{0,0} - \psi^+_{0,0} + q^* - q$ from (D.5) and inserting the result, as well as (4.35) and (4.39) in the $(0, 0)$ element of (D.8).

Appendix E

In this appendix we discuss the relation between the integral equation (2.1) in the case $\omega(k) = k^{-1}$, and the sine-Gordon equation and a shallow water wave equation.

Consider the Miura transformation which maps the function $v_{0,0}$ defined by (5.7) and (5.8), on the function $u_{0,0}$ defined by (2.1) and (2.6), i.e.

$$\partial_x u_{0,0} = \partial_x v_{0,0} + v_{0,0}^2\,, \qquad \text{(E.1)}$$

cf. eq. (2.29) with $u^-_{0,0} = -2v_{0,0}$ and $p = 0$.

Using eqs. (7.5), (3.26b) and (3.14b) for $p = -1$, and (6.16), with the minus sign, of ref. 18, it can be shown that

$$\partial_t u_{0,0} = \partial_t v_{0,0} - \tfrac{1}{2} + \tfrac{1}{2}[1 + 4(\partial_t v_{0,0})^2]^{1/2}\,. \qquad \text{(E.2)}$$

From (E.2), together with (5.9), we have

$$v_{0,0} = \frac{\partial_x \partial_t u_{0,0}}{1 + 2\partial_t u_{0,0}} . \tag{E.3}$$

Solving $\partial_t v_{0,0}$ from (E.2) and using (E.3) one can derive

$$\partial_t \left(\frac{\partial_x \partial_t u_{0,0}}{1 + 2\partial_t u_{0,0}} \right) = \frac{(\partial_t u_{0,0})^2 + \partial_t u_{0,0}}{1 + 2\partial_t u_{0,0}} . \tag{E.4}$$

Eq. (E.4) can be expressed as

$$2\partial_x \partial_t \ln[1 + 2\partial_t u_{0,0}] = [1 + 2\partial_t u_{0,0}] - \frac{1}{[1 + 2\partial_t u_{0,0}]} , \tag{E.5}$$

which is equivalent to the sine-Gordon equation (SG).

From (E.5) it can also be shown that the matrix element $u_{1,0}$, defined by (2.6), satisfies a PDE which is equivalent to the sine-Gordon equation. In fact, integrating (2.4) with $\mathbf{R} = \mathbf{J}^{\mathrm{T}} \cdot \mathbf{O} \cdot \mathbf{J}$, and taking the (0, 0) element, we immediately obtain the relation

$$2\mathrm{i}\partial_t u_{0,0} = 2u_{1,0} - \mathrm{i}u_{1,0}^2 . \tag{E.6}$$

The same fact can also be seen in the following way. Using eqs. (7.5) and (6.15) of ref. 18, we obtain the relation

$$u_{1,0} = v_{1,0} - \mathrm{i} + \mathrm{i}[1 - v_{1,0}^2]^{1/2} , \tag{E.7}$$

showing that the PDE's for $u_{1,0}$ and $v_{1,0}$ are equivalent, independent of the dispersion $\omega(k)$. For the special case $\omega(k) = k^{-1}$, $v_{1,0}$ satisfies an equation equivalent to the sine-Gordon equation, as follows from eq. (6.16) of ref. 18.

On the other hand, from (E.1) together with (E.3) one has

$$\partial_x \left(\frac{\partial_x \partial_t u_{0,0}}{1 + 2\partial_t u_{0,0}} \right) = \partial_x u_{0,0} - \left(\frac{\partial_x \partial_t u_{0,0}}{1 + 2\partial_t u_{0,0}} \right)^2 . \tag{E.8}$$

Substituting $s \equiv u_{0,0} + \frac{1}{2}t$, eq. (E.8) can be rewritten as

$$\tfrac{1}{2}(\partial_x^2 \partial_t s)(\partial_t s) = \tfrac{1}{4}(\partial_x \partial_t s)^2 + (\partial_t s)^2 (\partial_x s) , \tag{E.9}$$

which after differentiation with respect to x gives the shallow water wave equation$^{36,37})$

$$\tfrac{1}{2}\partial_x^3 \partial_t s = (\partial_x^2 s)(\partial_t s) + 2(\partial_x \partial_t s)(\partial_x s) . \tag{E.10}$$

Therefore, from the linear integral equation (2.1) with $\omega(k) = k^{-1}$ one can obtain solutions of the SG (E.5), as well as of the shallow water wave equation (E.10). This is not surprising, since on general grounds it can be shown that any solution of the SG leads to a solution of the shallow water wave equation. For that purpose we consider the equation

$$\frac{\partial_x\partial_t w}{w} - \frac{(\partial_t w)(\partial_x w)}{w^2} = w + \frac{g(t) - \frac{1}{4}}{w}, \tag{E.11}$$

where we have included an arbitrary function $g(t)$. For $g(t) = 0$, eq. (E.11) reduces to the SG eq. (E.5) with $w = \frac{1}{2} + \partial_t u_{0,0}$, and in the case $g(t) = \frac{1}{4}$, eq. (E.11) is equivalent to the Liouville equation $\partial_x\partial_t p = e^p$.

It is now easy to show that any solution of (E.11) for arbitrary $g(t)$ leads to a solution of (E.9). In fact, from (E.11) we have

$$\partial_x^2\partial_t w - \frac{(\partial_t w)(\partial_x^2 w)}{w} - \frac{(\partial_x\partial_t w)\partial_x w}{w} + \frac{(\partial_t w)(\partial_x w)^2}{w^2} = 2w\partial_x w\,, \tag{E.12}$$

and (E.12) is the compatibility condition for a variable $s(x, t)$ defined by

$$\partial_t s = w\,, \tag{E.13}$$

$$\partial_x s = \frac{1}{2}\frac{\partial_x^2 w}{w} - \frac{1}{4}\frac{(\partial_x w)^2}{w^2}\,. \tag{E.14}$$

Inserting (E.13) in the right-hand side of (E.14), one immediately obtains (E.9).

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