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FINITE-TEMPERATURE CORRELATIONS FOR THE ISING CHAIN IN A TRANSVERSE FIELD

J.H.H. PERK

Institute for Theoretical Physics, State University of New York at Stony Brook, NY 11794, USA

and

H.W. CAPEL, G.R.W. QUISPEL and F.W. NIJHOFF

Instituut-Lorentz voor Theoretische Natuurkunde, Nieuwsteeg 18, 2311 SB Leiden, The Netherlands

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Two sets of nonlinear differential equations are derived and discussed for the time-dependent correlations between x-components of spins ($S = 1/2$) in an Ising chain in the presence of a transverse magnetic field. The equations are independent of temperature which enters only through the initial conditions for the correlations. The equations are valid for the (general) inhomogeneous case in which the exchange coupling as well as the magnetic field depend on the sites in the chain. In the derivation use is made of a general formulation of the thermodynamic Wick theorem. For the homogeneous case a nonlinear differential–difference equation is derived, generalizing the Painlevé III equation found previously at zero temperature in the scaling limit. The finite-temperature field theory limit is discussed also.

1. Introduction

The one-dimensional XY-model with spin 1/2 has received a lot of interest as a nontrivial exactly-solvable quantum mechanical many-body problem. The equilibrium properties for the ground state have been evaluated by Lieb et al.[1]) and Katsura[2]) has derived the free energy per particle at finite temperature. The time correlations between z-components of spins have been obtained first by Niemeijer[3]), see also ref. 4. In general the zz-correlations at all temperatures show a very slow (power-law) decay vs. time.

For the time correlations between x-components of spins the situation is more complicated. The exact results at infinite temperature for the autocorrelation between x-components of a spin show in general a predominantly Gaussian decay vs. time[5–11]). (The correlations between x-components of different spins vanish at infinite temperature[12]).) These results have been derived using various methods including numerical calculations[5]), an approach based on Toeplitz determinants[6,7]), cf. also appendix B of ref. 9 and ref. 13 and also a general formulation[8–11]) of the thermodynamic Wick theorem[14–16]). For the (general)

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inhomogeneous XY-model the correlations can be derived from a Lagrangian corresponding to a classical lattice problem[10]) which in special cases such as the Ising chain in a transverse field, the isotropic XY-model, and the anisotropic XY-model in zero-field reduces to the Toda problem[17,18]). Detailed results for the frequency-Fourier transform of the autocorrelation of one spin in the homogeneous case can be found in ref. 11.

At zero temperature the correlations show in general a power-law decay $\sim t^{-\lambda}$ as a function of time, where λ can have several values depending on the specific regime for the interaction parameters. Explicit results in the scaling limit have been given for the correlations at large distances by Vaidya and Tracy[19,20]) using an approach based on Toeplitz determinants similar to the one developed by Wu et al.[21]) for the correlations of the two-dimensional Ising model, cf. also refs. 22–24. In refs. 25 and 26 an extensive treatment has been given of the impenetrable ($c = \infty$) Bose gas, see also refs. 27 and 28 for expansions in powers of $1/c$. Results for the autocorrelation of one spin have been given by Pesch and Mikeska[29]), based on a numerical evaluation of the block-Toeplitz determinant, and very recently by McCoy et al.[72]) in terms of a Fredholm determinant which satisfies a Painlevé differential equation.

For finite temperature not much is known apart from an explicit evaluation of the asymptotic behaviour of the autocorrelation in the special case of an Ising chain with nearest-neighbour exchange J^x in the presence of a transverse magnetic field $b = J^x$, or equivalently in the isotropic homogeneous XY-model in zero field[30]).

In the present paper we shall derive two nonlinear differential equations for the correlations in the case of the (inhomogeneous) Ising chain in which the exchange interactions and the (transverse) magnetic field depend in an arbitrary way on the sites in the chain. The equations are independent of the temperature, but temperature enters via the initial conditions. For finite temperatures the initial conditions are such that explicit solutions cannot be easily obtained, even in the homogeneous case with $b = J^x$, for which the correlations are determined by a Toda equation[31]) which, however, should not be compared with the one derived in the high-temperature limit[9–11]). The derivation of the Toda equation uses a general identity first given in ref. 31 which amounts to a general formulation of the thermodynamic Wick theorem, see also ref. 32. With this formulation one can derive also closed differential equations for the general inhomogeneous XY-model[31,33]), but in the present paper we restrict ourselves to the case of the Ising chain in a transverse field.

The general formulation of the thermodynamic Wick theorem has also been used to derive the nonlinear partial difference equations[34]) for the two-point correlations of the two-dimensional Ising model on a general planar lattice which in the homogeneous case leads to the lattice version of the Painlevé equation first obtained by McCoy and Wu[35–37]) and which at the critical temperature reduces to Hirota's discrete-time Toda equation[38]). Very recently, quadratic difference equations have also been obtained for n-point functions[39,40]) which generalize the nonlinear differential equations obtained by Sato et al., see ref. 41 for a review.

In section 2 of the present paper we present some preliminaries regarding the Ising chain in a transverse field, and in section 3 we give a general formulation of the thermodynamic Wick theorem for fermions, implying in particular a theorem for compound Pfaffians. By introducing anticommuting c-numbers, a bosonic counterpart can be derived, as will be shown in section 4. In section 5 we consider a slightly extended class of correlations containing not only the (physical) spin correlations and we shall derive differential equations for the generating functions of these extended correlations. By elementary integration two closed sets of equations for the spin correlations are derived in section 6, where also a discussion of the results is given.

In particular, in the case of site-independent interactions, an additional differential–difference equation is derived for the ratio of the spin correlations on the lattice and on the dual lattice, which in the scaling limit becomes the sinh-Gordon equation, independent of temperature. But, at zero temperature the boundary conditions have to be such that the solution is Lorentz invariant; the sinh-Gordon equation reduces then to the Painlevé III differential equation of refs. 19 and 21. In the special case of critical magnetic field, $b = J^x$, and in the scaling limit, the logarithm of the correlation function satisfies the free massless Klein–Gordon equation, also for the finite-temperature field theory. This shows that the crossover behaviour of the autocorrelation, from a power law at zero temperature to a Gaussian at high temperatures, is rather complicated, going almost certainly through at least one intermediate regime of exponential decay.

2. Preliminaries of the transverse Ising chain

In this paper we consider the (inhomogeneous) spin 1/2 Ising chain described by the Hamiltonian

$$\mathscr{H} = \sum_{j=1}^{N-1} 2J_j^x S_j^x S_{j+1}^x - \sum_{j=1}^{N} b_j S_j^z . \tag{2.1}$$

Here J_j^x denotes the exchange interaction between the x-components of neighbouring spins at the sites j and $j+1$ and b_j is the magnetic field (in the z-direction) acting on the spin at site j. Eq. (2.1) is the Hamiltonian for a finite Ising chain in a transverse field with N spins and open boundary conditions ($J_N^x = 0$).

The spin components satisfy Paulion-(anti)commutation relations, i.e. spin components S_j^ξ, $S_{j'}^{\xi'}$ referring to different sites ($j' \neq j$) commute, whereas different spin components S_j^ξ, $S_j^{\xi'}$, ($\xi' \neq \xi$), belonging to the same site j anticommute.

This feature can be overcome using the Jordan–Wigner transformation

$$S_j^x = \frac{1}{\sqrt{2}}\left(\prod_{k=1}^{j-1} P_k\right)\gamma_{2j-1}, \quad S_j^y = \frac{1}{\sqrt{2}}\left(\prod_{k=1}^{j-1} P_k\right)\gamma_{2j}, \quad S_j^z = -\tfrac{1}{2}P_j = -\mathrm{i}\gamma_{2j-1}\gamma_{2j}, \tag{2.2}$$

in which the operators γ are (Hermitean) fermion operators[42]) satisfying the anticommutation relations

$$[\gamma_k, \gamma_l]_+ = \delta_{kl}. \tag{2.3}$$

In terms of the operators γ the Hamiltonian (2.1) is given by

$$\mathscr{H} = \mathrm{i}\sum_{k=1}^{2N-1} J_k\gamma_k\gamma_{k+1} \tag{2.4}$$

with†

$$J_{2j-1} = b_j, \quad J_{2j} = J_j^x. \tag{2.5}$$

The time-dependent correlation between x-components of spins at the sites $n+1$ and $n+p+1$ is given by

$$\langle S_{n+1}^x(t) S_{n+1+p}^x\rangle = \frac{1}{2}\left\langle\left(\prod_{k=1}^{n} P_k(t)\right)\gamma_{2n+1}(t)\left(\prod_{k=1}^{n+p} P_k\right)\gamma_{2n+1+2p}\right\rangle, \tag{2.6}$$

where (t) and $\langle\cdots\rangle$ denote the time evolution and the canonical average with respect to the Hamiltonian (2.1), i.e.

$$\begin{aligned} A(t) &= \mathrm{e}^{\mathrm{i}\mathscr{H}t} A\, \mathrm{e}^{-\mathrm{i}\mathscr{H}t}, \\ \langle A\rangle &= \mathrm{Tr}\, A\, \mathrm{e}^{-\beta\mathscr{H}}/\mathrm{Tr}\, \mathrm{e}^{-\beta\mathscr{H}}, \end{aligned} \tag{2.7}$$

for an arbitrary operator A.

The right-hand side of (2.6) contains a large number of time-dependent Jordan–Wigner factors, i.e. the P_k, $k = 1, \ldots, n$, at the left, sandwiched between $\mathrm{e}^{\mathrm{i}\mathscr{H}t}$ and $\mathrm{e}^{-\mathrm{i}\mathscr{H}t}$, and a large number of time-independent Jordan–Wigner factors P_k, $k = 1, \ldots, n+p$, at the right.

This feature can be overcome using the commutation rule

$$\left(\prod_{k=1}^{m} \gamma_k\right)\mathscr{H} = (\mathscr{H} + h_m)\left(\prod_{k=1}^{m}\gamma_k\right), \tag{2.8}$$

where

$$h_m = -2\mathrm{i}\, J_m\gamma_m\gamma_{m+1}. \tag{2.9}$$

† Note that in ref. 31 a slightly different notation has been used.

The proof of (2.8) is straightforward using (2.3). Note that the Hamiltonian $\mathscr{H} + h_m$ can be obtained from (2.4) changing the sign of the interaction J_m between the operators γ_m and γ_{m+1}.

From (2.8) we have

$$\prod_{k=1}^{m} \gamma_k(t) = O_{m,t}\prod_{k=1}^{m}\gamma_k \tag{2.10}$$

with

$$O_{m,t} = \mathrm{e}^{\mathrm{i}\mathscr{H}t}\,\mathrm{e}^{-\mathrm{i}(\mathscr{H}+h_m)t} \tag{2.11}$$

and the time-dependent spin operator $S_{n+1}^x(t)$ can be expressed as

$$S_{n+1}^x(t) = \frac{1}{\sqrt{2}}\left(\prod_{k=1}^{n} P_k(t)\right)\gamma_{2n+1}(t) = \frac{1}{\sqrt{2}} O_{2n+1,t}\left(\prod_{k=1}^{n} P_k\right)\gamma_{2n+1}. \tag{2.12}$$

Inserting (2.12) into (2.6) and using the trivial relations $P_k^2 = 1$, we obtain the spin correlations, ($p = 0, 1, 2, \ldots$),

$$\begin{aligned} \langle S_{n+1}^x(t) S_{n+1+p}^x\rangle &= \tfrac{1}{4}(2\mathrm{i})^p A_{2n+1,2n+1+2p}, \\ \langle S_{n+1}^x(t) S_{n+1-p}^x\rangle &= \tfrac{1}{4}(-2\mathrm{i})^p A_{2n+1,2n+1-2p}, \end{aligned} \tag{2.13}$$

with

$$\begin{aligned} A_{m,m+s} &= \langle O_{m,t}\gamma_{m+1}\cdots\gamma_{m+s}\rangle, \quad s = 2, 4, \ldots, \\ A_{m,m-s} &= \langle O_{m,t}\gamma_{m}\cdots\gamma_{m-s+1}\rangle, \quad s = 2, 4, \ldots, \\ A_{m,m} &= \langle O_{m,t}\rangle. \end{aligned} \tag{2.14}$$

In view of (2.8) and (2.11) the right-hand side of (2.14) can also be expressed as the average of a string of (m) time-dependent γ-operators multiplied by a string of ($m+s$, resp. $m-s$, resp. m) time-independent γ-operators. In fact, from (2.10), we have, cf. also (2.3),

$$O_{m,t} = 2^m(-1)^{\frac{1}{2}m(m-1)}\gamma_1(t)\cdots\gamma_m(t)\gamma_1\cdots\gamma_m, \tag{2.15}$$

so that

$$\begin{aligned} A_{m,m+s} &= 2^m(-1)^{\frac{1}{2}m(m-1)}\langle\gamma_1(t)\cdots\gamma_m(t)\gamma_1\cdots\gamma_{m+s}\rangle, \\ A_{m,m-s} &= 2^{m-s}(-1)^{\frac{1}{2}m(m-1)}\langle\gamma_1(t)\cdots\gamma_m(t)\gamma_1\cdots\gamma_{m-s}\rangle, \quad s = 0, 2, \ldots. \end{aligned} \tag{2.16}$$

The correlations (2.13) with $m = \text{odd} = 2n+1$ are the correlations in the Ising chain (2.1) with

$$\begin{aligned} b_j &= J_{2j-1}, \quad j = 1, \ldots, N, \\ J_j^x &= J_{2j}, \quad j = 1, \ldots, N-1. \end{aligned} \tag{2.17}$$

For even $m = 2n$, the correlations $A_{m,m+s}$, $A_{m,m-s}$ can be expressed in terms of correlations of a "dual" Ising chain. In fact, introducing formally (Hermitean) fermion operators γ_0, γ_{2N+1}, in addition to the operators $\gamma_1, \ldots, \gamma_{2N}$ in (2.2), (2.3), one can define the "disorder variables"[43])

$$\begin{aligned} S^x_{j+\frac{1}{2}} &= \frac{1}{\sqrt{2}}\left(\prod_{k=1}^{j} 2\mathrm{i}\gamma_{2k-2}\gamma_{2k-1}\right)\gamma_{2j} = \frac{\gamma_0}{\sqrt{2}}\prod_{k=1}^{j}(-2S^z_k)\,, \\ S^y_{j+\frac{1}{2}} &= \frac{1}{\sqrt{2}}\left(\prod_{k=1}^{j} 2\mathrm{i}\gamma_{2k-2}\gamma_{2k-1}\right)\gamma_{2j+1} = \frac{\gamma_0}{\sqrt{2}}\left(\prod_{k=1}^{j}(-2S^z_k)\right)(-4\mathrm{i}S^x_jS^x_{j+1})\,, \\ S^z_{j+\frac{1}{2}} &= -\mathrm{i}\gamma_{2j}\gamma_{2j+1} = -2S^x_jS^x_{j+1}\,, \end{aligned} \tag{2.18}$$

for $j = 0, 1, \ldots, N$†. Since the anticommutation relation (2.3) is invariant under the transformation $\gamma_k \to \gamma_{k-1}$, it is clear that the operators (2.8) satisfy the appropriate Paulion-(anti)commutation relations and therefore may be considered to be spin operators at the sites $\frac{1}{2}, \frac{3}{2}, \ldots, N+\frac{1}{2}$.

The Hamiltonian (2.1), (2.4) can also be expressed in terms of the spins of the dual chain

$$\begin{aligned} \mathscr{H} &= \mathrm{i}\sum_{k=1}^{N} b_k\gamma_{2k-1}\gamma_{2k} + \mathrm{i}\sum_{k=1}^{N-1} J^x_k\gamma_{2k}\gamma_{2k+1} \\ &= 2\sum_{k=1}^{N} b_kS^x_{k-\frac{1}{2}}S^x_{k+\frac{1}{2}} - \sum_{k=1}^{N-1} J^x_kS^z_{k+\frac{1}{2}} \\ &= 2\sum_{k=1}^{N} J^x_{k-\frac{1}{2}}S^x_{k-\frac{1}{2}}S^x_{k+\frac{1}{2}} - \sum_{k=1}^{N-1} b_{k-\frac{1}{2}}S^z_{k+\frac{1}{2}}\,, \end{aligned} \tag{2.19}$$

in which the J^x and b with half-integer subscripts are defined by

$$\begin{aligned} &J^x_{k-\frac{1}{2}} = b_k, \quad k = 1, \ldots, N\,, \\ &b_{\frac{1}{2}} = 0, \quad b_{k-\frac{1}{2}} = J^x_{k-1}, \quad k = 2, \ldots, N\,, \\ &b_{N+\frac{1}{2}} = 0\,. \end{aligned} \tag{2.20}$$

The correlations between the dual spins are given by

$$\begin{aligned} &\langle S^x_{n+\frac{1}{2}}(t)S^x_{n+\frac{1}{2}+p}\rangle = \tfrac{1}{4}(2\mathrm{i})^pA_{2n,2n+2p}\,, \\ &\langle S^x_{n+\frac{1}{2}}(t)S^x_{n+\frac{1}{2}-p}\rangle = \tfrac{1}{4}(-2\mathrm{i})^pA_{2n,2n-2p}\,. \end{aligned} \tag{2.21}$$

Note that the left-hand side of (2.21) can also be given as an expression involving a product of time-dependent and a product of time-independent operators S^z_k, in view of (2.18) and the relations $\gamma_0^2 = \frac{1}{2}$, $[\gamma_0, \mathscr{H}] = 0$.

† Eqs. (2.18) are equivalent to eq. (3) of ref. 31, apart from two minus signs, i.e. in eq. (3) of ref. 31 one should have $\Gamma_{2j-1}\Gamma_{2j} = -\mathrm{i}\tau^x_j\tau^x_{j+1}$ and $\Gamma_{2j-2}\Gamma_{2j-1} = -\mathrm{i}\sigma^x_{j-1}\sigma^x_j$.

As a conclusion the quantities $A_{m,m+s}$, $A_{m,m-s}$ for odd and even m, s = even, give the time-dependent pair correlations between the x-components of spins in the transverse Ising chain (2.1), as well as the dual chain (2.19).

From the xx-correlations, the xy- and yy-correlations of the Ising chain (2.1) can be obtained using the relations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle S^x_{n+1}(t)S^x_{n+1+p}\rangle &= b_{n+1}\langle S^y_{n+1}(t)S^x_{n+1+p}\rangle \\ &= -b_{n+1+p}\langle S^x_{n+1}(t)S^y_{n+1+p}\rangle\,, \\ \frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle S^x_{n+1}(t)S^x_{n+1+p}\rangle &= -b_{n+1}b_{n+1+p}\langle S^y_{n+1}(t)S^y_{n+1+p}\rangle\,, \end{aligned} \tag{2.22}$$

cf. ref. 44 and also section 7 of ref. 8.

3. Wick theorem for fermions

In this section we discuss a general formulation of the thermodynamic Wick theorem which provides as a special application the (exact) factorizations needed for the derivation of the nonlinear differential equations in sections 5,6 for the spin correlations in the transverse Ising chain. Other applications with respect to the one-dimensional XY-model and the two-dimensional Ising model have been given in refs. 31, 33, 34. As a special consequence of the general formulation one has a theorem on compound Pfaffians which will be discussed at the end of this section.

3.1. *General formulation of Wick theorem*

Consider a set of (Hermitean) fermion operators $\gamma_1, \ldots, \gamma_{2N}$, defined on a Hilbert space $\mathscr{h}$ satisfying the anticommutation relations

$$[\gamma_k, \gamma_l]_+ = \delta_{kl}\,. \tag{2.3}$$

From the operators γ one can construct operators Q_p consisting of a finite product of exponentials of bilinear expressions in the operators γ multiplied by products of linear combinations of the γ's, i.e.

$$Q_p = \prod_{\nu=1}^{r_p}\{(\exp B_{p\nu})\Gamma_{p\nu}\}\,, \tag{3.1}$$

with

$$B_{p\nu} = \sum_{m,n} A^{p\nu}_{mn}\gamma_m\gamma_n\,, \tag{3.2}$$

$$\Gamma_{p\nu} = \prod_{\mu=1}^{S_{p\nu}} \Big(\sum_m \lambda_{p\nu\mu m} \gamma_m \Big), \quad \sum_{p,\nu} S_{p\nu} = 2R = \text{even}, \tag{3.3}$$

where the $A^{p\nu}_{mn}$ and $\lambda_{p\nu\mu m}$ are arbitrary coefficients and where it is understood that $\Gamma_{p\nu} = 1$ for $S_{p\nu} = 0$.

For the traces of products of operators Q_p multiplied by fermion operators γ_{i_p}, we have the factorization property

$$\mathrm{Tr}\Big(\prod_{p=1}^{2s} Q_p \gamma_{i_p} \Big) \Big/ \mathrm{Tr}\Big(\prod_{p=1}^{2s} Q_p \Big) = \underset{\{k<l\}}{\mathrm{Pf}} \left(\{X_{kl}\} \right) \equiv \mathrm{Pf}(1, 2, \ldots, 2s), \tag{3.4}$$

where the right-hand side is the Pfaffian[45,46] for the triangular array of "pair averages" X_{kl}, $1 \leqslant k < l \leqslant 2s$, which can be derived from the left-hand side of (3.4) omitting all γ_{i_p} except γ_{i_k} and γ_{i_l}. More explicitly,

$$X_{kl} = \mathrm{Tr}\Big\{ \Big(\prod_{p \leqslant k} Q_p \Big) \gamma_{i_k} \Big(\prod_{k<p\leqslant l} Q_p \Big) \gamma_{i_l} \Big(\prod_{p>l} Q_p \Big) \Big\} \Big/ \mathrm{Tr}\Big(\prod_{p=1}^{2s} Q_p \Big). \tag{3.5}$$

The Pfaffian in eq. (3.4) is defined by

$$\underset{\{k<l\}}{\mathrm{Pf}} \left(\{X_{kl}\} \right) = \sum_P{}' (-1)^P \prod_{k=1}^{s} X_{P(2k-1)P(2k)}, \tag{3.6}$$

where Σ' is the sum over all permutations $P(k)$, $k = 1, \ldots, 2s$, with $P(2k) > P(2k-1)$, $P(2k+1) > P(2k-1)$ and $(-1)^P$ is the sign of the permutation.

From the definition (3.6) we have the recursion relations

$$\mathrm{Pf}(1, 2, \ldots, 2s) = \sum_{j=2}^{2s} (-1)^j X_{1j} \, \mathrm{Pf}(1, \ldots, \not{k}, \ldots, 2s), \tag{3.7}$$

$$\mathrm{Pf}(1, 2, \ldots, 2s) = s^{-1} \sum_{k<l} (-1)^{k+l-1} X_{kl} \, \mathrm{Pf}(1, \ldots, \not{k}, \ldots, \not{l}, \ldots, 2s), \tag{3.8}$$

in which $\mathrm{Pf}(1, \ldots, \not{k}, \ldots, \not{l}, \ldots, 2s)$ is the Pfaffian of the $(2s-2) \times (2s-2)$ triangular array which can be obtained from $\{X_{pq}\}$, $1 \leqslant p < q \leqslant 2s$, omitting all X_{pk}, X_{kp}, X_{ql}, X_{lq}, X_{kl}. Eq. (3.7) can be found e.g. in refs. 45, 46, and eq. (3.8) follows from the fact that for each pair $k < l$ we have under the summation in (3.8) the sum of all terms containing X_{kl} in the Pfaffian $\mathrm{Pf}(1, 2, \ldots, 2s)$.

In section 2 of ref. 9 we have discussed a less general formulation with $\Gamma_{p\nu} = 1$, taking into account only products of exponentials of bilinear expressions in the γ's. With this formulation we could derive the differential equations for the spin correlations of the one-dimensional XY-model at infinite temperature[8,10]. The formulation with $\Gamma_{p\nu} \neq 1$ is important for the extension to finite temperatures[31] and for the correlations of the two-dimensional Ising model[34,39].

The proof of eq. (3.4) proceeds essentially along the same lines as the one given

in section 2 of ref. 9. The proof is based on a "doubling" of the Hilbert space. As a first step one labels all operators $\gamma_1, \ldots, \gamma_{2N}$ acting on the Hilbert space $\mathscr{h}$ by a superscript x, so that the left-hand side of (3.4) is expressed as

$$\mathrm{Tr}_x\Big(\prod_{p=1}^{2s} Q^x_p \gamma^x_{i_p} \Big) \Big/ \mathrm{Tr}_x\Big(\prod_{p=1}^{2s} Q^x_p \Big) \tag{3.9}$$

with

$$Q^x_p = \prod_{\nu=1}^{r_p} \Big\{ \exp\Big(\sum_{m,n} A^{p\nu}_{mn} \gamma^x_m \gamma^x_n \Big) \prod_{\mu=1}^{S_{p\nu}} \Big(\prod_m \lambda_{p\nu\mu m} \gamma^x_m \Big) \Big\}, \tag{3.10}$$

where Tr_x denotes the trace over the Hilbert space $\mathscr{h}_x$.

We next introduce a second independent Hilbert space† $\mathscr{h}_y \cong \mathscr{h}_x$ with operators γ^y_k, $k = 1, \ldots, 2N$, satisfying the anticommutation relations

$$[\gamma^\xi_k, \gamma^{\xi'}_{k'}]_+ = \delta_{k,k'} \delta_{\xi,\xi'}, \quad (\xi, \xi' = x, y). \tag{3.11}$$

The introduction of this Hilbert space may be visualized by considering two identical chains $\boldsymbol{S}_1, \ldots, \boldsymbol{S}_N$ and $\boldsymbol{S}_{N+1}, \ldots, \boldsymbol{S}_{2N}$, resp., with spins 1/2 and by applying the Jordan–Wigner transformation to the composite chain with spins $\boldsymbol{S}_1, \ldots, \boldsymbol{S}_{2N}$. Since Pfaffians and determinants can be expressed as Gaussian integrals over Clifford (or Grassmann) algebras, see refs. 46, 47, the doubling of the Hilbert space is reminiscent of the well-known trick in the evaluation of the Gaussian integral, i.e.

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-x^2} \int_{-\infty}^{\infty} \mathrm{d}y \, \mathrm{e}^{-y^2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{d}y \, \mathrm{e}^{-(x^2+y^2)} = \int_0^{\infty} \int_0^{2\pi} \mathrm{d}r \, r \, \mathrm{d}\phi \, \mathrm{e}^{-r^2} = \pi. \tag{3.12}$$

Then one considers the general set of 2^{2s} traces

$$C_{\xi_1 \xi_2 \ldots \xi_{2s}} = \mathrm{Tr}_{x,y}\Big(\prod_{p=1}^{2s} Q^x_p Q^y_p \gamma^{\xi_p}_{i_p} \Big), \tag{3.13}$$

where $\xi_1, \ldots, \xi_{2s} = x, y$ and Q^y_p can be obtained from the right-hand side of (3.10) replacing the operators γ^x_k by γ^y_k. $\mathrm{Tr}_{x,y}$ is the trace over the direct-product space $\mathscr{h}_x \otimes \mathscr{h}_y$.

The operator $Q^x_p Q^y_p$ in (3.13) can be expressed as

$$\begin{aligned} Q^x_p Q^y_p &= (-1)^{n_p} \prod_{\nu=1}^{r_p} \Big\{ \exp\Big(\sum_{m,n} A^{p\nu}_{mn} (\gamma^x_m \gamma^x_n + \gamma^y_m \gamma^y_n) \Big) \\ &\quad \times \prod_{\mu=1}^{S_{p\nu}} \Big(\sum_{k<l} \lambda_{p\nu\mu k} \lambda_{p\nu\mu l} (\gamma^x_k \gamma^y_l + \gamma^x_l \gamma^y_k) \Big) \Big\}, \\ n_p &= \frac{1}{2} \Big(\sum_\nu S_{p\nu} \Big) \Big\{ \Big(\sum_\nu S_{p\nu} \Big) - 1 \Big\}. \end{aligned} \tag{3.14}$$

† Note that the superscripts x and y, which have been introduced to label the two Hilbert spaces, are completely unrelated to those indicating the components of the spins $\boldsymbol{S}_j$.

It is clear that the right-hand side of (3.14) and thus $Q_p^x Q_p^y$ is invariant under canonical transformations of the form

$$\begin{aligned} \gamma_k^x &\to \gamma_k^x \cos q - \gamma_k^y \sin q\,, \\ \gamma_k^y &\to \gamma_k^x \sin q + \gamma_k^y \cos q\,, \end{aligned} \tag{3.15}$$

leading to

$$\gamma_m^x \gamma_n^x + \gamma_m^y \gamma_n^y \to \gamma_m^x \gamma_n^x + \gamma_m^y \gamma_n^y\,, \tag{3.16}$$

$$\gamma_k^x \gamma_l^y + \gamma_l^x \gamma_k^y \to \gamma_k^x \gamma_l^y + \gamma_l^x \gamma_k^y\,, \tag{3.17}$$

i.e. all expressions of the form $\gamma_m^x\gamma_n^x + \gamma_m^y\gamma_n^y$ and $\gamma_k^x\gamma_l^y + \gamma_l^x\gamma_k^y$ are invariant under the transformation (3.15). Then the 2^{2s} traces $C_{\xi_1\ldots\xi_{2s}}$ defined by (3.13) transform as the elements of an invariant tensor of rank $2s$ in two dimensions, i.e.

$$C_{\xi_1\xi_2\ldots\xi_{2s}} = \sum_{\xi'_1,\ldots,\xi'_{2s}=(x,y)} \left(\prod_{\nu=1}^{2s} D_{\xi_\nu\xi'_\nu}(q) \right) C_{\xi'_1\xi'_2\ldots\xi'_{2s}} \tag{3.18}$$

with

$$D(q) = \begin{pmatrix} \cos q & -\sin q \\ \sin q & \cos q \end{pmatrix}. \tag{3.19}$$

From (3.18) and (3.19) we have

$$C^{(0)} = \sum_k C^{(k)} \cos^{2s-2k} q \sin^{2k} q\,, \tag{3.20}$$

where $C^{(k)}$ is the sum of all traces $C_{\xi_1\xi_2\ldots\xi_{2s}}$ containing $2s-2k$ superscripts x and $2k$ superscripts y. (Terms with an odd number of superscripts y do not contribute in view of the canonical transformation $\gamma_k^y \to -\gamma_k^y$, $\gamma_k^x \to \gamma_k^x$.)

Eq. (3.20) implies in particular (for $q \to 0$) that

$$C^{(0)} = s^{-1} C^{(1)}\,, \tag{3.21}$$

as the coefficient of q^2 in (3.20) vanishes. Eq. (3.21) is a linear relation between traces which has been derived on the basis of a continuous symmetry by differentiation. This procedure is well known in non-Abelian gauge theory, e.g. in connection with the derivation of Ward–Takahashi identities using the BRS-transformation[48,49]).

When we insert (3.13) in (3.21), the traces in the right-hand side can be factorized, in view of the property

$$\mathrm{Tr}_{x,y}\, A^x B^y = \mathrm{Tr}_x\, A^x\, \mathrm{Tr}_y B^y = \mathrm{Tr}\, A\ \mathrm{Tr}\, B\,, \tag{3.22}$$

in which A and B are (analytic) expressions in the operators γ_k and A^x, B^x and A^y, B^y can be obtained from A, B replacing the operators γ_k by γ_k^x and by γ_k^y, respectively. As a result we obtain

$$\begin{aligned} &\mathrm{Tr}\left(\prod_{p=1}^{2s} Q_p \gamma_{i_p} \right) \mathrm{Tr}\left(\prod_{p=1}^{2s} Q_p \right) \\ &= s^{-1} \sum_{k<l} (-1)^{k+l-1}\, \mathrm{Tr}\left\{ \left(\prod_{p<k} Q_p \right) Q_k \gamma_{i_k} \left(\prod_{k<p<l} Q_p \right) Q_l \gamma_{i_l} \left(\prod_{p>l} Q_p \right) \right\} \\ &\quad \times \mathrm{Tr}\left\{ \left(\prod_{p<k} Q_p \gamma_{i_p} \right) Q_k \left(\prod_{k<p<l} Q_p \gamma_{i_p} \right) Q_l \left(\prod_{p>l} Q_p \gamma_{i_p} \right) \right\}. \end{aligned} \tag{3.23}$$

Assuming by induction that (3.4) has been proved for averages involving $(2s-2)$ operators γ_{i_p}, eq. (3.23), in combination with the recursion relation (3.8), leads immediately to (3.4) for averages involving $2s$ operators γ_{i_p}.

Note that the formulation (3.4) of the thermodynamic Wick theorem has been derived in the case of fermions. In fact, eq. (3.17) which has been used in the derivation of (3.4) is based on the anticommutation relations of fermions, in contrast to (3.16) which has been used also in the derivation of section 2 of ref. 9 which applies to the case of bosons as well. However, by introducing anticommuting c-numbers[47]) a bosonic counterpart of (3.4) can be derived as will be shown in section 4.

The Wick theorem (3.4) can be considered to be an extension of the usual formulation in which there is only one exponential of a bilinear expression of γ-operators and in which $\Gamma_{p\nu} = 1$[14-16]). Note that in the usual formulation the operators $\gamma_{i_1}, \ldots, \gamma_{i_{2s}}$ can also be replaced by time-dependent operators $\gamma_{i_1}(t_1), \ldots, \gamma_{i_{2s}}(t_{2s})$, since each $\gamma_k(t)$, under the evolution of a bilinear Hamiltonian, can be expressed as a linear combination of time-independent operators. The formulation given in ref. 9 may also be made plausible from the usual formulation, in view of the Baker–Hausdorff expansion and the fact that each $\gamma(t)$ under the evolution of a bilinear Hamiltonian is a linear combination of time-independent γ's.

3.2. *Theorem for compound Pfaffians*

In the preceding subsection, we have given an extension including products $\Gamma_{p\nu}$ of linear combinations of operators γ_k. Note that the left-hand side of (3.4) can be expressed as a linear combination of traces involving exponentials of bilinear forms and $2R+2s$, $2R = \Sigma_{p,\nu} S_{p\nu}$, operators γ. Such a trace can be expressed in terms of a $(2R+2s)\times(2R+2s)$ Pfaffian. The formulation of (3.4) implies a property for this $(2R+2s)\times(2R+2s)$ Pfaffian in terms of a $2s \times 2s$ compound Pfaffian. In fact, from (3.4) we have†

† This formulation is also very useful in connection with the two-dimensional Ising model[39,40,50]).

$$\mathrm{Pf}(1,2,\ldots,2s,2s+1,\ldots,2s+2R)/\mathrm{Pf}(2s+1,\ldots,2s+2R)$$
$$=\mathop{\mathrm{Pf}}_{1\leqslant i<j\leqslant 2s}(\{\mathrm{Pf}(i,j,2s+1,\ldots,2s+2R)/\mathrm{Pf}(2s+1,\ldots,2s+2R)\})\,, \tag{3.24}$$

or with the notation

$$A=\{1,2,\ldots,2s\},\quad B=\{2s+1,\ldots,2s+2R\}\,, \tag{3.25}$$

$$\frac{\mathrm{Pf}(A\cup B)}{\mathrm{Pf}\,B}=\mathop{\mathrm{Pf}}_{\{i,j\}\subset A}\left(\left\{\frac{\mathrm{Pf}(\{i,j\}\cup B)}{\mathrm{Pf}\,B}\right\}\right). \tag{3.26}$$

The compound Pfaffian theorem can also be inferred from the recursion relation

$$\mathrm{Pf}(1,\ldots,2s,2s+1,\ldots,2s+2R)\,\mathrm{Pf}(2s+1,\ldots,2s+2R)$$
$$=\sum_{j=2}^{2s}(-1)^j\,\mathrm{Pf}(\not{1},\ldots,\not{j},\ldots,2s,2s+1,\ldots,2s+2R)$$
$$\times\,\mathrm{Pf}(1,j,2s+1,\ldots,2s+2R)\,, \tag{3.27}$$

first discussed by Tanner[51]) and proved by Zajackowski[52]), see also refs. 53–56 for related expressions for determinants. [In fact, the theorem on compound determinants may be viewed as a special case of (3.26).]

As an application of (3.4) we consider the one-dimensional XY-model with open boundary conditions which contains the transverse Ising chain (2.1) as a special case. For the one-dimensional XY-model we have the factorization property (3.4), in which the operators γ_{i_p} and Q_p in spin language are given by

$$\gamma_{2n+1}=\sqrt{2}\left\{\prod_{j=1}^{n}(-2S_j^z)\right\}S_{n+1}^x,\quad \gamma_{2n+2}=\sqrt{2}\left\{\prod_{j=1}^{n}(-2S_j^z)\right\}S_{n+1}^y\,,$$

$$Q_p=\prod_{\nu=1}^{r_p}\left\{\exp\left(\sum_j\sum_{j'\geqslant j}\sum_{\xi,\xi'=x,y}C_{jj'\xi\xi'}^{p\nu}\,S_j^\xi S_{j+1}^z\ldots S_{j'-1}^z S_{j'}^{\xi'}\right)\right.$$
$$\left.\times f_{p\nu}(\{S_i^x\},\{S_i^y\},\{S_i^z\})\right\}, \tag{3.28}$$

where the $C_{jj'\xi\xi'}^{p\nu}$ are (arbitrary) coefficients and where $f_{p\nu}$ is an arbitrary product of spin components such that $\Pi_{p,\nu}f_{p\nu}$ is invariant under rotations over π around the z-axis with $S_j^x\to -S_j^x$, $S_j^y\to -S_j^y$, $S_j^z\to S_j^z$. Eq. (3.28) corresponds to the case that each $\Gamma_{p\nu}$ in (3.3) is a product of γ-operators. By taking products of linear combinations, like in (3.3), more general expressions may be derived. Special cases of (3.4) with (3.28) have been exploited in eq. (6) of ref. 31 and eq. (8) of ref. 34.

4. Wick theorem for bosons

In this section we discuss the bosonic counterpart of the treatment of the preceding section in terms of (Hermitean) operators satisfying the commutation relations

$$[\gamma_k,\gamma_l]=g_{kl}=-g_{lk}\,. \tag{4.1}$$

We define the operators

$$Q_p=\prod_{\nu=1}^{r_p}\{(\exp B_{p\nu})\Gamma_{p\nu}\}\,, \tag{4.2}$$

in which the

$$B_{p\nu}=\sum_{m,n}A_{mn}^{p\nu}\gamma_m\gamma_n \tag{4.3}$$

are bilinear expressions in the operators γ with arbitrary coefficients $A_{mn}^{p\nu}$. The $\Gamma_{p\nu}$ are products of linear combinations of the operators γ_m, i.e.

$$\Gamma_{p\nu}=\prod_{\mu=1}^{S_{p\nu}}\left(\sum_m\eta_{p\nu\mu m}\gamma_m\right),\quad \sum_{p,\nu}S_{p\nu}=2R=\text{even}\,, \tag{4.4}$$

where in contrast to section 3, the $\eta_{p\nu\mu m}$ are anticommuting c-numbers belonging to a Grassmann algebra[47]), i.e.

$$\eta_{p\nu\mu m}\eta_{p'\nu'\mu'm'}+\eta_{p'\nu'\mu'm'}\eta_{p\nu\mu m}=0\,. \tag{4.5}$$

It is understood that $\Gamma_{p\nu}=1$ for $S_{p\nu}=0$.

For the traces of products of operators Q_p multiplied by boson operators γ_{i_p}, we have the factorization property

$$\mathrm{Tr}\left(\prod_{p=1}^{2s}Q_p\gamma_{i_p}\right)\Big/\mathrm{Tr}\left(\prod_{p=1}^{2s}Q_p\right)=\mathop{\mathrm{Hf}}_{\{k<l\}}(\{X_{kl}\})\equiv\mathrm{Hf}(1,2,\ldots,2s)\,, \tag{4.6}$$

where the right-hand side is the Hafnian[45,46]) for the triangular array of "pair averages" X_{kl}, $1\le k<l\le 2s$, which can be derived from the left-hand side of (4.6) omitting all γ_{i_p} except γ_{i_k} and γ_{i_l}, cf. (3.5).

The Hafnian in (4.6) is defined by

$$\mathop{\mathrm{Hf}}_{\{k<l\}}(\{X_{kl}\})=\sum_P{}'\prod_{k=1}^{s}X_{P(2k-1)P(2k)}\,, \tag{4.7}$$

cf. (3.6) without the factor $(-1)^P$, where Σ' is the sum over all permutations $P(k)$, $k=1,\ldots,2s$, with $P(2k)>P(2k-1)$, $P(2k+1)>P(2k-1)$. For the Hafnian we have the recursion relation

$$\mathrm{Hf}(1,2,\ldots,2s) = s^{-1} \sum_{k<l} X_{kl}\, \mathrm{Hf}(1,\ldots,\not{k},\ldots,\not{l},\ldots,2s)\,, \tag{4.8}$$

cf. (3.8), where $\mathrm{Hf}(1,\ldots,\not{k},\ldots,\not{l},\ldots,2s)$ is the Hafnian of the $(2s-2)\times(2s-2)$ triangular array which can be obtained from $\{X_{pq}\}$, $1 \le p < q \le 2s$, omitting all elements X_{pk}, X_{kp}, X_{ql}, X_{lq}, X_{kl}.

The special case of (4.6) with $\Gamma_{p\nu} = 1$, $(S_{p\nu} = 0)$, was proved in ref. 9. The proof of (4.6) with $\Gamma_{p\nu} \neq 1$ is completely analogous to the one in section 3. As a first step we label all operators $\gamma_1, \ldots, \gamma_{2N}$ on the Hilbert space $\mathscr{h}$ by a superscript x, so that the left-hand side of (4.6) is given by (3.9) in which

$$Q_p^x = \prod_{\nu=1}^{r_p} \left\{ \exp\left(\sum_{m,n} A_{mn}^{p\nu} \gamma_m^x \gamma_n^x \right) \prod_{\mu=1}^{S_{p\nu}} \left(\sum_m \eta_{p\nu\mu m} \gamma_m^x \right) \right\}. \tag{4.9}$$

We next introduce a second (independent) Hilbert space $\mathscr{h}_y \cong \mathscr{h}_x$ with operators γ_k^y, $k = 1, \ldots, 2N$, satisfying the commutation relations

$$[\gamma_k^\xi, \gamma_{k'}^{\xi'}] = g_{kk'} \delta_{\xi,\xi'}, \quad (\xi, \xi' = x, y)\,. \tag{4.10}$$

Using (4.10) and (4.5) the product $Q_p^x Q_p^y$ can be expressed as

$$\begin{aligned} Q_p^x Q_p^y = (-1)^{n_p} \prod_{\nu=1}^{r_p} \Bigg\{ & \exp\left(\sum_{m,n} A_{mn}^{p\nu} (\gamma_m^x \gamma_n^x + \gamma_m^y \gamma_n^y) \right) \\ & \times \prod_{\mu=1}^{S_{p\nu}} \sum_{k<l} \eta_{p\nu\mu k} \eta_{p\nu\mu l} (\gamma_k^x \gamma_l^y - \gamma_l^x \gamma_k^y) \Bigg\}, \\ n_p = \frac{1}{2} \left(\sum_\nu S_{p\nu} \right) & \left\{ \left(\sum_\nu S_{p\nu} \right) - 1 \right\}. \end{aligned} \tag{4.11}$$

The right-hand side of (4.11) is invariant under the canonical transformation (3.15) leading with (4.10) to

$$\gamma_m^x \gamma_n^x + \gamma_m^y \gamma_n^y \to \gamma_m^x \gamma_n^x + \gamma_m^y \gamma_n^y\,, \tag{4.12}$$

$$\gamma_k^x \gamma_l^y - \gamma_l^x \gamma_k^y \to \gamma_k^x \gamma_l^y - \gamma_l^x \gamma_k^y\,. \tag{4.13}$$

As a consequence the 2^{2s} traces $C_{\xi_1 \xi_2 \ldots \xi_{2s}}$ defined by (3.13) transform as the elements of an invariant tensor of rank $2s$ in two dimensions. The properties (3.18)–(3.21) remain valid, and using (3.22) we arrive at

$$\begin{aligned} & \mathrm{Tr}\left(\prod_{p=1}^{2s} Q_p \gamma_{i_p} \right) \mathrm{Tr}\left(\prod_{p=1}^{2s} Q_p \right) \\ & = s^{-1} \sum_{k<l} \mathrm{Tr}\left\{ \left(\prod_{p<k} Q_p \right) Q_k \gamma_{i_k} \left(\prod_{k<p<l} Q_p \right) Q_l \gamma_{i_l} \left(\prod_{p>l} Q_p \right) \right\} \\ & \quad \times \mathrm{Tr}\left\{ \left(\prod_{p<k} Q_p \gamma_{i_p} \right) Q_k \left(\prod_{k<p<l} Q_p \gamma_{i_p} \right) Q_l \left(\prod_{p>l} Q_p \gamma_{i_p} \right) \right\}, \end{aligned} \tag{4.14}$$

cf. (3.23) without the factor $(-1)^{k+l-1}$. Eq. (4.14) corresponds to the proper recursion relation (4.8) for Hafnians, so that (4.6) has been proved. Eq. (4.6) is completely analogous to eq. (3.4) derived for the case of fermions and may be regarded as a compound theorem for Hafnians. The fact that the $\eta_{p\nu\mu m}$ in (4.5) are anticommuting c-numbers, however, gives rise to nontrivial problems of interpretation and also to limitations with regard to its physical applicability.

In fact, the applications in sections 5 and 6 for the transverse Ising chain will be derived from (3.4) for fermions, using for each operator $\Gamma_{p\nu}$ a product of γ operators, i.e.

$$\Gamma_{p\nu} = \prod_{\mu=1}^{S_{p\nu}} \gamma_{m(p,\nu,\mu)}, \quad \text{if } S_{p\nu} \neq 0\,. \tag{4.15}$$

The analogue of (4.15) in the bosonic case, which would amount to

$$\Gamma_{p\nu} = \sum_{\mu=1}^{S_{p\nu}} \{\eta_{p\nu\mu m(p,\nu,\mu)} \gamma_{m(p,\nu,\mu)}\}\,, \tag{4.16}$$

leads to the trivial result that both the left-hand side and the right-hand side of (4.14) are zero.

Factorization relations for the bosonic case can be derived assuming e.g. that each $\Gamma_{p\nu}$ with $S_{p\nu} \neq 0$ is a product of linear combinations of two γ-operators, i.e.

$$\Gamma_{p\nu} = \prod_{\mu=1}^{S_{p\nu}} \left\{ \sum_{\alpha(p,\nu,\mu)=\pm 1} \eta_{p\nu\mu m_\alpha(p,\nu,\mu)} \gamma_{m_\alpha(p,\nu,\mu)} \right\}. \tag{4.17}$$

Inserting (4.17) in (4.2) and (4.14) we find that every product of two traces in the left- and right-hand side of (4.14) is a linear combination of 4^{2R} terms, $2R$ being the number of factors (p, ν, μ) with $S_{p\nu} \neq 0$. From these terms only the ones which are proportional to

$$\prod_{p,\nu,\mu} \{\eta_{p\nu\mu m_+(p,\nu,\mu)} \eta_{p\nu\mu m_-(p,\nu,\mu)}\} \tag{4.18}$$

give a nonvanishing contribution, cf. (4.5). Dividing by this factor and working out the result one obtains

$$\begin{aligned} & \sum_{\{\alpha(p,\nu,\mu)=\pm 1\}} \mathrm{Tr}\left[\prod_{p=1}^{2s} \left(\prod_\nu \bar{B}_{p\nu} \prod_{\mu=1}^{S_{p\nu}} \gamma_{m_\alpha(p,\nu,\mu)} \alpha(p,\nu,\mu) \right) \gamma_{i_p} \right] \\ & \quad \times \mathrm{Tr}\left[\prod_{p=1}^{2s} \prod_\nu \bar{B}_{p\nu} \prod_{\mu=1}^{S_{p\nu}} \gamma_{m_{-\alpha}(p,\nu,\mu)} \right] \\ & = s^{-1} \sum_{k<l} \sum_{\{\alpha(p,\nu,\mu)=\pm 1\}} \mathrm{Tr}\Bigg[\left\{ \prod_{p<k} \left(\prod_\nu \bar{B}_{p\nu} \prod_{\mu=1}^{S_{p\nu}} \gamma_{m_\alpha(p,\nu,\mu)} \alpha(p,\nu,\mu) \right) \right\} \\ & \quad \times \left(\prod_\nu \bar{B}_{k\nu} \prod_{\mu=1}^{S_{k\nu}} \gamma_{m_\alpha(k,\nu,\mu)} \alpha(k,\nu,\mu) \right) \gamma_{i_k} \left\{ \prod_{k<p<l} \left(\prod_\nu \bar{B}_{p\nu} \prod_{\mu=1}^{S_{p\nu}} \gamma_{m_\alpha(p,\nu,\mu)} \alpha(p,\nu,\mu) \right) \right\} \end{aligned}$$

$$
\times\left(\prod_{\nu}\bar{B}_{l\nu}\prod_{\mu=1}^{S_{l\nu}}\gamma_{m_{\alpha}(l,\nu,\mu)}\alpha(l,\nu,\mu)\right)\gamma_{i_l}\left\{\prod_{p>l}\left(\prod_{\nu}\bar{B}_{p\nu}\prod_{\mu=1}^{S_{p\nu}}\gamma_{m_{\alpha}(p,\nu,\mu)}\alpha(p,\nu,\mu)\right)\right\}\Bigg]
$$

$$
\times\operatorname{Tr}\Bigg[\left\{\prod_{p<k}\left(\prod_{\nu}\bar{B}_{p\nu}\prod_{\mu=1}^{S_{p\nu}}\gamma_{m_{-\alpha}(p,\nu,\mu)}\right)\gamma_{i_p}\right\}
$$

$$
\times\left(\prod_{\nu}\bar{B}_{k\nu}\prod_{\mu=1}^{S_{k\nu}}\gamma_{m_{-\alpha}(k,\nu,\mu)}\right)\left\{\prod_{k<p<l}\left(\prod_{\nu}\bar{B}_{p\nu}\prod_{\mu=1}^{S_{p\nu}}\gamma_{m_{-\alpha}(p,\nu,\mu)}\right)\gamma_{i_p}\right\}
$$

$$
\times\left(\prod_{\nu}\bar{B}_{l\nu}\prod_{\mu=1}^{S_{l\nu}}\gamma_{m_{-\alpha}(l,\nu,\mu)}\right)\left\{\prod_{p>l}\left(\prod_{\nu}\bar{B}_{p\nu}\prod_{\mu=1}^{S_{p\nu}}\gamma_{m_{-\alpha}(p,\nu,\mu)}\right)\gamma_{i_p}\right\}\Bigg], \tag{4.19}
$$

with

$$
\bar{B}_{p\nu}\equiv\exp B_{p\nu}\,. \tag{4.20}
$$

Eq. (4.19) may be used in particular to derive expressions for a trace involving $2R$ fixed operators $\gamma_{m(p,\nu,\mu)}$ and $2s$ operators $\gamma_{i_1},\ldots,\gamma_{i_{2s}}$ in terms of traces involving $2R'$, $(R'\leqslant R)$, of the operators $\gamma_{m(p,\nu,\mu)}$, and $2s'$, $(s'<s)$, of the operators $\gamma_{i_1},\ldots,\gamma_{i_{2s}}$. This can be done applying (4.19) on an (extended) Hilbert space $h\otimes\hat{h}$, in which $\hat{h}$ is constructed from $2R$ operators $\hat{\gamma}_1,\ldots,\hat{\gamma}_{2R}$. In fact, choosing the operators $\gamma_{i_1},\ldots,\gamma_{i_{2s}}$ as well as $\gamma_{m_+(p,\nu,\mu)}$ in h, and identifying the operators $\gamma_{m_-(p,\nu,\mu)}$ with the operators $\hat{\gamma}_1,\ldots,\hat{\gamma}_{2R}$ in $\hat{h}$ and choosing furthermore $B_{p\nu}$ as a sum of bilinear expressions in h and $\hat{h}$, one can factorize all traces over $h\otimes\hat{h}$ in a trace over h and a trace over $\hat{h}$. After eliminating the traces over $\hat{h}$, one can arrive at explicit results containing only operators of h. In these results traces involving $2R$ fixed operators $\gamma_{m(p,\nu,\mu)}$ and $2s$ operators γ_{i_p} are expressed in traces involving $2R'$, $(R'\leqslant R)$, of the operators $\gamma_{m(p,\nu,\mu)}$ and $2s'$, $(s'<s)$, of the operators γ_{i_p}.

Obviously these relations have a much more complicated structure than the relation which can be inferred from (3.4) in the special case (4.15) for fermions and which expresses traces involving $2R$ operators $\gamma_{m(p,\nu,\mu)}$ and $2s$ operators γ_{i_p} in terms of traces involving the same $2R$ operators $\gamma_{m(p,\nu,\mu)}$ and only two, (or none), of the operators $\gamma_{i_1},\ldots,\gamma_{i_{2s}}$. We shall not write down the much more complicated relations for bosons, as not much can be said about their usefulness in practical applications at the present state of affairs.

5. Differential equations

In this section we consider the correlations

$$
\operatorname{Tr}(O_{n,t}\Gamma\gamma_l\gamma_k K)\,, \tag{5.1}
$$

where the trace is taken over the Hilbert space of the spins $\boldsymbol{S}_1,\ldots,\boldsymbol{S}_N$, or equivalently the Hilbert space of the fermion operators $\gamma_1,\ldots,\gamma_{2N}$, the operator $O_{n,t}$ has been defined by (2.11), and Γ is a product of different fermion operators. The operator K can be an arbitrary product of exponentials (of bilinear forms in the γ's) and linear combinations of γ's, as described in the previous section.

5.1. *Application of general Wick theorem*

Since our main interest lies in the spin correlations (2.13), (2.21), we shall mainly restrict ourselves to the case that

$$
K=\mathrm{e}^{-\beta\mathscr{H}}/\operatorname{Tr}\mathrm{e}^{-\beta\mathscr{H}}\,. \tag{5.2}
$$

For $O_{n,t}$ we have the equation of motion

$$
\frac{\mathrm{d}}{\mathrm{d}t}O_{n,t}=-2J_n\gamma_n(t)\gamma_{n+1}(t)O_{n,t}\,, \tag{5.3}
$$

cf. (2.11) and (2.9), and the equation of motion for (5.1) can be written

$$
\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}(O_{n,t}\Gamma\gamma_l\gamma_k K)=-2J_n\operatorname{Tr}(\gamma_n(t)\gamma_{n+1}(t)O_{n,t}\Gamma\gamma_l\gamma_k K)\,. \tag{5.4}
$$

The right-hand side of (5.4) can be decoupled into averages involving two operators γ besides Γ and K, in view of

$$
\begin{aligned}
&\operatorname{Tr}(\gamma_a(t)\gamma_b(t)O\Gamma\gamma_c\gamma_d K)\operatorname{Tr}(O\Gamma K)=\operatorname{Tr}(\gamma_a(t)\gamma_b(t)O\Gamma K)\operatorname{Tr}(O\Gamma\gamma_c\gamma_d K)\\
&\quad+\operatorname{Tr}(\gamma_a(t)O\Gamma\gamma_d K)\operatorname{Tr}(\gamma_b(t)O\Gamma\gamma_c K)-\operatorname{Tr}(\gamma_a(t)O\Gamma\gamma_c K)\operatorname{Tr}(\gamma_b(t)O\Gamma\gamma_d K)\,,
\end{aligned} \tag{5.5}
$$

where O is a product of exponentials of bilinear expressions of the operators γ. Eq. (5.5) is a special case with $s=2$ of the general relation (3.4), cf. also (3.6), with the operators

$$
Q_1=K\,\mathrm{e}^{\mathrm{i}\mathscr{H}t},\quad Q_2=\mathbb{1},\quad Q_3=\mathrm{e}^{-\mathrm{i}\mathscr{H}t}O\Gamma,\quad Q_4=\mathbb{1}\,. \tag{5.6}
$$

In view of (5.5), eq. (5.4) can be expressed as

$$
\begin{aligned}
&\left\{\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}(O_{n,t}\Gamma\gamma_l\gamma_k K)\right\}\operatorname{Tr}(O_{n,t}\Gamma K)\\
&\quad=-2J_n\operatorname{Tr}(\gamma_n(t)\gamma_{n+1}(t)O_{n,t}\Gamma K)\operatorname{Tr}(O_{n,t}\Gamma\gamma_l\gamma_k K)\\
&\qquad-2J_n\operatorname{Tr}(\gamma_n(t)O_{n,t}\Gamma\gamma_k K)\operatorname{Tr}(\gamma_{n+1}(t)O_{n,t}\Gamma\gamma_l K)\\
&\qquad+2J_n\operatorname{Tr}(\gamma_n(t)O_{n,t}\Gamma\gamma_l K)\operatorname{Tr}(\gamma_{n+1}(t)O_{n,t}\Gamma\gamma_k K)\,.
\end{aligned} \tag{5.7}
$$

(Note that (5.7) may also be derived from a more restricted version of (5.5) without the operator O, since the operator $O_{n,t}$ defined by (2.11) can be expressed as a product of time-dependent operators γ multiplied by a product of time-independent γ's.)

With the commutation relations, cf. (2.8) and (2.11),

$$\gamma_n(t)O_{n,t} = O_{n-1,t}\gamma_n\,,$$
$$\gamma_{n+1}(t)O_{n,t} = O_{n+1,t}\gamma_{n+1}\,, \tag{5.8}$$

eq. (5.7) can be rewritten

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}(O_{n,t}\Gamma\gamma_l\gamma_k K) - \left\{\frac{\mathrm{d}}{\mathrm{d}t}\ln\operatorname{Tr}(O_{n,t}\Gamma K)\right\}\operatorname{Tr}(O_{n,t}\Gamma\gamma_l\gamma_k K)$$
$$= -2J_n\operatorname{Tr}(O_{n-1,t}\gamma_n\Gamma\gamma_k K)\operatorname{Tr}(O_{n+1,t}\gamma_{n+1}\Gamma\gamma_l K)/\operatorname{Tr}(O_{n,t}\Gamma K)$$
$$+ 2J_n\operatorname{Tr}(O_{n-1,t}\gamma_n\Gamma\gamma_l K)\operatorname{Tr}(O_{n+1,t}\gamma_{n+1}\Gamma\gamma_k K)/\operatorname{Tr}(O_{n,t}\Gamma K)\,. \tag{5.9}$$

For different choices of Γ one can derive a variety of nonlinear differential equations. As the main interest lies in the spin correlations, we shall restrict ourselves to the following three choices of Γ and γ_l

$$\Gamma = \gamma_{n+1}\cdots\gamma_{n+s},\quad \gamma_l = \gamma_{n+s+1}\,, \tag{5.10a}$$

or

$$\Gamma = \gamma_n\cdots\gamma_{n-s+1},\quad \gamma_l = \gamma_{n-s},\quad (s = 2, 4, 6, \ldots)\,, \tag{5.10b}$$

and

$$\Gamma = \gamma_{n+1}\cdots\gamma_{n+s},\quad \gamma_l = \gamma_{n+s}\,, \tag{5.11a}$$

or

$$\Gamma = \gamma_n\cdots\gamma_{n-s+1},\quad \gamma_l = \gamma_{n-s+1},\quad (s = 2, 4, 6, \ldots)\,, \tag{5.11b}$$

and

$$\Gamma = \mathbb{1},\quad \gamma_l = \gamma_{n+1}\,, \tag{5.12a}$$

or

$$\Gamma = \mathbb{1},\quad \gamma_l = \gamma_n\,, \tag{5.12b}$$

corresponding to $s = 0$.

Proceeding from (5.10a), (5.10b) one obtains a closed set of differential equations for the correlations

$$A^{(s)}_{n,k} = \operatorname{Tr}(O_{n,t}\gamma_{n+1}\cdots\gamma_{n+s}\gamma_{n+s+1}\gamma_k K)\,, \tag{5.13a}$$

$$\tilde{A}^{(s)}_{n,k} = \operatorname{Tr}(O_{n,t}\gamma_n\cdots\gamma_{n-s+1}\gamma_{n-s}\gamma_k K)\,,$$
$$(s = 0, 2, 4, \ldots)\,. \tag{5.13b}$$

In fact, with (5.10a), (5.10b) inserted in (5.9), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}A^{(s)}_{n,k} - \left(\frac{\mathrm{d}}{\mathrm{d}t}\ln A_{n,n+s}\right)A^{(s)}_{n,k} = -J_n\left\{\frac{A_{n+1,n+s+1}}{A_{n,n+s}}A^{(s)}_{n-1,k} - \frac{A_{n-1,n+s+1}}{A_{n,n+s}}A^{(s-2)}_{n+1,k}\right\},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}^{(s)}_{n,k} - \left(\frac{\mathrm{d}}{\mathrm{d}t}\ln A_{n,n-s}\right)\tilde{A}^{(s)}_{n,k} = J_n\left\{\frac{A_{n-1,n-s-1}}{A_{n,n-s}}\tilde{A}^{(s)}_{n+1,k} - \frac{A_{n+1,n-s-1}}{A_{n,n-s}}\tilde{A}^{(s-2)}_{n-1,k}\right\},$$
$$(s = 2, 4, 6, \ldots)\,. \tag{5.14}$$

In (5.14) use has been made of the special values

$$A^{(s)}_{n,n+s+1} = \tfrac{1}{2}A_{n,n+s},\quad A^{(s)}_{n,n+s+2} = A_{n,n+s+2}\,,$$
$$\tilde{A}^{(s)}_{n,n-s} = \tfrac{1}{2}A_{n,n-s},\quad \tilde{A}^{(s)}_{n,n-s-1} = A_{n,n-s-2}\,,$$
$$(s = 0, 2, 4, \ldots)\,, \tag{5.15}$$

in which the A's are defined by

$$A_{n,n+s} = \operatorname{Tr}(O_{n,t}\gamma_{n+1}\cdots\gamma_{n+s}K)\,,$$
$$A_{n,n-s} = \operatorname{Tr}(O_{n,t}\gamma_n\cdots\gamma_{n-s+1}K)\,,$$
$$(s = 2, 4, \ldots)\,, \tag{5.16}$$
$$A_{n,n} = \operatorname{Tr}(O_{n,t}K)\,.$$

In the case (5.2) the A's in (5.16) reduce to the expressions (2.14) for the spin correlations (2.13), (2.21).

From (5.11a), (5.11b) and (5.9) it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}A^{(s-2)}_{n,k} - \left(\frac{\mathrm{d}}{\mathrm{d}t}\ln A_{n,n+s}\right)A^{(s-2)}_{n,k} = -J_n\left\{\frac{A_{n+1,n+s-1}}{A_{n,n+s}}A^{(s)}_{n-1,k} - \frac{A_{n-1,n+s-1}}{A_{n,n+s}}A^{(s-2)}_{n+1,k}\right\},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}^{(s-2)}_{n,k} - \left(\frac{\mathrm{d}}{\mathrm{d}t}\ln A_{n,n-s}\right)\tilde{A}^{(s-2)}_{n,k} = J_n\left\{\frac{A_{n-1,n-s+1}}{A_{n,n-s}}\tilde{A}^{(s)}_{n+1,k} - \frac{A_{n+1,n-s+1}}{A_{n,n-s}}\tilde{A}^{(s-2)}_{n-1,k}\right\},$$
$$(s = 2, 4, \ldots)\,, \tag{5.17}$$

and from (5.12a), (5.12b) and (5.9) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}A^{(0)}_{n,k} - \left(\frac{\mathrm{d}}{\mathrm{d}t}\ln A_{n,n}\right)A^{(0)}_{n,k} = -J_n\left\{\frac{A_{n+1,n+1}}{A_{n,n}}A^{(0)}_{n-1,k} - 2\frac{A_{n-1,n+1}}{A_{n,n}}\tilde{A}^{(0)}_{n+1,k}\right\},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}^{(0)}_{n,k} - \left(\frac{d}{dt}\ln A_{n,n}\right)\tilde{A}^{(0)}_{n,k} = J_n\left\{\frac{A_{n-1,n-1}}{A_{n,n}}\tilde{A}^{(0)}_{n+1,k} - 2\frac{A_{n+1,n-1}}{A_{n,n}}A^{(0)}_{n-1,k}\right\}. \tag{5.18}$$

Apart from (5.15), other relations for the $A^{(s)}_{n,k}$, $\tilde{A}^{(s)}_{n,k}$ can be derived from the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}O_{n,t} = \mathrm{i}\mathscr{H}O_{n,t} - \mathrm{i}O_{n,t}(\mathscr{H} + h_n)\,. \tag{5.19}$$

In fact, from (5.19) and the commutation property

$$(\mathcal{H}+h_n)\gamma_{n+1}\cdots\gamma_{n+s}=\gamma_{n+1}\cdots\gamma_{n+s}(\mathcal{H}+h_{n+s})\,, \tag{5.20}$$

we have, cf. (2.9),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A_{n,n+s} &= \mathrm{Tr}\,\{(\mathrm{i}K\mathcal{H}O_{n,t}-\mathrm{i}KO_{n,t}(\mathcal{H}+h_n))\gamma_{n+1}\cdots\gamma_{n+s}\}\\ &= -\mathrm{Tr}\{O_{n,t}\gamma_{n+1}\cdots\gamma_{n+s}\,\mathrm{i}[\mathcal{H},K]\}\\ &\quad +2J_{n+s}\,\mathrm{Tr}\{O_{n,t}\gamma_{n+1}\cdots\gamma_{n+s}\gamma_{n+s+1}\gamma_{n+s}K\}\,.\end{aligned} \tag{5.21}$$

In the case of (5.2) we obtain the relations

$$\begin{aligned}A^{(s)}_{n,n+s} &= (2J_{n+s})^{-1}\frac{\mathrm{d}}{\mathrm{d}t}A_{n,n+s}\,,\\ \tilde A^{(s)}_{n,n-s+1} &= -(2J_{n-s})^{-1}\frac{\mathrm{d}}{\mathrm{d}t}A_{n,n-s}\,,\\ A^{(s)}_{n,n+s+3} &= -2A^{(s+2)}_{n,n+s+2} = -(J_{n+s+2})^{-1}\frac{\mathrm{d}}{\mathrm{d}t}A_{n,n+s+2}\,,\\ \tilde A^{(s)}_{n,n-s-2} &= -2\tilde A^{(s+2)}_{n,n-s-1} = (J_{n-s-2})^{-1}\frac{\mathrm{d}}{\mathrm{d}t}A_{n,n-s-2}\,,\\ &\qquad (s=0,2,4,\ldots)\,,\end{aligned} \tag{5.22}$$

using (5.21) and a similar relation for $\mathrm{d}A_{n,n-s}/\mathrm{d}t$.

5.2. *Generating functions*

The results (5.14), (5.17) and (5.18) can be expressed equivalently in terms of generating functions, we define

$$\begin{aligned}X^{(s)\mathrm{L}}_n(\lambda,t) &= (A_{n,n+s})^{-1}\sum_{k\le n+s+1}A^{(s)}_{n,k}\lambda^{n+s-k}\,,\\ X^{(s)\mathrm{R}}_n(\lambda,t) &= (A_{n,n+s})^{-1}\sum_{k> n+s+1}A^{(s)}_{n,k}\lambda^{n+s-k}\,,\\ \tilde X^{(s)\mathrm{L}}_n(\lambda,t) &= (A_{n,n-s})^{-1}\sum_{k< n-s}\tilde A^{(s)}_{n,k}\lambda^{n-s-k}\,,\\ \tilde X^{(s)\mathrm{R}}_n(\lambda,t) &= (A_{n,n-s})^{-1}\sum_{k\ge n-s}\tilde A^{(s)}_{n,k}\lambda^{n-s-k}\,.\end{aligned} \tag{5.23}$$

In fact, from (5.14) and (5.23) it can be shown that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}X^{(s)\mathrm{L,R}}_n(\lambda,t) = -J_n\lambda\Bigg\{&\frac{A_{n+1,n+s+1}A_{n-1,n+s-1}}{A^2_{n,n+s}}X^{(s)\mathrm{L,R}}_{n-1}\\ &-\frac{A_{n-1,n+s+1}A_{n+1,n+s-1}}{A^2_{n,n+s}}X^{(s-2)\mathrm{L,R}}_{n+1}\Bigg\}\,,\end{aligned} \tag{5.24}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde X^{(s)\mathrm{L,R}}_n(\lambda,t) = J_n\lambda^{-1}\Bigg\{&\frac{A_{n+1,n-s+1}A_{n-1,n-s-1}}{A^2_{n,n-s}}\tilde X^{(s)\mathrm{L,R}}_{n+1}\\ &-\frac{A_{n+1,n-s-1}A_{n-1,n-s+1}}{A^2_{n,n-s}}\tilde X^{(s-2)\mathrm{L,R}}_{n-1}\Bigg\}\,,\\ &(s=2,4,6,\ldots)\,.\end{aligned} \tag{5.24}$$

The derivation of (5.24) is straightforward, but somewhat tedious. Note that there is a complete separation with respect to the left generating functions $X^{(s)\mathrm{L}}_n$, $\tilde X^{(s)\mathrm{L}}_n$ on the one hand, and the right generating functions $X^{(s)\mathrm{R}}_n$, $\tilde X^{(s)\mathrm{R}}_n$ on the other hand, implying that $X^{(s)\mathrm{L}}_n$ and $X^{(s)\mathrm{R}}_n$, (and also $\tilde X^{(s)\mathrm{L}}_n$ and $\tilde X^{(s)\mathrm{R}}_n$), satisfy the same differential equation. This feature may not be obvious at first sight from the definition, but can be checked explicitly. When one evaluates e.g. the right-hand side of $\mathrm{d}X^{(s)\mathrm{L}}_n/\mathrm{d}t$ using (5.23) and (5.14), one finds that the unwanted correlations $A^{(s)}_{m,k}$ with $k>m+s+1$, i.e. $A^{(s)}_{n-1,n+s+1}$, $A^{(s-2)}_{n+1,n+s+1}$, which are not contained in $X^{(s)\mathrm{L}}_{n-1}$, $X^{(s-2)\mathrm{L}}_{n+1}$, cancel in view of (5.15). This cancellation may be related to the Wiener–Hopf factorization exploited e.g. within the (block)-Toeplitz determinant approach in refs. 19–23.

In a similar way one can derive from (5.17), (5.23) and from (5.18), (5.23), respectively,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}X^{(s-2)\mathrm{L,R}}_n-\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{A_{n,n+s}}{A_{n,n+s-2}}\right)X^{(s-2)\mathrm{L,R}}_n\\ &\quad=-J_n\lambda^{-1}\frac{A_{n+1,n+s-1}A_{n-1,n+s-1}}{A_{n,n+s-2}A_{n,n+s}}\left(X^{(s)\mathrm{L,R}}_{n-1}-X^{(s-2)\mathrm{L,R}}_{n+1}\right),\\ &\frac{\mathrm{d}}{\mathrm{d}t}\tilde X^{(s-2)\mathrm{L,R}}_n-\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{A_{n,n-s}}{A_{n,n-s+2}}\right)\tilde X^{(s-2)\mathrm{L,R}}_n\\ &\quad=J_n\lambda\frac{A_{n-1,n-s+1}A_{n+1,n-s+1}}{A_{n,n-s+2}A_{n,n-s}}\left(\tilde X^{(s)\mathrm{L,R}}_{n+1}-\tilde X^{(s-2)\mathrm{L,R}}_{n-1}\right),\\ &\quad(s=2,4,\ldots)\,,\end{aligned} \tag{5.25}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}X^{(0)\mathrm{L,R}}_n &= -J_n\left(\lambda\frac{A_{n+1,n+1}A_{n-1,n-1}}{A^2_{n,n}}X^{(0)\mathrm{L,R}}_{n-1}-2\lambda^{-1}\frac{A_{n+1,n+1}A_{n-1,n+1}}{A^2_{n,n}}\tilde X^{(0)\mathrm{L,R}}_{n+1}\right),\\ \frac{\mathrm{d}}{\mathrm{d}t}\tilde X^{(0)\mathrm{L,R}}_n &= J_n\left(\lambda^{-1}\frac{A_{n-1,n-1}A_{n+1,n+1}}{A^2_{n,n}}\tilde X^{(0)\mathrm{L,R}}_{n+1}-2\lambda\frac{A_{n-1,n-1}A_{n+1,n-1}}{A^2_{n,n}}X^{(0)\mathrm{L,R}}_{n-1}\right),\\ &\quad(s=0)\,,\end{aligned} \tag{5.26}$$

again the unwanted terms cancelling upon using (5.15).

It may be noted that the coefficients of X in the right-hand sides of (5.24)–(5.26) are of the form $A_{n-1,\ldots}A_{n+1,\ldots}/A_{n,\ldots}A_{n,\ldots}$, which is reminiscent of the exponential interactions in the classical Toda chain[17,18]. In the following section we shall derive nonlinear differential equations for the spin correlations which in special cases reduce to the equation of motion of the Toda chain.

Using (5.15), (5.22), and the definition (5.23) one has the integral relations

$$
\begin{aligned}
&\frac{1}{\pi i}\int_C d\lambda X_n^{(s)L}(\lambda,t) = 1\,,\\
&\frac{1}{\pi i}\int_C d\lambda X_n^{(s)L}(\lambda,t)\lambda^{-1} = (J_{n+s})^{-1}\frac{d}{dt}\ln A_{n,n+s}\,,\\
&\frac{1}{2\pi i}\int_C d\lambda X_n^{(s)R}(\lambda,t)\lambda = (A_{n,n+s})^{-1}A_{n,n+s+2}\,,\\
&\frac{1}{2\pi i}\int_C d\lambda X_n^{(s)R}(\lambda,t)\lambda^{2} = -(J_{n+s+2}A_{n,n+s})^{-1}\frac{d}{dt}A_{n,n+s+2}\,,
\end{aligned}
\tag{5.27a}
$$

and

$$
\begin{aligned}
&\frac{1}{\pi i}\int_C d\lambda \tilde{X}_n^{(s)R}(\lambda,t)\lambda^{-1} = 1\,,\\
&\frac{1}{\pi i}\int_C d\lambda \tilde{X}_n^{(s)R}(\lambda,t) = -(J_{n-s})^{-1}\frac{d}{dt}\ln A_{n,n-s}\,,\\
&\frac{1}{2\pi i}\int_C d\lambda \tilde{X}_n^{(s)L}(\lambda,t)\lambda^{-2} = (A_{n,n-s})^{-1}A_{n,n-s-2}\,,\\
&\frac{1}{2\pi i}\int_C d\lambda \tilde{X}_n^{(s)L}(\lambda,t)\lambda^{-3} = (J_{n-s-2}A_{n,n-s})^{-1}\frac{d}{dt}A_{n,n-s-2}\,,
\end{aligned}
\tag{5.27b}
$$

where C is an arbitrary contour surrounding the origin in the complex λ plane. Finally, from (5.23) we have

$$
\begin{aligned}
&\int_C d\lambda X_n^{(s)L}(\lambda,t)\lambda^{p} = 0, \quad p \geq 1\,,\\
&\int_C d\lambda X_n^{(s)R}(\lambda,t)\lambda^{p} = 0, \quad p \leq 0\,,\\
&\int_C d\lambda \tilde{X}_n^{(s)R}(\lambda,t)\lambda^{p} = 0, \quad p \leq -2\,,\\
&\int_C d\lambda \tilde{X}_n^{(s)L}(\lambda,t)\lambda^{p} = 0, \quad p \geq -1\,.
\end{aligned}
\tag{5.28}
$$

Remarks

i) In this section we have derived a number of differential equations for the quantities $A_{n,k}^{(s)}$ and $X_n^{(s)L,R}(\lambda,t)$, cf. (5.14), (5.17), (5.18) and (5.24)–(5.26), and also for the corresponding quantities with tildes, cf. (5.13) and (5.23) for the definitions. By means of the transformation $\gamma_m \to (-1)^m\gamma_{2N-m+1}$ the equations for the quantities with tildes can be inferred from the corresponding equations for the quantities without tildes. The details of the derivation are somewhat tedious and are presented in appendix A.

ii) The differential equations in this section have been derived using the equation of motion (5.3), the thermodynamic Wick theorem (3.4), and the anticommutation relation (2.3). The order of the different steps in the derivation is not unique and as an example a slightly different derivation is given in appendix B.

6. Applications

In this section we shall derive closed sets of nonlinear differential–difference equations for the spin correlations, using the results of section 5. An alternative, more direct, derivation will be presented in appendix C. Then we shall discuss the necessary initial conditions, (which are only simple in the infinite-temperature limit). For the homogeneous Ising chain in transverse field an additional equation will be derived and the scaling limit will be discussed. Finally, a few remarks will be made about the case of critical transverse field.

6.1. *Spin correlations*

In section 5 we have derived the nonlinear differential equations (5.24)–(5.26) for the generating functions (5.23) of the correlations (5.13). By multiplying the left- and right-hand sides of these equations by a suitable factor λ^p, by integrating over λ over a contour C surrounding the origin, and by using the integral relations

(5.27), one may attempt to derive closed differential equations for the parameters $A_{m,m+s}$. (Of course the same information can be obtained by considering (5.14), (5.17) and (5.18) for suitable values of k, but the equations for the generating functions may be more convenient to use.)

Since the main physical interest lies in the spin correlations (2.13), (2.21), we shall restrict ourselves to the case (5.2), in which all the traces involving the operator K are thermal averages with respect to the Hamiltonian (2.1).

From the first equation (5.24) one obtains the closed sets of differential equations

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln A_{n,n+s} = -J_nJ_{n+s}\left(\frac{A_{n+1,n+s+1}A_{n-1,n+s-1}}{A^2_{n,n+s}} - \frac{A_{n+1,n+s-1}A_{n-1,n+s+1}}{A^2_{n,n+s}}\right), \tag{6.1}$$

and

$$A^2_{n,n+s}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A_{n,n+s+2}}{A_{n,n+s}}\right) = \frac{J_n}{J_{n+s+1}}A^2_{n+1,n+s+1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A_{n-1,n+s+1}}{A_{n+1,n+s+1}}\right), \tag{6.2}$$

$$(s = 2, 4, 6, \ldots).$$

Eq. (6.1) follows multiplying the first equation (5.24) for $X^{(s)\mathrm{L}}_n$ by λ^{-1} and integrating over a contour C surrounding the origin, cf. the first two integral relations (5.27a). Eq. (6.2) follows from the first equation (5.24) for $X^{(s)\mathrm{R}}_n$ after multiplication of both sides by λ and integrating over a contour C surrounding the origin, cf. the last two integral relations (5.27a).

From the second equation (5.24) it can be shown that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln A_{n,n-s} = -J_nJ_{n-s}\left(\frac{A_{n-1,n-s-1}A_{n+1,n-s+1}}{A^2_{n,n-s}} - \frac{A_{n-1,n-s+1}A_{n+1,n-s-1}}{A^2_{n,n-s}}\right) \tag{6.3}$$

and

$$A^2_{n,n-s}\frac{\mathrm{d}}{\mathrm{d}t}\frac{A_{n,n-s-2}}{A_{n,n-s}} = \frac{J_n}{J_{n-s-1}}A^2_{n-1,n-s-1}\frac{\mathrm{d}}{\mathrm{d}t}\frac{A_{n+1,n-s-1}}{A_{n-1,n-s-1}}, \tag{6.4}$$

$$(s = 2, 4, 6, \ldots),$$

cf. (5.27b).

Furthermore, from the first equation (5.26) it follows that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln A_{n,n} = -J_n^2\left(\frac{A_{n+1,n+1}A_{n-1,n-1}}{A^2_{n,n}} - 4\frac{A_{n+1,n-1}A_{n-1,n+1}}{A^2_{n,n}}\right) \tag{6.5}$$

and

$$A^2_{n,n}\frac{\mathrm{d}}{\mathrm{d}t}\frac{A_{n,n+2}}{A_{n,n}} = \frac{J_n}{J_{n+1}}A^2_{n+1,n+1}\frac{\mathrm{d}}{\mathrm{d}t}\frac{A_{n-1,n+1}}{A_{n+1,n+1}}, \tag{6.6}$$

and from the second equation (5.26) we have (6.5) and

$$A^2_{n,n}\frac{\mathrm{d}}{\mathrm{d}t}\frac{A_{n,n-2}}{A_{n,n}} = \frac{J_n}{J_{n-1}}A^2_{n-1,n-1}\frac{\mathrm{d}}{\mathrm{d}t}\frac{A_{n+1,n-1}}{A_{n-1,n-1}}. \tag{6.7}$$

Using eqs. (2.13) and (2.21) for the spin correlations in the Ising chain (2.1) and the dual chain (2.19), (2.20), eqs. (6.1), (6.3), and (6.5) can be combined to

$$\begin{aligned}&\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\langle S^x_{\frac{1}{2}n+\frac{1}{2}}(t)S^x_{\frac{1}{2}n+\frac{1}{2}+p}\rangle\right\}\langle S^x_{\frac{1}{2}n+\frac{1}{2}}(t)S^x_{\frac{1}{2}n+\frac{1}{2}+p}\rangle^2\\ &\quad = -J_nJ_{n+2p}\{\langle S^x_{\frac{1}{2}n+1}(t)S^x_{\frac{1}{2}n+1+p}\rangle\langle S^x_{\frac{1}{2}n}(t)S^x_{\frac{1}{2}n+p}\rangle\\ &\qquad -\langle S^x_{\frac{1}{2}n+1}(t)S^x_{\frac{1}{2}n+p}\rangle\langle S^x_{\frac{1}{2}n}(t)S^x_{\frac{1}{2}n+1+p}\rangle\},\end{aligned} \tag{6.8}$$

for integral values of p and n.

From (6.2), (6.4), (6.6) and (6.7) it follows in a similar way that

$$\begin{aligned}&\langle S^x_{\frac{1}{2}n+\frac{1}{2}}(t)S^x_{\frac{1}{2}n+\frac{1}{2}+p}\rangle^2\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\langle S^x_{\frac{1}{2}n+\frac{1}{2}}(t)S^x_{\frac{1}{2}n+\frac{3}{2}+p}\rangle}{\langle S^x_{\frac{1}{2}n+\frac{1}{2}}(t)S^x_{\frac{1}{2}n+\frac{1}{2}+p}\rangle}\right\}\\ &\quad = \frac{J_n}{J_{n+1+2p}}\langle S^x_{\frac{1}{2}n+1}(t)S^x_{\frac{1}{2}n+1+p}\rangle^2\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\langle S^x_{\frac{1}{2}n}(t)S^x_{\frac{1}{2}n+1+p}\rangle}{\langle S^x_{\frac{1}{2}n+1}(t)S^x_{\frac{1}{2}n+1+p}\rangle}\right\},\end{aligned} \tag{6.9}$$

for integral values of p and n. Both eqs. (6.8) and (6.9) are duality relations. If n is odd, the correlations in the left-hand sides are spin correlations of the Ising chain, whereas the right-hand sides contain the correlations of the dual chain. For even n, it is the other way around. Eq. (6.8) was already given in ref. 31. In fact, if one eliminates the time derivatives in (6.8), (6.9) using (2.22), the resulting equations can be inferred from the Wick theorem (3.4) directly, in view of (2.2) and (2.18), see appendix C for some details. Although the spin correlations are completely determined by (6.8) and the proper initial conditions, additional relations like (6.9) may be helpful, as it is not easy to find general solutions of (6.8). So far we did not use eq. (5.25). Although (5.25) is obviously different from (5.24), eq. (5.25) together with (5.27) does not lead to new closed differential equations for the spin correlations. In fact, (6.1) follows from the first equation (5.25) for $X^{(s)\mathrm{R}}_n$ after multiplying by λ^2 and integration over a contour C surrounding the origin, whereas (6.2) follows from the first equation (5.25) for $X^{(s)\mathrm{L}}_n$ by integration over C. In a similar way (6.3) and (6.4) can be derived from the second equation (5.25).

Remarks

So far we have shown that both choices (5.10) and (5.11) for Γ and γ_l, cf. (5.14) and (5.17), and (5.24) and (5.25) respectively, lead to the same equations (6.1)–(6.7), or (6.8), (6.9) for the spin correlations. This does not mean, however, that eqs. (5.24) and (5.25) are equivalent. In fact, both equations can be combined to derive new equations for the generating functions $X_n^{(s)L,R}(\lambda, t)$. In particular one can derive "algebraic relations", which contain only the derivatives of the spin correlations $A_{n,n+s}$, $A_{n,n-s}$, but which are linear in the generating functions $X_n^{(s)L,R}(\lambda, t)$, and also (second-order) differential equations which do not depend explicitly on λ. Further details are given in appendix D.

6.2. *Initial conditions*

The nonlinear differential equations (6.8) and (6.9) must be solved under well-defined initial conditions at $t=0$. Restricting ourselves to the case (5.2), the initial conditions at $t=0$ are given by thermal averages with respect to the Ising Hamiltonian (2.1). For eqs. (5.14), (5.17) and (5.18) we then have the initial conditions

$$A_{n,k}^{(s)}\big|_{t=0} = \langle \gamma_{n+1}\cdots\gamma_{n+s}\gamma_{n+s+1}\gamma_k\rangle = \det C_{\alpha\beta}, \tag{6.10}$$

where

$$C_{\alpha\beta} = \begin{cases} \langle \gamma_{n+2\alpha-1}\gamma_{n+2\beta}\rangle, & \alpha = 1,\dots,\tfrac12 s+1, \quad \beta = 1,\dots,\tfrac12 s, \\ \langle \gamma_{n+2\alpha-1}\gamma_k\rangle, & \alpha = 1,\dots,\tfrac12 s+1, \quad \beta = \tfrac12 s+1, \end{cases} \tag{6.11}$$

and

$$\tilde{A}_{n,k}^{(s)}\big|_{t=0} = \langle \gamma_n\cdots\gamma_{n-s+1}\gamma_{n-s}\gamma_k\rangle = \det \tilde{C}_{\alpha\beta}, \tag{6.12}$$

with

$$\tilde{C}_{\alpha\beta} = \begin{cases} \langle \gamma_{n+2-2\alpha}\gamma_{n+1-2\beta}\rangle, & \alpha = 1,\dots,\tfrac12 s+1, \quad \beta = 1,\dots,\tfrac12 s, \\ \langle \gamma_{n+2-2\alpha}\gamma_k\rangle, & \alpha = 1,\dots,\tfrac12 s+1, \quad \beta = \tfrac12 s+1, \end{cases} \qquad (s = 0, 2, 4, \dots). \tag{6.13}$$

In the derivation of (6.10)–(6.13) use has been made of the Wick theorem[14–16] together with the property

$$\langle \gamma_m\gamma_{m+2k}\rangle = 0, \quad k \neq 0. \tag{6.14}$$

Eq. (6.14) follows from eqs. (2.3) and (2.4). In fact, from eq. (2.3) and the relation $\langle A\rangle = \langle A^\dagger\rangle^*$ for the operator $A = \gamma_m\gamma_{m+2k}$, it follows that $\langle\gamma_m\gamma_{m+2k}\rangle$ is imaginary for $k\neq 0$, whereas on the other hand from (2.4) it can be shown that $\langle\gamma_m\gamma_{m+2k}\rangle$ is real. (In the standard representation, γ_{2j-1} is real and γ_{2j} is imaginary, see (2.2).)

The equilibrium pair correlations are given by[9]

$$\langle\gamma_k\gamma_l\rangle = \{\mathbf{1} + \mathrm{e}^{-\mathrm{i}\beta\mathbf{a}}\}^{-1}_{kl}, \tag{6.15}$$

where $\mathbf{1}$ is the $2N\times 2N$ unit matrix and the matrix elements of $\mathbf{a}$ for the Ising chain (2.4) are given by

$$a_{kl} = J_k\delta_{l,k+1} - J_l\,\delta_{k,l+1}. \tag{6.16}$$

The initial conditions for the spin correlations in (6.8) and (6.9) are given by, cf. (2.13), (2.21), (5.15), (5.22) and (6.10)–(6.14),

$$\begin{aligned} \langle S^x_{n+\frac12}(t)S^x_{n+\frac12+p}\rangle\big|_{t=0} &= \langle S^x_{n+\frac12}S^x_{n+\frac12+p}\rangle = \tfrac14(2\mathrm{i})^p\langle\gamma_{2n+1}\cdots\gamma_{2n+2p}\rangle, \\ \langle S^x_{n+\frac12}(t)S^x_{n+\frac12-p}\rangle\big|_{t=0} &= \langle S^x_{n+\frac12}S^x_{n+\frac12-p}\rangle = \tfrac14(-2\mathrm{i})^p\langle\gamma_{2n}\cdots\gamma_{2n-2p+1}\rangle, \\ & p = 0,1,2,\dots, \end{aligned} \tag{6.17}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle S^x_{n+\frac12}(t)S^x_{n+\frac12+p}\rangle\big|_{t=0} = -\tfrac12 J_{2n}\langle\gamma_{2n}\gamma_{2n+1}\rangle\delta_{p,0}, \tag{6.18}$$

(p integer, n integer or half-integer).

In the uniform case $J_k = J$, (k = even), $J_k = b$, (k = odd), the determinants (6.10)–(6.13) are translationally invariant, apart from one row or column. In particular, the equal-time correlations are Toeplitz determinants[57–59].

A further characterization of the equal-time correlations has been given in the scaling limit[20,25,26] leading to the reduced density matrix of the impenetrable Bose system. Following the monodromy treatment of Jimbo and Miwa[60] for the two-point function on the diagonal in the two-dimensional Ising model one can also derive a recursion relation for the two-point function as a function of the distance at zero temperature in the uniform case. An alternative derivation[61] can be given by studying the Toeplitz determinants[57,58] generated by a kernel corresponding to (6.15) and kernels with a linear factor added in the numerator or the denominator. The recursion relations then follow from theorems on compound determinants and the linear recursion relations satisfied by (6.15).

For arbitrary temperature the initial conditions (6.17) and (6.18) are such that explicit solutions of (6.8) and (6.9) are not easily obtained. Of course, we do not mean here the formal solution (2.16), which with the help of the Wick theorem[14–16] can be expressed as a $(2m+s)\times(2m+s)$ Pfaffian consisting of pair correlations

$$\langle\gamma_k(t)\gamma_l\rangle = \sum_m (\mathrm{e}^{\mathbf{a}t})_{km}\langle\gamma_m\gamma_l\rangle, \tag{6.19}$$

in which $\mathbf{a}$ and $\langle\gamma_m\gamma_l\rangle$ have been given by (6.16) and (6.15), respectively. From a formal point of view one might argue that (2.16) provides a complete linearization of the nonlinear differential equations (6.8) and (6.9) in the sense that the appropriate solutions can be found solving linear problems. This feature, however, does not seem to be particularly useful to obtain explicit results for the spin correlations apart from special cases in which the Pfaffians or related expressions in terms of block-Toeplitz determinants can be evaluated explicitly. In connection with this a direct investigation of the differential equations (6.8) and (6.9) may prove to be useful. In the remaining part of this section we shall discuss two special cases: the infinite temperature limit and the homogeneous Ising chain in a transverse field.

6.3. *Infinite temperature limit*

At infinite temperature substantial simplifications occur, in view of the property

$$\begin{aligned}\operatorname{Tr} O_{n,t}\gamma_i\gamma_j = 0, \quad &\text{if } i,j \leqslant n,\\ &\text{or if } i,j > n, \quad i \neq j.\end{aligned} \tag{6.20}$$

Eq. (6.20) is an immediate consequence of the reality of the trace, cf. (2.4), (2.9) and (2.11), and of the canonical transformation

$$\gamma_k \to \begin{cases} -\gamma_k, & (k \leqslant n),\\ \gamma_k, & (k > n).\end{cases} \tag{6.21}$$

which implies that the trace is imaginary.

From the Wick theorem (3.4), cf. also section 2 of ref. 9, it follows that

$$A_{n,n+s} = A_{n,n-s} = 0, s \neq 0, \quad \text{for } \beta = 0, \tag{6.22}$$

or equivalently

$$\langle S^x_{\frac{1}{2}n+\frac{1}{2}}(t) S^x_{\frac{1}{2}n+\frac{1}{2}+p}\rangle = 0, p \neq 0, \quad \text{for } \beta = 0, \tag{6.23}$$

so that all spin correlations, apart from the autocorrelation function vanish at all times at infinite temperature, as noted previously[1,8,9,12].

In view of (6.23), eq. (6.9) becomes trivial and eq. (6.8) reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln\langle S^x_{\frac{1}{2}n+\frac{1}{2}}(t) S^x_{\frac{1}{2}n+\frac{1}{2}}\rangle = -J_n^2 \frac{\langle S^x_{\frac{1}{2}n+1}(t) S^x_{\frac{1}{2}n+1}\rangle\langle S^x_{\frac{1}{2}n}(t) S^x_{\frac{1}{2}n}\rangle}{\langle S^x_{\frac{1}{2}n+\frac{1}{2}}(t) S^x_{\frac{1}{2}n+\frac{1}{2}}\rangle^2}. \tag{6.24}$$

Introducing the variables

$$\mathrm{e}^{-r_n(t)} = J_n^2 \frac{\langle S^x_{\frac{1}{2}n+1}(t) S^x_{\frac{1}{2}n+1}\rangle\langle S^x_{\frac{1}{2}n}(t) S^x_{\frac{1}{2}n}\rangle}{\langle S^x_{\frac{1}{2}n+\frac{1}{2}}(t) S^x_{\frac{1}{2}n+\frac{1}{2}}\rangle^2}, \tag{6.25}$$

eq. (6.24) leads to

$$\ddot{r}_n = \mathrm{e}^{-r_{n+1}} + \mathrm{e}^{-r_{n-1}} - 2\mathrm{e}^{-r_n}, \tag{6.26}$$

with the initial conditions

$$r_n(0) = -\ln J_n^2, \quad \dot{r}_n(0) = 0, \tag{6.27}$$

cf. also ref. 9. Eq. (6.26) is the equation of motion for the Toda lattice[17,18]), apart from a minus sign, which can be repaired introducing an imaginary time $\tau = it$. For the homogeneous Ising model with $J_{2j-1} = b_j = b$, $J_{2j} = J^x_j = J^x$, independent of the site j, explicit solutions have been obtained in terms of periodic elliptic functions[8,9]).

The spin correlations can be obtained from

$$\langle S^x_{\frac{1}{2}n+\frac{1}{2}}(t) S^x_{\frac{1}{2}n+\frac{1}{2}}\rangle = \frac{1}{4}\exp\left[-\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \exp\{-r_n(t_2)\}\right], \tag{6.28}$$

and in the homogeneous case the result can be expressed in terms of a Gaussian multiplied by an elliptic theta function[8,9]), see also refs. 10, 11 for more general results on the XY-model. The results at infinite temperature are quite different from the ones at zero temperature, given in refs. 19, 20, 29, 72, which show in general power-law decay $\sim t^{-\lambda}$, where λ can have several values depending on the specific regime for the exchange J^x and the magnetic field b[19,22]).

6.4. *Homogeneous Ising chain*

For the homogeneous Ising chain we have

$$J_{2j-1} = b_j \equiv b, \quad J_{2j} = J^x_j \equiv J^x, \tag{6.29}$$

cf. (2.5), independent of the site j. From (6.29) it follows that

$$\begin{aligned} C_p &\equiv \langle S^x_{j+\frac{1}{2}}(t) S^x_{j+\frac{1}{2}+p}\rangle,\\ D_p &\equiv \langle S^x_j(t) S^x_{j+p}\rangle,\end{aligned} \tag{6.30}$$

independent of (the integer) j, for spins in the bulk of the chain. Here D_p is the correlation function between two x components of spins at distance p in the Ising chain with exchange J^x, in the presence of a magnetic field b, whereas C_p is the correlation between two x components of spins at distance p in the dual Ising chain with exchange b in a magnetic field J^x.

Eqs. (6.8) and (6.9) can be rewritten as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln C_p = J^{x2}\left(\frac{D_{p+1}D_{p-1} - D_p^2}{C_p^2}\right), \tag{6.31}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln D_p = b^2\left(\frac{C_{p+1}C_{p-1} - C_p^2}{D_p^2}\right), \tag{6.32}$$

and

$$J^x D_p D_{p+1} \frac{\mathrm{d}}{\mathrm{d}t} \ln\frac{D_{p+1}}{D_p} = b C_p C_{p+1} \frac{\mathrm{d}}{\mathrm{d}t} \ln\frac{C_{p+1}}{C_p}. \tag{6.33}$$

From eqs. (6.31)–(6.33) one can derive the relation

$$\frac{bJ^x(D_{p-1}C_{p+1} + D_{p+1}C_{p-1}) - (D_p\ddot{C}_p + C_p\ddot{D}_p - 2\dot{C}_p\dot{D}_p)}{C_p D_p} = (J^{x2} + b^2), \tag{6.34}$$

in which the dots denote differentiations with respect to the time t. The derivation of (6.34), which may be regarded as an integral of motion of (6.31)–(6.33), or as another application of (3.4) is not entirely trivial and will be given in appendix E. Inserting (6.31) and (6.32) in (6.34) we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{C_p}{D_p}\right)^2 = b^2\left[\left(\frac{C_p}{D_p}\right)^2 - 1\right] + J^{x2}\left[\left(\frac{D_p}{C_p}\right)^2 - 1\right] - \left[b\frac{C_{p-1}}{D_p} - J^x\frac{D_{p-1}}{C_p}\right]\left[b\frac{C_{p+1}}{D_p} - J^x\frac{D_{p+1}}{C_p}\right]. \tag{6.35}$$

In the high-temperature limit, where $C_p = D_p = 0$ for $p \neq 0$ and the last term in the right-hand side is 0, eq. (6.35) has been used to express C_0/D_0 in terms of periodic elliptic functions[8,9]) and the spin correlations follow from (6.31) and (6.32) by direct integration[8,9]). For finite β, the last term in (6.35) gives rise to complications but it is possible to rewrite (6.35), with the help of (6.31), (6.32), as an equation for just the ratio of the two-spin correlation function on the lattice and the one on the dual lattice,

$$G_p \equiv \left(\frac{|b|}{|J^x|}\right)^{1/2}\{\mathrm{sign}(bJ^x)\}^p\frac{C_p}{D_p}. \tag{6.36}$$

We have

$$(1 - G_p^2G_{p+1}G_{p-1})\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln G_p\right)^2 + (1 - G_pG_{p+1})(1 - G_pG_{p-1})\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln G_p + (|J^x| - |b|)^2(1 - G_p^2G_{p+1}G_{p-1}) = |bJ^x|(G_p - G_p^{-1}) \times [(2G_p - G_{p+1} - G_{p-1}) + G_p^2G_{p+1}G_{p-1}(2G_p^{-1} - G_{p+1}^{-1} - G_{p-1}^{-1})], \tag{6.37}$$

see appendix F for the details of the derivation. It should be noted that eq. (6.37), for zero temperature, can be seen as the Wick-rotated continuum-time limit of the lattice-Painlevé equation of McCoy and Wu for the two-dimensional Ising model, eq. (20) of ref. 35, but also that eq. (6.37) is valid for arbitrary temperature, the temperature only entering through the boundary conditions. Given the solution of (6.37), the correlation functions can be calculated by quadratures from (6.31)–(6.33), (6.36), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln F_p = \frac{G_p\dot{G}_{p+1} - G_{p+1}\dot{G}_p}{1 - G_pG_{p+1}}, \tag{6.38a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln H_p = \frac{G_p\dot{G}_{p+1} - G_{p+1}\dot{G}_p}{G_pG_{p+1}(1 - G_pG_{p+1})}, \tag{6.38b}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln C_p = |bJ^x|G_p^{-2}\left(\frac{F_p}{F_{p-1}} - 1\right), \tag{6.39a}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln D_p = |bJ^x|G_p^2\left(\frac{H_p}{H_{p-1}} - 1\right), \tag{6.39b}$$

where

$$F_p \equiv D_{p+1}/D_p, \tag{6.40a}$$

$$H_p \equiv C_{p+1}/C_p, \tag{6.40b}$$

At zero temperature, in the ordered phase $|b| < |J^x|$, one then has the explicit solution

$$D_p = \frac{1}{4}(-\mathrm{sign}\,J^x)^p\left(1 - \frac{b^2}{J^{x2}}\right)^{1/4} \times \exp\sum_{n=p}^{\infty}\int_t^{\infty}\mathrm{d}\tau\,\frac{G_n(\tau)\dot{G}_{n+1}(\tau) - G_{n+1}(\tau)\dot{G}_n(\tau)}{1 - G_n(\tau)G_{n+1}(\tau)}, \tag{6.41}$$

$$C_p = \left(\frac{|J^x|}{|b|}\right)^{1/2}\{\mathrm{sign}(bJ^x)\}^pG_pD_p.$$

Here use is made of the fact that, for $t\to\infty$, D_p tends to the square of the (nonvanishing) long-range order, as calculated in refs. 57–59.

6.5. *Scaling limits*

In order to obtain a field-theory limit, where the details of the lattice disappear, it is necessary to find a limit in which the correlation lengths, both on the lattice and the dual lattice, diverge in the same way. Then it is possible to construct a rescaling of the lattice spacing, the time variable, and the correlation functions, i.e.

$$n \to x_1 \equiv n/\xi, \quad t \to x_0 \equiv t/\tau, \tag{6.42}$$

$$D_n(t) \to D(x_1, x_0) = \lim \epsilon_nD_n(t)/\delta, \quad C_n(t) \to C(x_1, x_0) = \lim \bar{\epsilon}_nC_n(t)/\gamma, \tag{6.43}$$

with a nontrivial result in the so-called scaling limit. Here we have added sign factors,

$$\epsilon_n \equiv (-\mathrm{sign}\,J^x)^n, \quad \bar{\epsilon}_n \equiv (-\mathrm{sign}\,b)^n, \tag{6.44}$$

to take care of the alternation in sign of the correlations with distance in the antiferromagnetic case $J^x > 0$, respectively $b > 0$ on the dual lattice.

We want to have some feeling of possible scaling limits; so we shall list a few existing results[57–59]) for the asymptotic fall-off of the correlations for equal time.

(We used the equivalence of the zero-field XY model with two identical Ising models in transverse field when consulting ref. 57.) For very low, but fixed, temperature there are three different asymptotic behaviours all characterized by exponential fall-off with correlation length ξ:

a) $|b|<|J^x|$

$$D_n = \tfrac{1}{4}\epsilon_n\{1-(b/J^x)^2\}^{1/4}\,\mathrm{e}^{-n/\xi}\,\mathrm{e}^K\,, \tag{6.45a}$$

$$\xi^{-1} = \left\{\frac{2(|J^x|-|b|)}{\pi\beta|bJ^x|}\right\}^{1/2}\mathrm{e}^{-\beta(|J^x|-|b|)}(1+\mathcal{O}(\beta^{-1}))\,, \tag{6.46a}$$

$$K = \frac{4}{\pi}\,\mathrm{e}^{-2\beta(|J^x|-|b|)}(1+\mathcal{O}(\beta^{-1}))\,; \tag{6.47a}$$

b) $|b|=|J^x|$,

$$D_n = \tfrac{1}{4}\epsilon_n(\beta|J^x|/2)^{-1/4}\,\mathrm{e}^{-n/\xi}\,\mathrm{e}^K\,, \tag{6.45b}$$

$$\xi^{-1} = \frac{\pi}{4\beta|J^x|}+\mathcal{O}(\beta^{-3})\,, \tag{6.46b}$$

$$K = \frac{-\pi^2}{128\beta^2 J^{x2}}+\mathcal{O}(\beta^{-4})\,; \tag{6.47b}$$

c) $|b|>|J^x|$,

$$D_n = \tfrac{1}{4}\epsilon_n(\tfrac{1}{2}\beta|J^x|)^{-1}\{1-(J^x/b)^2\}^{-3/4}\,\mathrm{e}^{-(n+1)/\xi}\,\mathrm{e}^K\,, \tag{6.45c}$$

$$\xi^{-1} = \ln\left(\frac{|b|}{|J^x|}\right)+\left\{\frac{2(|b|-|J^x|)}{\pi\beta|bJ^x|}\right\}^{1/2}\mathrm{e}^{-\beta(|b|-|J^x|)}(1+\mathcal{O}(\beta^{-1}))\,, \tag{6.46c}$$

$$K = \frac{4}{\pi}\,\mathrm{e}^{-2\beta(|b|-|J^x|)}-\left\{\frac{2|J^x|}{\pi\beta|b|(|b|-|J^x|)}\right\}^{1/2}\mathrm{e}^{-\beta(|b|-|J^x|)}(1+\mathcal{O}(\beta^{-1}))\,. \tag{6.47c}$$

We see that the correlation length ξ, given by (6.46a), diverges in the zero temperature limit ($\beta\to\infty$), provided $|b|\leqslant|J^x|$. So, "finite-temperature field theory"† can exist only in the limit

$$\beta\to\infty,\quad |b|\to|J^x|,\quad \beta(|J^x|-|b|)\to\theta=\mathrm{const}\,, \tag{6.48}$$

in which case the ξ in both (6.46a) and (6.46c), the latter with b and J^x interchanged, diverge in the same way. [The expansions given in refs. 57, 58 are the best available at present although they are in the strictest sense not meant to be used in the limit (6.48).]

The more usual scaling limits[19,21]) correspond to the case of zero temperature, for which there are again three different behaviours, but with two of them characterized by power-law decay[57–59]):

a) $|b|<|J^x|$,

$$D_n = \tfrac{1}{4}\epsilon_n\{1-(b/J^x)^2\}^{1/4}\left\{1+\frac{1}{2\pi n^2}(b/J^x)^{2n}\left(\frac{J^x}{b}-\frac{b}{J^x}\right)^{-2}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\right\}; \tag{6.49a}$$

b) $|b|=|J^x|$,

$$D_n = \tfrac{1}{4}\epsilon_n\,\mathrm{e}^{1/4}2^{1/12}C_{\mathrm{G}}^{-3}n^{-1/4}\left(1-\frac{1}{64n^2}+\cdots\right); \tag{6.49b}$$

c) $|b|>|J^x|$

$$D_n = \frac{1}{4}\epsilon_n\frac{|J^x/b|^n}{(\pi n)^{1/2}}\{1-(J^x/b)^2\}^{-1/4}\left\{1-\frac{1}{8n}\frac{b^2+J^{x2}}{b^2-J^{x2}}+\mathcal{O}(n^{-2})\right\}. \tag{6.49c}$$

Here C_{G} is Glaisher's constant, given by

$$\ln C_{\mathrm{G}} = \frac{1}{12}-\zeta'(-1) = \lim_{N\to\infty}\left\{\sum_{n=1}^{N-1} n\ln n-\frac{1}{2}\left(N^2-N+\frac{1}{6}\right)\ln N+\frac{1}{4}N^2\right\}. \tag{6.50}$$

At zero temperature, therefore, a scaling limit is only possible in the limits $|b|\downarrow|J^x|$ and $|b|\uparrow|J^x|$, which can be seen as special cases of (6.48) with $\theta=\pm\infty$. These two limits are also known as the high-temperature and the low-temperature phases of the Ising field theory (2-dimensional ϕ^4-theory), see e.g. refs. 66, 67 for further information. In general we expect a one-parameter family of finite-temperature field theories to exist, given by (6.42), (6.48) with $-\infty<\theta<+\infty$, for which the Green's functions will not be Lorentz invariant. Only in the limit $\theta=\pm\infty$ we have a more conventional field theory with Lorentz-invariant Green's functions†[19,21]).

We shall now take the scaling limit in eqs. (6.31)–(6.34) and (6.37). In order to do so, we substitute the expansions

$$\begin{aligned}\bar{\epsilon}_{n\pm1}C_{n\pm1}(t) &= \gamma C(n/\xi,t/\tau)\pm(\gamma/\xi)\frac{\partial}{\partial x_1}C(n/\xi,t/\tau)\\ &\quad+(\gamma/2\xi^2)\frac{\partial^2}{\partial x_1^2}C(n/\xi,t/\tau)+\cdots,\end{aligned} \tag{6.51a}$$

$$\begin{aligned}\epsilon_{n\pm1}D_{n\pm1}(t) &= \delta D(n/\xi,t/\tau)\pm(\delta/\xi)\frac{\partial}{\partial x_1}D(n/\xi,t/\tau)\\ &\quad+(\delta/2\xi^2)\frac{\partial^2}{\partial x_1^2}D(n/\xi,t/\tau)+\cdots,\end{aligned} \tag{6.51b}$$

† Recently the concept of temperature has been introduced in gauge theories and more general quantum-field theories, see for instance refs. 62–65.

† We are grateful to Professor B.M. McCoy for some useful comments concerning this and some other aspects of the present subsection.

and

$$\bar{\epsilon}_n \dot{C}_n = (\gamma/\tau)\frac{\partial}{\partial x_0} C(n/\xi, t/\tau)\,, \tag{6.51c}$$

etc. We then obtain, in the limit $\xi \to \infty$,

$$C\frac{\partial^2}{\partial x_0^2}C - \left(\frac{\partial C}{\partial x_0}\right)^2 = D\frac{\partial^2}{\partial x_1^2}D - \left(\frac{\partial D}{\partial x_1}\right)^2, \tag{6.52}$$

$$D\frac{\partial^2}{\partial x_0^2}D - \left(\frac{\partial D}{\partial x_0}\right)^2 = C\frac{\partial^2}{\partial x_1^2}C - \left(\frac{\partial C}{\partial x_1}\right)^2, \tag{6.53}$$

$$C\frac{\partial^2}{\partial x_0 \partial x_1}C - \frac{\partial C}{\partial x_0}\frac{\partial C}{\partial x_1} = D\frac{\partial^2}{\partial x_0 \partial x_1}D - \frac{\partial D}{\partial x_0}\frac{\partial D}{\partial x_1}, \tag{6.54}$$

and

$$\left(C\frac{\partial^2 D}{\partial x_1^2} - 2\frac{\partial C}{\partial x_1}\frac{\partial D}{\partial x_1} + D\frac{\partial^2 C}{\partial x_1^2}\right) - \left(C\frac{\partial^2 D}{\partial x_0^2} - 2\frac{\partial C}{\partial x_0}\frac{\partial D}{\partial x_0} + D\frac{\partial^2 C}{\partial x_0^2}\right) = \alpha CD\,. \tag{6.55}$$

Here we have set

$$\delta = (|b|/|J^x|)^{1/2}\gamma\,, \tag{6.56}$$

$$\tau = \xi/(|bJ^x|)^{1/2}\,, \tag{6.57}$$

$$\alpha \equiv (|J^x| - |b|)^2\xi^2/|bJ^x|\,. \tag{6.58}$$

Finally, from eq. (6.37) we have

$$\frac{1-G^2}{1+G^2}\Box \ln G + \left(\frac{\partial}{\partial x_1}\ln G\right)^2 - \left(\frac{\partial}{\partial x_0}\ln G\right)^2 = \alpha\,, \tag{6.59}$$

with

$$\Box \equiv \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_0^2}$$

and

$$G(x_1, x_0) \equiv C(x_1, x_0)/D(x_1, x_0)\,. \tag{6.60}$$

With the substitution

$$G = \tanh(\tfrac{1}{2}\psi)\,, \tag{6.61}$$

eq. (6.59) becomes the sinh-Gordon equation

$$\Box\psi = \tfrac{1}{2}\alpha \sinh(2\psi)\,. \tag{6.62}$$

(For $|b| \neq |J^x|$, we may set $\alpha = 1$ by proper rescaling of x_1 and x_0.) Given the

appropriate solution of eq. (6.59) or (6.62), the scaled correlation function can be directly obtained from (6.54), (6.60), (6.61), i.e.

$$\frac{\partial^2}{\partial x_1 \partial x_0}\ln D = \sinh^2(\tfrac{1}{2}\psi)\frac{\partial}{\partial x_1}\left\{(\sinh\psi)^{-1}\frac{\partial\psi}{\partial x_0}\right\}. \tag{6.63}$$

We consider now first the case of critical field, $|b| = |J^x|$, with $C \equiv D$, $G \equiv 1$, $\alpha = 0$. In this case, (6.54) and (6.59) are trivial and (6.52), (6.53) and (6.55) can all be rewritten as

$$\Box \ln D = 0\,. \tag{6.64}$$

Therefore, the logarithm of the spin–spin correlation function satisfies the free massless Klein–Gordon equation in this case, with the solution

$$D(x_1, x_0) = f_1(x_1 - x_0)f_2(x_1 + x_0)\,. \tag{6.65}$$

It is now very tempting to assume $f_1 \equiv f_2$ and to use (6.49b) or (6.45b) to determine $f_1(x_1)^2 = D(x_1, 0)$. As a consequence one obtains at zero temperature the Lorentz-invariant result

$$D(x_1, x_0) = \text{const} \times (x_1^2 - x_0^2)^{-1/8}\,, \tag{6.66}$$

in agreement with the prediction of Luther and Peschel[68]), correcting a misprint, see also refs. 69, 29, 72. At nonzero temperature one would be tempted to expect[70,71])†

$$\begin{aligned} D(x_1, x_0) &\sim \{\sinh[4(x_1 - x_0)]\sinh[4(x_1 + x_0)]\}^{-1/8} \\ &\sim \exp[-\tfrac{1}{2}(|x_1 - x_0| + |x_1 + x_0|)]\,, \end{aligned} \tag{6.67}$$

which is of the form, respectively, predicted by Luther and Peschel, for the Luttinger–Tomonaga model, [see eq. (25) of ref. 70 with x_0 small, the zero-field isotropic XY-model corresponding to the case $g = 0$, an extra power 1/8 being needed here as in ref. 68], and by Nelson and Fisher[71]), using a "fixed-length" hydrodynamic description for the classical XY-model at low temperatures. Note that the zero-temperature result is Lorentz-invariant, whereas the finite-temperature scaling result is not. Also, the general result for the time-dependent correlation functions of the transverse Ising chain have to be rather complicated. Not only is there a cross-over from the Gaussian decay at infinite (or high) temperature to the power-law behaviour at zero temperature, but in view of (6.67),

† The result of ref. 70, as reproduced on the first line of eq. (6.67), cannot be correct close to the light cone, where exponentially small terms may no longer be ignored. In fact, the fall-off of the first form in (6.67) is qualitatively correct for the equal-time correlation function, both at low- and at zero temperature, but eqs. (6.45b) and (6.49b) lead to different values of the numerical coefficient. The second line of (6.67) is always real, whereas the initial conditions involve an imaginary part, cf. (6.18). Accordingly not much can be said for sure about the behaviour of $D(x_1, x_0)$ in the time-like regime.

which is based on the assumption $f_1 = f_2$, one may expect an intermediate regime with exponential decay. At very low temperature one expects the power-law behaviour (6.66), apart from some deviations on the light cone, within a certain length and time scale. On a larger scale one would expect a behaviour like (6.67), whereas on some time scale one still might expect to see the Gaussian decay.

We finally consider the case of zero temperature and $|b| \neq |J^x|$ in some more detail. In this case, the scaling limit results are Lorentz invariant[19]), i.e.

$$C = C(s), \quad D = D(s), \quad s^2 \equiv x_1^2 - x_0^2 . \tag{6.68}$$

Therefore the sinh-Gordon equation (6.62) reduces to the Painlevé differential equation of the third kind

$$\frac{d^2\eta}{ds^2} - \frac{1}{\eta}\left(\frac{d\eta}{ds}\right)^2 + \frac{1}{s}\frac{d\eta}{ds} + \frac{1-\eta^4}{4\eta} = 0, \quad \eta \equiv e^{-\psi}, \tag{6.69}$$

which is precisely one of the main results of ref. 19, see in particular their subsection 2.3.3. Also in this case eq. (6.63) can be solved explicitly, cf. eq. (6.41), giving

$$C(s) = \sinh(\tfrac{1}{2}\psi(s)) \exp(-\tfrac{1}{4}I(s)) ,$$

$$D(s) = \cosh(\tfrac{1}{2}\psi(s)) \exp(-\tfrac{1}{4}I(s)) ,$$

$$I(s) \equiv \int_s^\infty dx\, x\left[\left(\frac{d\psi}{dx}\right)^2 - \sinh^2\psi(x)\right]. \tag{6.70}$$

where the scaling δ has been chosen such that $D(s) \to 1$ for $s \to \infty$, if $|b| < |J^x|$.

In conclusion, we have generalized in this subsection the scaling results of ref. 19 for zero temperature to finite temperatures, cf. (6.62), (6.63) for $|b| \neq |J^x|$ and (6.64) for $|b| = |J^x|$. In doing so, we have seen that the differential equations determining the Green's functions are basically unchanged. The effect of temperature is restricted to the boundary conditions. These boundary conditions ensure the Lorentz invariance of the Green's functions at zero temperature, but at finite temperatures the Lorentz invariance will not hold anymore and a more detailed investigation of the boundary conditions would be of interest.

6.6. *Homogeneous Ising chain in critical transverse field*

The most simple case is the homogeneous Ising chain at the special value $b = J^x$ of the magnetic field, corresponding to the critical field at zero temperature. In this case we have

$$J_k = b = J^x , \tag{6.71}$$

independent of $k = 1, \ldots, 2N-1$ in (2.4), and therefore

$$\langle S^x_{j+\frac{1}{2}}(t) S^x_{j+\frac{1}{2}+p}\rangle = C_p = \langle S^x_j(t) S^x_{j+p}\rangle = D_p , \tag{6.72}$$

for spins in the bulk of the chain. Eqs. (6.33), (6.35) and (6.37) are trivial and eqs. (6.31), (6.32) and (6.34) can be rewritten as

$$\frac{d^2}{dt^2} \ln D_p = J^{x2}\left(\frac{D_{p+1}D_{p-1}}{D_p^2} - 1\right). \tag{6.73}$$

Using the substitution

$$e^{-\rho_p(t)} = J^{x2}\frac{D_{p+1}D_{p-1}}{D_p^2} , \tag{6.74}$$

eq. (6.73) may be rewritten as

$$\ddot{\rho}_p = 2\, e^{-\rho_p} - e^{-\rho_{p-1}} - e^{-\rho_{p+1}} , \tag{6.75}$$

with the initial conditions, cf. (6.18), (2.2),

$$\rho_p(0) = -\ln\left(J^{x2}\frac{\langle S^x_j S^x_{j+p+1}\rangle\langle S^x_j S^x_{j+p-1}\rangle}{\langle S^x_j S^x_{j+p}\rangle^2}\right), \tag{6.76}$$

$$\dot{\rho}_p(0) = 2iJ^x(\delta_{p+1,0} + \delta_{p-1,0} - 2\delta_{p,0})\langle S^z_j\rangle . \tag{6.77}$$

Eq. (6.75), which has been first obtained in ref. 31, is the equation of motion for the Toda lattice with the correct sign. The interpretation of eq. (6.75) is different from that of the Toda equation in (6.26). Eq. (6.26) refers to the inhomogeneous Ising chain at infinite temperature and the index n labels the autocorrelations of spins at different sites n and $n+\frac{1}{2}$ of the Ising chain and the dual Ising chain. Eq. (6.75) deals with the homogeneous Ising chain with $b = J^x$ at finite temperature and the index p labels the correlation function of spins at distance p. From the solutions of (6.75) one can obtain the spin correlations, using the integral

$$\langle S^x_j(t) S^x_{j+p}\rangle = \langle S^x_j S^x_{j+p}\rangle \exp[-2iJ^x t\delta_{0,p}\langle S^z_j\rangle]$$
$$\times \exp\left[-\int_0^t dt_1 \int_0^{t_1} dt_2 \{J^{x2} - \exp(-\rho_p(t_2))\}\right]. \tag{6.78}$$

The initial conditions $\langle S^x_j S^x_{j+p}\rangle$ can be evaluated using (6.17), (6.18) and (6.10)–(6.14), in which the pair correlations $\langle\gamma_k\gamma_l\rangle$ in (6.15), are explicitly given by

$$\langle\gamma_k\gamma_l\rangle = \frac{1}{2}\delta_{kl} + \frac{1}{4\pi}\int_0^{2\pi} d\phi\, \tanh(\beta J^x \sin\phi)\, e^{i(l-k)\phi} . \tag{6.79}$$

Also in this case it is not easy to solve the differential equation explicitly for finite β. There is, however, a treatment by Mohan[30]), in which use has been made of an explicit expression for the operator $O_{m,t}$ derived in ref. 9. After an extensive analysis of the leading terms in the limit $t \to \infty$, he finds that the predominantly Gaussian decay of the autocorrelation ($p = 0$) will persist without any change in the width of the Gaussian. Another treatment is ref. 72, in which the autocorrelation has been expressed as a product of the Gaussian in the high-temperature limit and a Fredholm determinant which is temperature dependent. Inserting this result for the autocorrelation in (6.73), the nearest-neighbour and the more distant correlations can be found in a recursive way, without solving the initial-value problem for the Toda equation.

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Appendix A

In this appendix we show that the equations derived in section 5 for the quantities $\tilde{X}_n^{(s)\mathrm{L,R}}$, $\tilde{A}_{n,k}^{(s)}$ with tildes follow from the corresponding equations for the quantities without tildes. For that purpose we consider the transformation

$$\gamma_m \to \gamma_m^{\mathrm{T}} = (-1)^m \gamma_{2N-m+1}, \tag{A.1}$$

which transforms the Ising Hamiltonian (2.4) into

$$\mathscr{H} \to \mathscr{H}^{\mathrm{T}} = \mathrm{i} \sum_{m=1}^{2N-1} J_m^{\mathrm{T}} \gamma_m \gamma_{m+1}, \quad J_m^{\mathrm{T}} = J_{2N-m}. \tag{A.2}$$

Furthermore,

$$h_m = -2\mathrm{i}J_m \gamma_m \gamma_{m+1} \to -2\mathrm{i}J_m \gamma_{2N-m} \gamma_{2N-m+1} = h_{2N-m}^{\mathrm{T}} \tag{A.3}$$

and therefore

$$O_{m,t} = \mathrm{e}^{\mathrm{i}\mathscr{H}t}\, \mathrm{e}^{-\mathrm{i}(\mathscr{H}+h_m)t} \to \mathrm{e}^{\mathrm{i}\mathscr{H}^{\mathrm{T}}t}\, \mathrm{e}^{-\mathrm{i}(\mathscr{H}^{\mathrm{T}}+h_{2N-m}^{\mathrm{T}})t} = O_{2N-m,t}^{\mathrm{T}}. \tag{A.4}$$

We define the quantities

$$\begin{aligned} A_{m,k}^{(s)\mathrm{T}} &= \mathrm{Tr}\{O_{m,t}^{\mathrm{T}} \gamma_{m+1} \cdots \gamma_{m+s} \gamma_{m+s+1} \gamma_k K^{\mathrm{T}}\}, \\ \tilde{A}_{m,k}^{(s)\mathrm{T}} &= \mathrm{Tr}\{O_{m,t}^{\mathrm{T}} \gamma_m \cdots \gamma_{m-s+1} \gamma_{m-s} \gamma_k K^{\mathrm{T}}\}, \\ &(s = 0, 2, 4, \ldots), \end{aligned} \tag{A.5}$$

cf. (5.13), in which K^{T} can be obtained from K, cf. (5.2), replacing all operators γ_m by the corresponding γ_m^{T}. Starting from (A.5) and (5.15), one can define the parameters $A_{m,m+s}^{\mathrm{T}}$, $A_{m,m-s}^{\mathrm{T}}$, $A_{m,m}^{\mathrm{T}}$, cf. (5.16) with $O_{m,t}^{\mathrm{T}}$ and K^{T} instead of $O_{m,t}$ and K. We can also define generating functions $X_m^{(s)\mathrm{L,R}^{\mathrm{T}}}(\lambda, t)$, $\tilde{X}_m^{(s)\mathrm{L,R}^{\mathrm{T}}}(\lambda, t)$ replacing the $A_{m,k}^{(s)}$ and $\tilde{A}_{m,k}^{(s)}$ in (5.23) by the corresponding $A_{m,k}^{(s)\mathrm{T}}$ and $\tilde{A}_{m,k}^{(s)\mathrm{T}}$.

From the definitions (5.13), (A.5) and the transformation (A.1), it can be shown that

$$A_{m,k}^{(s)} = (-1)^{m-k+1+s/2} \tilde{A}_{2N-m,2N-k+1}^{(s)\mathrm{T}}, \tag{A.6}$$

implying that

$$A_{m,m+s} = (-1)^{s/2} A_{2N-m,2N-m-s}^{\mathrm{T}}. \tag{A.7}$$

For the generating functions, we have the relation

$$X_m^{(s)\mathrm{L,R}}(\lambda, t) = \tilde{X}_{2N-m}^{(s)\mathrm{R,L}^{\mathrm{T}}}(-\lambda^{-1}, t)\lambda^{-1}. \tag{A.8}$$

The derivation of (A.8) is straightforward, but somewhat tedious. For example,

$$\begin{aligned} X_m^{(s)\mathrm{L}}(\lambda, t) &= \frac{1}{A_{m,m+s}} \sum_{k \leqslant m+s+1} A_{m,k}^{(s)} \lambda^{m+s-k} \\ &= \frac{(-1)^{1+s/2}}{(-1)^{s/2}} \frac{(-1)^s}{A_{2N-m,2N-m-s}^{\mathrm{T}}} \sum_{k \leqslant m+s+1} \tilde{A}_{2N-m,2N-k+1}^{(s)\mathrm{T}} (-\lambda)^{m+s-k} \\ &= \frac{-1}{A_{2N-m,2N-m-s}^{\mathrm{T}}} \sum_{l \geqslant 2N-m-s} \tilde{A}_{2N-m,l}^{(s)\mathrm{T}} (-\lambda^{-1})^{2N-m-s-l+1} \\ &= \lambda^{-1} \tilde{X}_{2N-m}^{(s)\mathrm{R}^{\mathrm{T}}}(-\lambda^{-1}, t). \end{aligned} \tag{A.9}$$

Using (A.6)–(A.8) it is straightforward to verify that eqs. (5.14), (5.17), (5.18), (5.24)–(5.26) with tildes follow from the corresponding equations without tildes. For example, when one applies (A.8) and (A.7) to the first equation (5.24) for $X_n^{(s)\mathrm{L,R}}(\lambda, t)$ one finds that the $\tilde{X}_m^{(s)\mathrm{R,L}^{\mathrm{T}}}(\lambda', t)$, (with $m = n$, $n+1$, $n-1$), satisfy the second equation (5.24) for $\tilde{X}_n^{(s)\mathrm{R,L}^{\mathrm{T}}}(\lambda', t)$ with $\lambda' = -\lambda^{-1}$. Since the first equation (5.24) has been derived for any Ising chain of type (2.1), one can infer that the second equation (5.24) is also satisfied by the generating functions $\tilde{X}_m^{(s)\mathrm{R,L}}(\lambda, t)$ of the original Ising chain (2.1). This line of reasoning can also be applied to the other equations with tildes.

Appendix B

In section 5, we have derived differential equations (5.14), (5.17), (5.18) for the quantities $A^{(s)}_{n,k}$, $\tilde{A}^{(s)}_{n,k}$, using the equation of motion (5.4) for $\mathrm{Tr}(O_{n,t}\Gamma\gamma_l\gamma_k K)$. The equations (5.14), (5.17) and (5.18) may be derived in several ways and in this appendix we present a slightly different derivation.

Using (5.8) and (2.3) we have the identity

$$\begin{aligned}\mathrm{Tr}(O_{n,t}\Gamma\gamma_l\gamma_k K) &= -4\,\mathrm{Tr}(O_{n,t}\gamma_{n-1}\gamma_n\gamma_{n-1}\gamma_n\Gamma\gamma_l\gamma_k K)\\ &= -4\,\mathrm{Tr}(\gamma_{n-1}(t)\gamma_n(t)O_{n-2,t}\gamma_{n-1}\gamma_n\Gamma\gamma_l\gamma_k K)\,. \end{aligned}\tag{B.1}$$

We now apply the Wick theorem (3.4) with $s=2$ to the right-hand side of (B.1), cf. also (5.5) with $\gamma_a(t)\to\gamma_{n-1}(t)$, $\gamma_b(t)\to\gamma_n(t)$, $O\to O_{n-2,t}$, $\Gamma\to\gamma_{n-1}\gamma_n\Gamma$, $\gamma_c\to\gamma_l$, $\gamma_d\to\gamma_k$. Then

$$\begin{aligned}&\mathrm{Tr}(O_{n,t}\Gamma\gamma_l\gamma_k K)\,\mathrm{Tr}(O_{n-2,t}\gamma_{n-1}\gamma_n\Gamma K)\\ &\quad= -4\{\mathrm{Tr}(\gamma_{n-1}(t)\gamma_n(t)O_{n-2,t}\gamma_{n-1}\gamma_n\Gamma K)\,\mathrm{Tr}(O_{n-2,t}\gamma_{n-1}\gamma_n\Gamma\gamma_l\gamma_k K)\\ &\qquad+\mathrm{Tr}(\gamma_{n-1}(t)O_{n-2,t}\gamma_{n-1}\gamma_n\Gamma\gamma_k K)\,\mathrm{Tr}(\gamma_n(t)O_{n-2,t}\gamma_{n-1}\gamma_n\Gamma\gamma_l K)\\ &\qquad-\mathrm{Tr}(\gamma_{n-1}(t)O_{n-2,t}\gamma_{n-1}\gamma_n\Gamma\gamma_l K)\,\mathrm{Tr}(\gamma_n(t)O_{n-2,t}\gamma_{n-1}\gamma_n\Gamma\gamma_k K)\}\,. \end{aligned}\tag{B.2}$$

In view of (5.8), (2.3), and the equation of motion (5.3), eq. (B.2) can be rewritten

$$\begin{aligned}&\mathrm{Tr}(O_{n,t}\Gamma\gamma_l\gamma_k K)\,\mathrm{Tr}(O_{n-2,t}\gamma_{n-1}\gamma_n\Gamma K)\\ &\quad=\mathrm{Tr}(O_{n,t}\Gamma K)\,\mathrm{Tr}(O_{n-2,t}\gamma_{n-1}\gamma_n\Gamma\gamma_l\gamma_k K)\\ &\qquad-(J_{n-1})^{-1}\,\mathrm{Tr}(O_{n-1,t}\gamma_n\Gamma\gamma_k K)\,\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}(O_{n-1,t}\gamma_n\Gamma\gamma_l K)\\ &\qquad+(J_{n-1})^{-1}\,\mathrm{Tr}(O_{n-1,t}\gamma_n\Gamma\gamma_l K)\,\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}(O_{n-1,t}\gamma_n\Gamma\gamma_k K)\,. \end{aligned}\tag{B.3}$$

Eq. (B.3) is different from (5.9), but with the special choice (5.10a), eq. (B.3) reduces to

$$\begin{aligned}A^{(s)}_{n,k}A_{n-2,n+s} &= A^{(s+2)}_{n-2,k}A_{n,n+s}-(J_{n-1})^{-1}A^{(s)}_{n-1,k}\frac{\mathrm{d}}{\mathrm{d}t}A_{n-1,n+s+1}\\ &\quad+(J_{n-1})^{-1}A_{n-1,n+s+1}\frac{\mathrm{d}}{\mathrm{d}t}A^{(s)}_{n-1,k}\,, \end{aligned}\tag{B.4}$$

which after substituting $n-1\to n$, $s\to s-2$ becomes identical with the first equation (5.17)

For the special choice (5.11a), eq. (B.3) reduces to

$$\begin{aligned}\tfrac{1}{2}A^{(s-2)}_{n,k}A_{n-2,n+s} &= \tfrac{1}{2}A^{(s)}_{n-2,k}A_{n,n+s}-(2J_{n-1})^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}A_{n-1,n+s-1}\right)A^{(s)}_{n-1,k}\\ &\quad+(2J_{n-1})^{-1}A_{n-1,n+s-1}\frac{\mathrm{d}}{\mathrm{d}t}A^{(s)}_{n-1,k}\,, \end{aligned}\tag{B.5}$$

which after substituting $n-1\to n$ becomes identical with the first equation (5.14) for $s=2,4,\ldots$. The first equation (5.18) can be derived from (B.3) using (5.12b).

Appendix C

In this appendix we shall give an alternative direct derivation of eqs. (6.8) and (6.9). Starting with the l.h.s. of (6.8) one has

$$\begin{aligned}&\langle S^x_{\frac12 n+\frac12}(t)S^x_{\frac12 n+\frac12+p}\rangle^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\langle S^x_{\frac12 n+\frac12}(t)S^x_{\frac12 n+\frac12+p}\rangle\\ &\quad=\langle S^x_{\frac12 n+\frac12}(t)S^x_{\frac12 n+\frac12+p}\rangle\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle S^x_{\frac12 n+\frac12}(t)S^x_{\frac12 n+\frac12+p}\rangle\\ &\qquad-\left\{\frac{\mathrm{d}}{\mathrm{d}t}\langle S^x_{\frac12 n+\frac12}(t)S^x_{\frac12 n+\frac12+p}\rangle\right\}^2\\ &\quad=-J_nJ_{n+2p}\{\langle S^x_{\frac12 n+\frac12}(t)S^x_{\frac12 n+\frac12+p}\rangle\langle S^y_{\frac12 n+\frac12}(t)S^y_{\frac12 n+\frac12+p}\rangle\\ &\qquad-\langle S^x_{\frac12 n+\frac12}(t)S^y_{\frac12 n+\frac12+p}\rangle\langle S^y_{\frac12 n+\frac12}(t)S^x_{\frac12 n+\frac12+p}\rangle\}\,, \end{aligned}\tag{C.1}$$

where use has been made of eq. (2.22) and the companion equation for the dual chain, cf. also (2.17). Eq. (6.8) follows then immediately from the special form of the general Wick theorem (3.4)

$$\begin{aligned}\mathrm{Tr}(Q_1\gamma_a\gamma_bQ_3\gamma_c\gamma_d)\,\mathrm{Tr}(Q_1Q_3) &= \mathrm{Tr}(Q_1\gamma_a\gamma_bQ_3)\,\mathrm{Tr}(Q_1Q_3\gamma_c\gamma_d)\\ &\quad-\mathrm{Tr}(Q_1\gamma_aQ_3\gamma_c)\,\mathrm{Tr}(Q_1\gamma_bQ_3\gamma_d)+\mathrm{Tr}(Q_1\gamma_aQ_3\gamma_d)\,\mathrm{Tr}(Q_1\gamma_bQ_3\gamma_c)\,, \end{aligned}\tag{C.2}$$

identifying

$$Q_1=\mathrm{e}^{-\beta\mathscr{H}}\,\mathrm{e}^{\mathrm{i}\mathscr{H}t}S^x_{\frac12 n}\,,\qquad Q_3=\mathrm{e}^{-\mathrm{i}\mathscr{H}t}S^x_{\frac12 n+p}\,,\tag{C.3}$$

$$\gamma_a=\gamma_n\,,\quad \gamma_b=\gamma_{n+1}\,,\qquad \gamma_c=\gamma_{n+2p}\,,\quad \gamma_d=\gamma_{n+2p+1}\,,\tag{C.4}$$

and

$$
\begin{aligned}
&S^x_{\frac12 m+\frac12} = 2\alpha_m \gamma_0 S^x_{\frac12 m}\gamma_m\,,\\
&S^y_{\frac12 m+\frac12} = 2\alpha_m \gamma_0 S^x_{\frac12 m}\gamma_{m+1}\,,\\
&S^x_{\frac12 m+1} = 2\mathrm{i} S^x_{\frac12 m}\gamma_m\gamma_{m+1}\,,\\
&\alpha_m = \mathrm{i},\quad m\ \text{even},\quad \alpha_m = 1,\quad m\ \text{odd}\,,
\end{aligned}
\tag{C.5}
$$

see eqs. (2.2), (2.18). In a similar fashion one can rewrite eq. (6.9), using (2.22), as

$$
\begin{aligned}
&\langle S^y_{\frac12 n+\frac12}(t) S^x_{\frac12 n+\frac32+p}\rangle\langle S^x_{\frac12 n+\frac12}(t) S^x_{\frac12 n+\frac12+p}\rangle\\
&\quad -\langle S^x_{\frac12 n+\frac12}(t) S^x_{\frac12 n+\frac32+p}\rangle\langle S^y_{\frac12 n+\frac12}(t) S^x_{\frac12 n+\frac12+p}\rangle\\
&= -\langle S^x_{\frac12 n}(t) S^y_{\frac12 n+1+p}\rangle\langle S^x_{\frac12 n+1}(t) S^x_{\frac12 n+1+p}\rangle\\
&\quad +\langle S^x_{\frac12 n}(t) S^x_{\frac12 n+1+p}\rangle\langle S^x_{\frac12 n+1}(t) S^y_{\frac12 n+1+p}\rangle\,.
\end{aligned}
\tag{C.6}
$$

This again holds in view of (C.2), but now with the identifications

$$
\begin{aligned}
&Q_1 = \mathrm{e}^{-\beta\mathscr{H}}\,\mathrm{e}^{\mathrm{i}\mathscr{H}t} S^x_{\frac12 n}\,,\\
&Q_3 = \mathrm{e}^{-\mathrm{i}\mathscr{H}t} S^x_{\frac12 n+1+p}\,,
\end{aligned}
\tag{C.7}
$$

$$
\begin{aligned}
&\gamma_a = \gamma_n,\quad \gamma_b = \gamma_{n+1}\,,\\
&\gamma_c = \gamma_{n+1+2p},\quad \gamma_d = \gamma_{n+2+2p}\,,
\end{aligned}
\tag{C.8}
$$

eq. (C.5), and

$$
\begin{aligned}
&S^y_{\frac12 n+1+p} = 2 S^x_{\frac12 n+1+p}\gamma_{n+1+2p}\gamma_{n+2+2p}\,,\\
&S^x_{\frac12 n+\frac12+p} = 2\alpha^{-1}_{n+1}\gamma_0 S^x_{\frac12 n+1+p}\gamma_{n+1+2p}\,,\\
&S^x_{\frac12 n+\frac32+p} = 2\mathrm{i}\alpha^{-1}_{n+1}\gamma_0 S^x_{\frac12 n+1+p}\gamma_{n+2+2p}\,.
\end{aligned}
\tag{C.9}
$$

In both cases (6.8) and (6.9), the γ_0's cancel out trivially since γ_0 does not appear in the Hamiltonian (2.4).

Appendix D

In this appendix we give algebraic relations which contain the time derivatives of the spin correlations, but which are linear in the generating functions

$X^{(s)\mathrm{L,R}}_n(\lambda, t)$. From (5.24) and (5.25) we have

$$
\begin{aligned}
&-\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{A_{n,n+s}}{A_{n,n+s-2}}\right) X^{(s-2)\mathrm{L,R}}_n\\
&= J_n\lambda\left[\frac{A_{n+1,n+s-1}A_{n-1,n+s-3}}{A^2_{n,n+s-2}} X^{(s-2)\mathrm{L,R}}_{n-1} - \frac{A_{n-1,n+s-1}A_{n+1,n+s-3}}{A^2_{n,n+s-2}} X^{(s-4)\mathrm{L,R}}_{n+1}\right]\\
&\quad - J_n\lambda^{-1}\left(\frac{A_{n+1,n+s-1}A_{n-1,n+s-1}}{A_{n,n+s-2}A_{n,n+s}}\right)\left(X^{(s)\mathrm{L,R}}_{n-1} - X^{(s-2)\mathrm{L,R}}_{n+1}\right)\,.\\
&\qquad (s = 4, 6, \ldots)\,,
\end{aligned}
\tag{D.1}
$$

and

$$
\begin{aligned}
&-\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{A_{n,n+2}}{A_{n,n}}\right) X^{(0)\mathrm{L,R}}_n\\
&= J_n\left[\lambda\frac{A_{n+1,n+1}A_{n-1,n-1}}{A^2_{n,n}} X^{(0)\mathrm{L,R}}_{n-1} - 2\lambda^{-1}\frac{A_{n-1,n+1}A_{n+1,n+1}}{A^2_{n,n}} \tilde{X}^{(0)\mathrm{L,R}}_{n+1}\right]\\
&\quad - J_n\lambda^{-1}\left(\frac{A_{n+1,n+1}A_{n-1,n+1}}{A_{n,n}A_{n,n+2}}\right)\left(X^{(2)\mathrm{L,R}}_{n-1} - X^{(0)\mathrm{L,R}}_{n+1}\right)\,,
\end{aligned}
\tag{D.2}
$$

and two similar relations for $\tilde{X}^{(s-2)\mathrm{L,R}}_n$ and $\tilde{X}^{(2)\mathrm{L,R}}_n$, which can be inferred from (D.1), (D.2), using the transformation formulae of appendix A.

On the other hand, one may derive from (5.24) and (5.25) differential equations which are independent of λ. We have

$$
\begin{aligned}
&\left[-J_n\lambda^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(X^{(s)\mathrm{L,R}}_{n-1} - X^{(s-2)\mathrm{L,R}}_{n+1}\right) = \right]\\
&\quad \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{A^2_{n,n+s}}{A_{n+1,n+s-1}A_{n-1,n+s-1}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A_{n,n+s-2}}{A_{n,n+s}} X^{(s-2)\mathrm{L,R}}_n\right)\right\}\\
&\quad = J_nJ_{n-1}\left(\frac{A_{n,n+s}A_{n-2,n+s-2}}{A^2_{n-1,n+s-1}} X^{(s)\mathrm{L,R}}_{n-2} - \frac{A_{n-2,n+s}A_{n,n+s-2}}{A^2_{n-1,n+s-1}} X^{(s-2)\mathrm{L,R}}_n\right)\\
&\qquad - J_nJ_{n+1}\left(\frac{A_{n+2,n+s}A_{n,n+s-2}}{A^2_{n+1,n+s-1}} X^{(s-2)\mathrm{L,R}}_n - \frac{A_{n,n+s}A_{n+2,n+s-2}}{A^2_{n+1,n+s-1}} X^{(s-4)\mathrm{L,R}}_{n+2}\right)\,.
\end{aligned}
\tag{D.3}
$$

Eqs. (D.1)–(D.3) [and also (5.24)–(5.26)] may be regarded as relations of a linear problem associated with the differential equations (6.8) and (6.9) for the spin correlations. Eqs. (D.1)–(D.3) are linear in the "wave functions" $X^{(s)\mathrm{L,R}}_n$ belonging to the "eigenvalues" λ and the coefficients contain the potentials $A_{n,n+s}$, $A_{n,n-s}$, i.e. the solutions for the spin correlations. Eqs. (D.1) and (D.2) which are algebraic and linear in the $X^{(s)\mathrm{L,R}}_n$ contain the eigenvalues λ, whereas in (D.3) which contains the time derivatives of $X^{(s)\mathrm{L,R}}_n$, the λ's do not occur.

Finally, one can write down differential equations for $X^{(s)\mathrm{L,R}}_n$ at constant s which contain only different values $s' = s \pm 2$ through the $A_{m,m+s}$, in the coefficients. As an example we give the equation for $s=0$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A_{n-1,n-1}^{2}}{A_{n,n}A_{n-2,n}}\frac{\mathrm{d}}{\mathrm{d}t}X_{n-1}^{(0)\mathrm{L,R}}\right)+J_{n-1}\lambda\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A_{n-2,n-2}}{A_{n-2,n}}X_{n-2}^{(0)\mathrm{L,R}}\right)$$
$$=J_{n-1}\lambda^{-1}\frac{A_{n-1,n-1}}{A_{n-1,n+1}}\frac{\mathrm{d}}{\mathrm{d}t}X_{n}^{(0)\mathrm{L,R}}$$
$$+J_{n-1}J_{n}\frac{A_{n-1,n-1}}{A_{n-1,n+1}}\left(\frac{A_{n+1,n+1}A_{n-1,n-1}-4A_{n-1,n+1}A_{n+1,n-1}}{A_{n,n}^{2}}\right)X_{n-1}^{(0)\mathrm{L,R}},\quad\text{(D.4)}$$

cf. (5.26). Here the $s'=-2$ contribution, i.e. the $A_{n+1,n-1}$, can be eliminated using (6.5).

Appendix E

In this appendix we give two different derivations of eq. (6.34). In the first derivation, we consider the function

$$h(p,t)\equiv\frac{bJ^{x}(D^{-}C^{+}+D^{+}C^{-})-(C\ddot{D}+D\ddot{C}-2\dot{C}\dot{D})}{CD},\qquad\text{(E.1)}$$

where

$$D^{\pm}=\langle S_{j}^{x}(t)S_{j+p\pm1}^{x}\rangle,\quad D=\langle S_{j}^{x}(t)S_{j+p}^{x}\rangle,$$
$$C^{\pm}=\langle S_{j+\frac12}^{x}(t)S_{j+\frac12+p\pm1}^{x}\rangle,\quad C=\langle S_{j+\frac12}^{x}(t)S_{j+\frac12+p}^{x}\rangle.\qquad\text{(E.2)}$$

Eqs. (6.31)–(6.33) can be written in the form

$$C\ddot{C}-\dot{C}^{2}=J^{x2}(D^{+}D^{-}-D^{2}),\qquad\text{(E.3)}$$
$$D\ddot{D}-\dot{D}^{2}=b^{2}(C^{+}C^{-}-C^{2}),\qquad\text{(E.4)}$$
$$J^{x}(D\dot{D}^{+}-D^{+}\dot{D})=b(C\dot{C}^{+}-C^{+}\dot{C}).\qquad\text{(E.5)}$$

Taking the time derivative of (E.1) and using (E.3), (E.4), and

$$C\dddot{C}-\dot{C}\ddot{C}=J^{x2}(D^{+}\dot{D}^{-}+D^{-}\dot{D}^{+}-2D\dot{D}),\qquad\text{(E.6)}$$
$$D\dddot{D}-\dot{D}\ddot{D}=b^{2}(C^{+}\dot{C}^{-}+C^{-}\dot{C}^{+}-2C\dot{C}),\qquad\text{(E.7)}$$

we find

$$\begin{aligned}\dot{h}(p,t)(CD)^{2}&=bJ^{x}\{CD(C^{+}\dot{D}^{-}+D^{-}\dot{C}^{+}+C^{-}\dot{D}^{+}+D^{+}\dot{C}^{-})\\&\quad-(C\dot{D}+D\dot{C})(D^{-}C^{+}+D^{+}C^{-})\}-C^{2}(D\dddot{D}-\dot{D}\ddot{D})\\&\quad-D^{2}(C\dddot{C}-\dot{C}\ddot{C})+2C\dot{C}(D\ddot{D}-\dot{D}^{2})+2D\dot{D}(C\ddot{C}-\dot{C}^{2})\\&=bJ^{x}\{CC^{-}(D\dot{D}^{+}-D^{+}\dot{D})+CC^{+}(D\dot{D}^{-}-D^{-}\dot{D})\\&\quad+DD^{-}(C\dot{C}^{+}-C^{+}\dot{C})+DD^{+}(C\dot{C}^{-}-C^{-}\dot{C})\}\\&\quad-b^{2}\{CC^{-}(C\dot{C}^{+}-C^{+}\dot{C})+CC^{+}(C\dot{C}^{-}-C^{-}\dot{C})\}\\&\quad-J^{x2}\{DD^{-}(D\dot{D}^{+}-D^{+}\dot{D})+DD^{+}(D\dot{D}^{-}-D^{-}\dot{D})\}\end{aligned}\qquad\text{(E.8)}$$

and from (E.5), it is obvious that

$$\dot{h}(p,t)=0.\qquad\text{(E.9)}$$

We write (E.5) in the form

$$D\dot{D}^{+}-kC\dot{C}^{+}=D^{+}\dot{D}-kC^{+}\dot{C},\quad(k=b/J^{x}).\qquad\text{(E.10)}$$

Differentiating (E.10) we have

$$D\ddot{D}^{+}-kC\ddot{C}^{+}=D^{+}\ddot{D}-kC^{+}\ddot{C}.\qquad\text{(E.11)}$$

From (E.10) and (E.11) it follows that

$$(DD^{+}-kCC^{+})(D\ddot{D}^{+}-kC\ddot{C}^{+})-(D\dot{D}^{+}-kC\dot{C}^{+})^{2}$$
$$=(DD^{+}-kCC^{+})(D^{+}\ddot{D}-kC^{+}\ddot{C})-(D^{+}\dot{D}-kC^{+}\dot{C})^{2}.\qquad\text{(E.12)}$$

Working out (E.12) we have

$$D^{2}(D^{+}\ddot{D}^{+}-\dot{D}^{+2})+k^{2}C^{2}(C^{+}\ddot{C}^{+}-\dot{C}^{+2})-kC\,D(C^{+}\ddot{D}^{+}+D^{+}\ddot{C}^{+}-2\dot{C}^{+}\dot{D}^{+})$$
$$=D^{+2}(D\ddot{D}-\dot{D}^{2})+k^{2}C^{+2}(C\ddot{C}-\dot{C}^{2})-kC^{+}D^{+}(C\ddot{D}+D\ddot{C}-2\dot{C}\dot{D}).\qquad\text{(E.13)}$$

Inserting (E.3) and (E.4), and also (E.3) and (E.4) with $p+1$ instead of p, we obtain

$$b^{2}(D^{2}CC^{++}+C^{2}DD^{++})-kCD(C^{+}\ddot{D}^{+}+D^{+}\ddot{C}^{+}-2\dot{C}^{+}\dot{D}^{+})$$
$$=b^{2}(D^{+2}C^{-}C^{+}+C^{+2}D^{-}D^{+})-kC^{+}D^{+}(C\ddot{D}+D\ddot{C}-2\dot{C}\dot{D}),\qquad\text{(E.14)}$$

where

$$C^{++}=\langle S_{j+\frac12}^{x}(t)S_{j+\frac12+p+2}^{x}\rangle,\quad D^{++}=\langle S_{j}^{x}(t)S_{j+p+2}^{x}\rangle.\qquad\text{(E.15)}$$

Dividing (E.14) by $CDC^{+}D^{+}$, we immediately have

$$\frac{b}{J^{x}}h(p+1,t)=\frac{b}{J^{x}}h(p,t),\qquad\text{(E.16)}$$

so that

$$h(p,t)=h(0,0),\qquad\text{(E.17)}$$

i.e. $h(p,t)$ is independent of p and t, cf. (E.9). For $p=0$, $t=0$, we have

$$D=C=\tfrac14,\quad D^{-}=D^{+}=\tfrac12\mathrm{i}\langle\gamma_{2m}\gamma_{2m+1}\rangle,$$
$$C^{-}=C^{+}=-\tfrac12\mathrm{i}\langle\gamma_{2m}\gamma_{2m-1}\rangle,\quad\dot{D}=\tfrac12 b\langle\gamma_{2m}\gamma_{2m-1}\rangle,$$
$$\dot{C}=-\tfrac12 J^{x}\langle\gamma_{2m}\gamma_{2m+1}\rangle,\quad\ddot{D}=-\tfrac14 b^{2},\quad\ddot{C}=-\tfrac14 J^{x2},\qquad\text{(E.18)}$$

and from (E.18) and (E.1)

$$h(0,0)=b^{2}+J^{x2},\qquad\text{(E.19)}$$

leading with (E.17) to (6.34).

The second derivation is based on an application of the general Wick theorem (3.4). From (2.2) and (2.18), using also (2.3), it is straightforward to show that

$$S_j^x = 2\gamma_0 S_{j-\frac12}^x \gamma_{2j-1}, \quad S_{j+\frac12}^x = 2\mathrm{i} S_{j-\frac12}^x \gamma_{2j-1}\gamma_{2j},$$
$$S_{j+1}^x = 4\mathrm{i}\gamma_0 S_{j-\frac12}^x \gamma_{2j-1}\gamma_{2j}\gamma_{2j+1}, \quad S_k^x = -2\mathrm{i}\gamma_0 S_{k+\frac12}^x \gamma_{2k}, \tag{E.20}$$

for integer j and k, so that

$$\langle S_{j-\frac12}^x(t) S_{k+\frac12}^x\rangle\langle S_{j+1}^x(t) S_k^x\rangle$$
$$= -4\langle S_{j-\frac12}^x(t) S_{k+\frac12}^x\rangle\langle S_{j-\frac12}^x(t)\gamma_{2j-1}(t)\gamma_{2j}(t)\gamma_{2j+1}(t) S_{k+\frac12}^x\gamma_{2k}\rangle. \tag{E.21}$$

Applying (3.4) with $s=2$ and

$$Q_1 = \mathrm{e}^{-\beta\mathscr{H}}(\mathrm{Tr}\,\mathrm{e}^{-\beta\mathscr{H}})^{-1} S_{j-\frac12}^x(t), \quad Q_2 = Q_3 = \mathbb{1}, \quad Q_4 = S_{k+\frac12}^x,$$
$$\gamma_{i_1} = \gamma_{2j-1}(t), \quad \gamma_{i_2} = \gamma_{2j}(t), \quad \gamma_{i_3} = \gamma_{2j+1}(t), \quad \gamma_{i_4} = \gamma_{2k}, \tag{E.22}$$

we find

$$\langle S_{j-\frac12}^x(t) S_{k+\frac12}^x\rangle\langle S_{j+1}^x(t) S_k^x\rangle$$
$$= -4\langle S_{j-\frac12}^x(t)\gamma_{2j-1}(t)\gamma_{2j}(t) S_{k+\frac12}^x\rangle\langle S_{j-\frac12}^x(t)\gamma_{2j+1}(t) S_{k+\frac12}^x\gamma_{2k}\rangle$$
$$+4\langle S_{j-\frac12}^x(t)\gamma_{2j-1}(t)\gamma_{2j+1}(t) S_{k+\frac12}^x\rangle\langle S_{j-\frac12}^x(t)\gamma_{2j}(t) S_{k+\frac12}^x\gamma_{2k}\rangle$$
$$-4\langle S_{j-\frac12}^x(t)\gamma_{2j-1}(t) S_{k+\frac12}^x\gamma_{2k}\rangle\langle S_{j-\frac12}^x(t)\gamma_{2j}(t)\gamma_{2j+1}(t) S_{k+\frac12}^x\rangle. \tag{E.23}$$

Inserting the relations, cf. (2.2), (2.18) and (2.3),

$$S_{k+\frac12}^x\gamma_{2k} = \mathrm{i}\gamma_0 S_k^x, \quad S_{j-\frac12}^x\gamma_{2j-1} = \gamma_0 S_j^x, \quad S_{j-\frac12}^x\gamma_{2j} = \gamma_0 S_j^y,$$
$$S_{j-\frac12}^x\gamma_{2j+1} = -2\gamma_0 S_{j+1}^x S_j^z, \quad S_{j-\frac12}^x\gamma_{2j-1}\gamma_{2j} = -\tfrac12\mathrm{i} S_{j+\frac12}^x, \tag{E.24}$$
$$S_{j-\frac12}^x\gamma_{2j-1}\gamma_{2j+1} = -\tfrac12\mathrm{i} S_{j+\frac12}^y, \quad S_{j-\frac12}^x\gamma_{2j}\gamma_{2j+1} = \mathrm{i} S_{j-\frac12}^x S_{j+\frac12}^z,$$

in (E.23) we obtain the factorization property

$$\langle S_{j-\frac12}^x(t) S_{k+\frac12}^x\rangle\langle S_{j+1}^x(t) S_k^x\rangle + 2\langle S_{j+\frac12}^x(t) S_{k+\frac12}^x\rangle\langle S_j^z(t) S_{j+1}^x(t) S_k^x\rangle$$
$$+2\langle S_j^x(t) S_k^x\rangle\langle S_{j-\frac12}^x(t) S_{j+\frac12}^z(t) S_{k+\frac12}^x\rangle + \langle S_{j+\frac12}^y(t) S_{k+\frac12}^x\rangle\langle S_j^y(t) S_k^x\rangle = 0, \tag{E.25}$$

which is valid for the general inhomogeneous Ising chain as described by (2.1).

For the homogeneous Ising chain we can take into account eq. (6.30) and a similar relation for the three-spin correlation functions involving S^z. From (E.25) and an analogous relation which can be inferred from (E.25) replacing j and k by $j+\frac12$ and $k+\frac12$, respectively, we then obtain

$$bJ^x\{\langle S_{j+\frac12}^x(t) S_{k+\frac32}^x\rangle\langle S_j^x(t) S_{k-1}^x\rangle + \langle S_{j+\frac12}^x(t) S_{k-\frac12}^x\rangle\langle S_j^x(t) S_{k+1}^x\rangle\}$$
$$+2bJ^x\langle S_{j+\frac12}^x(t) S_{k+\frac12}^x\rangle\{\langle S_j^z(t) S_{j+1}^x(t) S_k^x\rangle + \langle S_{j-1}^x(t) S_j^z(t) S_k^x\rangle\}$$
$$+2bJ^x\langle S_j^x(t) S_k^x\rangle\{\langle S_{j+\frac12}^z(t) S_{j+\frac32}^x(t) S_{k+\frac12}^x\rangle + \langle S_{j-\frac12}^x(t) S_{j+\frac12}^z(t) S_{k+\frac12}^x\rangle\}$$
$$+2bJ^x\langle S_j^y(t) S_k^x\rangle\langle S_{j+\frac12}^y(t) S_{k+\frac12}^x\rangle = 0. \tag{E.26}$$

The averages involving S^y and S^z can be evaluated using the first and second time derivatives of $S^x(t)$ which for the homogeneous Ising model are given by

$$\frac{\mathrm{d}}{\mathrm{d}t} S_j^x(t) = bS_j^y(t), \quad \frac{\mathrm{d}}{\mathrm{d}t} S_{j+\frac12}^x(t) = J^x S_{j+\frac12}^y(t),$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} S_j^x(t) = -b^2 S_j^x(t) - 2J^x b\{S_j^z(t) S_{j+1}^x(t) + S_{j-1}^x(t) S_j^z(t)\}, \tag{E.27}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} S_{j+\frac12}^x(t) = -J^{x2} S_{j+\frac12}^x(t) - 2J^x b\{S_{j+\frac12}^z(t) S_{j+\frac32}^x(t) + S_{j-\frac12}^x(t) S_{j+\frac12}^z(t)\}.$$

Inserting (E.27) in (E.26) we obtain

$$J^x b\{\langle S_{j+\frac12}^x(t) S_{k+\frac32}^x\rangle\langle S_j^x(t) S_{k-1}^x\rangle + \langle S_{j+\frac12}^x(t) S_{k-\frac12}^x\rangle\langle S_j^x(t) S_{k+1}^x\rangle\}$$
$$-\langle S_{j+\frac12}^x(t) S_{k+\frac12}^x\rangle\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle S_j^x(t) S_k^x\rangle - \langle S_j^x(t) S_k^x\rangle\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle S_{j+\frac12}^x(t) S_{k+\frac12}^x\rangle$$
$$+2\left\{\frac{\mathrm{d}}{\mathrm{d}t}\langle S_j^x(t) S_k^x\rangle\right\}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\langle S_{j+\frac12}^x(t) S_{k+\frac12}^x\rangle\right\}$$
$$= (b^2 + J^{x2})\langle S_j^x(t) S_k^x\rangle\langle S_{j+\frac12}^x(t) S_{k+\frac12}^x\rangle, \tag{E.28}$$

which is equivalent to (6.34)

Appendix F

In this appendix we derive eq. (6.37). For convenience, introduce the short-hand notations, for terms appearing in (6.35),

$$Q(\pm) \equiv J^{x2}\frac{D_{p+1}D_{p-1}}{C_p^2} \pm b^2\frac{C_{p+1}C_{p-1}}{D_p^2}, \tag{F.1}$$

$$R(\pm) \equiv J^{x2}\frac{D_p^2}{C_p^2} \pm b^2\frac{C_p^2}{D_p^2}, \tag{F.2}$$

$$S \equiv bJ^x\frac{(C_{p+1}D_{p-1} + C_{p-1}D_{p+1})}{C_p D_p}, \tag{F.3}$$

Then one has

$$R(\pm) = |bJ^x|(G_p^{-2} \pm G_p^2), \tag{F.4}$$

$$\frac{Q(+)}{Q(-)} = \frac{1 + G_p^2 G_{p+1}G_{p-1}}{1 - G_p^2 G_{p+1}G_{p-1}}, \tag{F.5}$$

$$\frac{S}{Q(-)} = \frac{G_p(G_{p+1} + G_{p-1})}{1 - G_p^2 G_{p+1}G_{p-1}}, \tag{F.6}$$

$$Q(-) = R(-) + J^{x2}\left(\frac{D_{p+1}D_{p-1} - D_p^2}{C_p^2}\right) - b^2\left(\frac{C_{p+1}C_{p-1} - C_p^2}{D_p^2}\right)$$
$$= R(-) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln G_p\,, \tag{F.7}$$

where in the last step use is made of (6.31), (6.32) and (6.36). Therefore, all terms in (6.35) can be expressed in terms of the ratio G_p, i.e.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln G_p\right)^2 = S - (J^{x2} + b^2) - Q(+) + R(+)$$
$$= R(+) - J^{x2} - b^2 + \frac{(S - Q(+))}{Q(-)}\left\{R(-) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln G_p\right\}$$
$$= |bJ^x|(G_p^{-2} + G_p^2) - J^{x2} - b^2$$
$$+ \frac{\{G_p(G_{p+1} + G_{p-1}) - 1 - G_p^2G_{p+1}G_{p-1}\}\left\{|bJ^x|(G_p^{-2} - G_p^2) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln G_p\right\}}{(1 - G_p^2G_{p+1}G_{p-1})}\,, \tag{F.8}$$

which is eq. (6.37).

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